Some formulas for π

Orges Leka

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1 Introduction

In this note, we will describe a way to generate formulas for the circle number π by using positive definite kernels on the natural numbers as an example.

Motivation: Let $a = \cos(\alpha), b = \cos(\beta), c = \cos(\gamma)$ where α, β, γ are the angles in a triangle. Then by

$$\alpha + \beta + \gamma = \pi$$

and using $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, $\sin(\text{acos}(x) = \sqrt{1 - x^2}$ we find that the a, b, c are points on the surface:

$$a^2 + b^2 + c^2 + 2abc - 1 = 0$$

which is the Cayley's nodal cubic surface.

Since every three point metric space can be isometrically emdedded in $\mathbb{R}^2$, we can build the possibly to a line degenerated triangle from these three points:

Using the law of cosines to define angles, given distances, we find that the quantities:

$$a := \frac{d(y,z)^2 + d(y,x)^2 - d(x,z)^2}{2d(y,z)d(x,y)},$$

$$b := \frac{d(y,z)^2 + d(z,x)^2 - d(x,y)^2}{2d(y,z)d(z,x)},$$

$$c := \frac{d(x,z)^2 + d(y,x)^2 - d(y,z)^2}{2d(x,z)d(x,y)}$$

satisfy by what was given above the following equation:

$$a^2 + b^2 + c^2 + 2abc - 1 = 0$$

hence are points on the Cayley's nodal cubic surface.
For instance for the metric on natural numbers , where σ is the sum of divisors function,

$$d(x,y)=\sqrt{\sigma(x)+\sigma(y)-2\sigma(\gcd(x,y))}$$

and for three pairwise distinct primes p,q,r we get the following nice formula:

$$\pi=\mathrm{acos}\left(\frac{r}{\sqrt{(p+r)(q+r)}}\right)+\mathrm{acos}\left(\frac{q}{\sqrt{(p+q)(q+r)}}\right)+\mathrm{acos}\left(\frac{p}{\sqrt{(p+r)(p+q)}}\right)$$

Setting

$$a=\frac{r}{\sqrt{(p+r)(q+r)}},b=\frac{q}{\sqrt{(p+q)(q+r)}},c=\frac{p}{\sqrt{(p+r)(p+q)}}$$

we see that those are points on the Cayley nodal cubic surface.
Let $0<x<y<z$. Then we have:

$$\pi=\arccos\left(-\frac{\frac{x^2}{y^2}+\frac{x^2}{z^2}-\frac{y^2}{z^2}-1}{2\sqrt{-\frac{x^2}{y^2}+1}\sqrt{-\frac{x^2}{z^2}+1}}\right)+\arccos\left(-\frac{\frac{x^2}{y^2}-\frac{x^2}{z^2}+\frac{y^2}{z^2}-1}{2\sqrt{-\frac{x^2}{y^2}+1}\sqrt{-\frac{y^2}{z^2}+1}}\right)+\arccos\left(\frac{\frac{x^2}{y^2}-\frac{x^2}{z^2}-\frac{y^2}{z^2}+1}{2\sqrt{-\frac{x^2}{z^2}+1}\sqrt{-\frac{y^2}{z^2}+1}}\right)$$

Here are some formulas I derived using the technique above, which I find amusing:

$$\pi=\arccos\left(\frac{1}{12}\sqrt{5}\sqrt{3}\right)+\arccos\left(\frac{5}{32}\sqrt{5}\sqrt{2}\right)+\arccos\left(\frac{13}{48}\sqrt{3}\sqrt{2}\right)$$

$$\pi=\arccos\left(\frac{1}{24}\sqrt{7}\sqrt{5}\right)+\arccos\left(\frac{13}{108}\sqrt{7}\sqrt{3}\right)+\arccos\left(\frac{25}{144}\sqrt{5}\sqrt{3}\right)$$

