Some inequalities similar to Lagarias inequality

Orges Leka
Holzheimerstrasse 25
65549 Limburg
Germany
orges.leka@gmail.com

Abstract

Let $\sigma_r(n)$ denote the sum of d^r divisors of n and ϕ the Euler totient function. We prove the following $\sigma_{r-1}(n) = \frac{1}{n}\sum_{d|n}\sigma_r(d)\phi(\frac{n}{d})$. Furthermore we show using the inequality of Lagarias, that some other inequalities are equivalent to Riemann Hypothesis.

1 Introduction

Let $H_n = \sum_{k=1}^{n} 1/k$ be the n-th harmonic number. Let $\sigma_r(n) = \sum_{d|n} d^r$ be a sum of divisors of n. Let $U_1(n) := H_n + \exp(H_n)\log(H_n)$ and for $s = 0, 1, 2, 3\cdots$ and let

$$U_{-s}(n) := \frac{1}{n}(U_{-s+1}(n) + \sum_{d<n, d|n} \sigma_{-s+1}(d)\phi(\frac{n}{d})) \quad (1)$$

We show in the following that for all $s = -1, 0, 1, 2, \cdots$ we have

$$\sigma_{-s}(n) \le U_{-s}(n) \forall n \in \mathbb{N} \iff \text{Riemann Hypothesis is true} \quad (2)$$

Theorem 1. *We have* $\sigma_{r-1}(n) = \frac{1}{n}\sum_{d|n}\sigma_r(d)\phi(\frac{n}{d})$ *for every natural number* n *and every* $r \in \mathbb{R}$.

Proof. First notice that $\frac{1}{n}\sum_{d|n}\sigma_r(d)\phi(\frac{n}{d}) = \sum_{1\le l\le n}\sigma_r(\gcd(n,l))$ Then it follows that

$$\sum_{1\le l\le n}\sigma_r(\gcd(n,l)) = \sum_{1\le l\le n}\sum_{d|\gcd(n,r)} d^r = \sum_{d|n} d^r\left(\sum_{1\le l\le n, l \text{ multiple of } d} 1\right) = \sum_{d|n} d^r\cdot\frac{n}{d} = n\sigma_{r-1}(n) \tag{3}$$

□

Theorem 2. *For all $s=-1,0,1,2,\cdots$ the following equivalence is true*

$$\sigma_{-s}(n)\le U_{-s}(n)\forall n\in\mathbb{N} \iff \textit{Riemann Hypothesis is true} \tag{4}$$

Proof. We prove this by induction on s. From Lagarias [1] it is known, that Riemann Hypothesis is equivalent to

$$\sigma_1(n)\le H_n+\exp(H_n)\log(H_n)=U_1(n)\forall n\in\mathbb{N} \tag{5}$$

hence the case $s=-1$ is shown. Suppose by induction on s that we have shown

$$\sigma_{-s+1}(n)\le U_{-s+1}(n)\forall n\in\mathbb{N} \iff \text{Riemann Hypothesis is true} \tag{6}$$

We have to show that

$$\sigma_{-s}(n)\le U_{-s}(n)\forall n\in\mathbb{N} \iff \text{Riemann Hypothesis is true} \tag{7}$$

From Theorem 1 we have

$$\sigma_{-s}(n)=\frac{1}{n}\sum_{d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d})=\frac{\sigma_{-s+1}(n)}{n}+\frac{1}{n}\sum_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d}) \tag{8}$$

which is equivalent to

$$\sigma_{-s+1}(n)=n\sigma_{-s}(n)-\sum_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d}) \tag{9}$$

" $\implies$ " Let $\sigma_{-s}(n)\le U_{-s}(n)\forall n\in\mathbb{N}$. Then we have by (9) the following

$$\sigma_{-s+1}(n)=n\sigma_{-s}(n)-\sum_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d})\le nU_{-s}(n)-\sum_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d}) \tag{10}$$

which is equivalent to

$$U_{-s}(n)\ge\frac{1}{n}\left(\sigma_{-s+1}(n)+\sigma_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d})\right) \tag{11}$$

But by definition (1) of $U_{-s}(n)$ we have

$$U_{-s}(n)=\frac{1}{n}(U_{-s+1}(n)+\sum_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d})) \tag{12}$$

which means that by (11)

$$\frac{1}{n}(U_{-s+1}(n)+\sum_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d})) = U_{-s}(n) \geq \frac{1}{n}\left(\sigma_{-s+1}(n)+\sum_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d})\right) \quad (13)$$

Solving for $U_{-s+1}(n)$ and $\sigma_{-s+1}(n)$ we find that $U_{-s+1}(n) \geq \sigma_{-s+1}(n) \forall n \in \mathbb{N}$ from which by (6) the Riemann Hypothesis follows.

" $\Longleftarrow$ " Let the Riemann Hypothesis be true, which means that by (6) we have $\sigma_{-s+1}(n) \leq U_{-s+1}(n) \forall n \in \mathbb{N}$. We have to show that $\sigma_{-s}(n) \leq U_{-s}(n) \forall n \in \mathbb{N}$. By (9) we have with $U_{-s+1}(n) \geq \sigma_{-s+1}(n)$ also

$$U_{-s+1}(n) \geq \sigma_{-s+1}(n) = n\sigma_{-s}(n) - \sum_{d<n,d|n}\sigma_{-s+1}(n)\phi(\frac{n}{d}) \quad (14)$$

Solving (14) for $\sigma_{-s}(n)$ we get

$$\sigma_{-s}(n) \leq \frac{1}{n}\left(U_{-s+1}(n