$$\pi=\arccos\left(\frac{75}{128}\right)+\arccos\left(\frac{41}{160}\sqrt{7}\right)+\arccos\left(\frac{3}{40}\sqrt{7}\right)$$

$$\pi=\arccos\left(\frac{41}{500}\sqrt{11}\sqrt{5}\right)+\arccos\left(\frac{1}{20}\sqrt{11}\right)+\arccos\left(\frac{61}{200}\sqrt{5}\right)$$

$$\pi=\arccos\left(\frac{1}{84}\sqrt{13}\sqrt{11}\right)+\arccos\left(\frac{61}{864}\sqrt{13}\sqrt{6}\right)+\arccos\left(\frac{85}{1008}\sqrt{11}\sqrt{6}\right)$$

$$\pi = \arccos\left(\frac{1}{112}\sqrt{15}\sqrt{13}\right) + \arccos\left(\frac{85}{1372}\sqrt{15}\sqrt{7}\right) + \arccos\left(\frac{113}{1568}\sqrt{13}\sqrt{7}\right)$$

$$\pi = \arccos\left(\frac{1}{144}\sqrt{17}\sqrt{15}\right) + \arccos\left(\frac{113}{1024}\sqrt{17}\sqrt{2}\right) + \arccos\left(\frac{145}{1152}\sqrt{15}\sqrt{2}\right)$$

$$\pi = \arccos\left(\frac{1}{180}\sqrt{19}\sqrt{17}\right) + \arccos\left(\frac{145}{972}\sqrt{19}\right) + \arccos\left(\frac{181}{1080}\sqrt{17}\right)$$

$$\pi = \arccos\left(\frac{1}{220}\sqrt{21}\sqrt{19}\right) + \arccos\left(\frac{181}{4000}\sqrt{21}\sqrt{10}\right) + \arccos\left(\frac{221}{4400}\sqrt{19}\sqrt{10}\right)$$

The proof is based on noticing that every 3 point metric space can be embedded in $\mathbb{R}^2$ as a triangle and then using trigonometry.
The metric space I am considering is a Hilbert space:

$$k(x,y) = \frac{\min(x,y)^2}{\max(x,y)^2}$$

with metric:

$$d(x,y) = \sqrt{2(1-k(x,y))}$$

For three points x, y, z in a metric space, we can define (using the law of cosines) the following quantity:

$$s(x,y,z) = \frac{d(x,y)^2 + d(y,z)^2 - d(x,z)^2}{2d(x,y)d(y,z)}$$

Then we can embedd $X \times X \times X$ to the Cayleys surface through the mapping:

$$f(x,y,z) = (s(x,y,z), s(z,x,y), s(y,z,x))$$

We then have:

$$\pi = \arccos(s(x,y,z)) + \arccos(s(z,x,y)) + \arccos(s(y,z,x))$$

which proves the claim.
Some formulas generated this way are:

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\left(-\frac{4}{65}\sqrt{13}\sqrt{5}\right)^{2k+1} + \left(\frac{17}{26}\sqrt{2}\right)^{2k+1} + \left(\frac{9}{130}\sqrt{13}\sqrt{5}\sqrt{2}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\left(-\frac{261}{47965}\sqrt{181}\sqrt{53}\sqrt{2}\right)^{2k+1} + \left(\frac{2071}{47965}\sqrt{53}\sqrt{5}\sqrt{2}\right)^{2k+1} + \left(\frac{1309}{47965}\sqrt{181}\sqrt{5}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{931}{96050}\sqrt{113}\sqrt{85}\right)^{2k+1}+\left(\frac{2061}{19210}\sqrt{85}\right)^{2k+1}+\left(\frac{1781}{19210}\sqrt{113}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{19}{2210}\sqrt{85}\sqrt{65}\sqrt{2}\right)^{2k+1}+\left(\frac{32}{1105}\sqrt{85}\sqrt{13}\right)^{2k+1}+\left(\frac{53}{2210}\sqrt{65}\sqrt{13}\sqrt{2}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{96}{12665}\sqrt{149}\sqrt{85}\right)^{2k+1}+\left(\frac{437}{25330}\sqrt{85}\sqrt{17}\sqrt{2}\right)^{2k+1}+\left(\frac{351}{25330}\sqrt{149}\sqrt{17}\sqrt{2}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{283}{3770}\sqrt{145}\right)^{2k+1}+\left(\frac{1037}{3770}\sqrt{13}\right)^{2k+1}+\left(\frac{413}{18850}\sqrt{145}\sqrt{13}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{464}{3485}\sqrt{41}\right)^{2k+1}+\left(\frac{1161}{6970}\sqrt{17}\sqrt{2}\right)^{2k+1}+\left(\frac{889}{34850}\sqrt{41}\sqrt{17}\sqrt{2}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{704}{40001}\sqrt{181}\sqrt{13}\right)^{2k+1}+\left(\frac{3781}{80002}\sqrt{17}\sqrt{13}\sqrt{2}\right)^{2k+1}+\left(\frac{925}{80002}\sqrt{181}\sqrt{17}\sqrt{2}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{320}{42601}\sqrt{145}\sqrt{113}\right)^{2k+1}+\left(\frac{697}{85202}\sqrt{113}\sqrt{65}\sqrt{2}\right)^{2k+1}+\left(\frac{3069}{426010}\sqrt{145}\sqrt{65}\sqrt{2}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{18751}{1380634}\sqrt{113}\sqrt{41}\right)^{2k+1}+\left(\frac{10323}{1380634}\sqrt{149}\sqrt{113}\right)^{2k+1}+\left(\frac{17475}{1380634}\sqrt{149}\sqrt{41}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{21131}{3505970}\sqrt{181}\sqrt{65}\sqrt{2}\right)^{2k+1}+\left(\frac{25023}{3505970}\sqrt{149}\sqrt{65}\sqrt{2}\right)^{2k+1}+\left(\frac{10272}{1752985}\sqrt{181}\sqrt{149}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{12879}{2152090}\sqrt{181}\sqrt{145}\right)^{2k+1}+\left(\frac{27721}{2152090}\sqrt{145}\sqrt{41}\right)^{2k+1}+\left(\frac{24769}{2152090}\sqrt{181}\sqrt{41}\right)^{2k+1}}{4^k(2k+1)}$$

Other formulas are:

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{1}{21}\sqrt{7}\sqrt{3}\right)^{2k+1}+\left(\frac{4}{21}\sqrt{7}\sqrt{3}\right)^{2k+1}+\left(\frac{2}{3}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{1}{26}\right)^{2k+1}+\left(\frac{3}{26}\sqrt{13}\sqrt{3}\right)^{2k+1}+\left(\frac{3}{26}\sqrt{13}\sqrt{3}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(\frac{1}{133}\sqrt{19}\sqrt{3}\right)^{2k+1}+\left(\frac{31}{399}\sqrt{21}\sqrt{3}\right)^{2k+1}+\left(\frac{5}{133}\sqrt{21}\sqrt{19}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2\cdot\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\left(-\frac{1}{91}\sqrt{13}\sqrt{7}\right)^{2k+1}+\left(\frac{9}{91}\sqrt{13}\sqrt{7}\right)^{2k+1}+\left(\frac{3}{7}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\left(\frac{1}{399}\sqrt{19}\sqrt{7}\right)^{2k+1} + \left(\frac{4}{57}\sqrt{21}\sqrt{7}\right)^{2k+1} + \left(\frac{10}{399}\sqrt{21}\sqrt{19}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\left(-\frac{1}{273}\sqrt{21}\sqrt{13}\right)^{2k+1} + \left(\frac{16}{273}\sqrt{21}\sqrt{13}\right)^{2k+1} + \left(\frac{4}{13}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\left(-\frac{31}{546}\sqrt{13}\sqrt{7}\right)^{2k+1} + \left(\frac{109}{546}\sqrt{7}\sqrt{3}\right)^{2k+1} + \left(\frac{73}{546}\sqrt{13}\sqrt{3}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\left(-\frac{4}{133}\sqrt{19}\sqrt{7}\right)^{2k+1} + \left(\frac{9}{133}\sqrt{19}\sqrt{7}\right)^{2k+1} + \left(\frac{6}{7}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\left(-\frac{37}{798}\sqrt{21}\sqrt{3}\right)^{2k+1} + \left(\frac{101}{798}\sqrt{19}\sqrt{3}\right)^{2k+1} + \left(\frac{25}{798}\sqrt{21}\sqrt{19}\right)^{2k+1}}{4^k(2k+1)}$$

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\left(-\frac{223}{5187}\sqrt{21}\sqrt{13}\right)^{2k+1} + \left(\frac{311}{5187}\sqrt{19}\sqrt{13}\right)^{2k+1} + \left(\frac{235}{5187}\sqrt{21}\sqrt{19}\right)^{2k+1}}{4^k(2k+1)}$$

Collection of similarities and positive definite kernels over the natural numbers

Let $k = s$ be a positive definite symmetric function and a simililarity over the natural numbers such that $k(a, a) = 1, k(ac, bc) = k(a, b)$ for all natural numbers a, b, c.

A similarity $s : X \times X \to \mathbb{R}$ is defined in the Encyclopedia of Distances as:

1) $s(x, y) \geq 0 \forall x, y \in X$
2) $s(x, y) = s(y, x) \forall x, y \in X$
3) $s(x, y) \leq s(x, x) \forall x, x \in X$
4) $s(x, y) = s(x, x) \iff x = y$

For example $k(a, b) = \frac{2\gcd(a,b)}{a+b}$ or $k(a, b) = \frac{\gcd(a,b)^2}{ab}$ and other examples are given below:

For each natural number n let X_n be a finite subset of the natural numbers, such that $X_n = X_m \iff n = m$. Such subsets are given for example through:

$$X_n := \{(d, i) : d|n, 0 \leq i \leq d - 1\}$$

with $|X_n| = \sigma(n)$ and $X_n \cap X_m = X_{\gcd(m,n)}$. Another subsets are given by:

$$X_a := \{k/a | 1 \leq k \leq a\}$$

with $|X_a| = a, X_a \cap X_b = X_{\gcd(a,b)}$.

- $s(x, y) := k(x, y) := \frac{\min(x,y)}{\max(x,y)} =$ a Jaccard similarity

- $s_{Si}(a,b) = \frac{\gcd(a,b)}{\min(a,b)}$ = Simpson similarity
- $s_{BB}(a,b) = \frac{\gcd(a,b)}{\max(a,b)}$ = Braun,Blanquet similarity
- $s_J(a,b) = \frac{\gcd(a,b)}{a+b-\gcd(a,b)}$ = Jaccard similarity
- $s_S(a,b) = \frac{2\gcd(a,b)}{a+b}$ = Sorensen similarity
- $s_{Cos}(a,b) = \frac{\gcd(a,b)^2}{ab}$ = Squared Cosine similarity
- $k(a,b) = \frac{\sigma(\gcd(a,b))}{\max(\sigma(a),\sigma(b))}$ = a Braun,Blanquet similarity
- $k(a,b) = \frac{|X_a \cap X_b|}{\max(|X_a|,|X_b|)}$ = a Braun,Blanquet similarity
- $k(a,b) = 2\frac{|X_a \cap X_b|}{|X_a|+|X_b|}$ = a Sorensen similarity
- $k(a,b) = \frac{|X_a \cap X_b|^2}{|X_a||X_b|}$ = a squared Cosine similarity
- $k(a,b) = \frac{|X_a \cap X_b|^2}{\min(|X_a|,|X_b|)}$ = an overlap / Simpson similarity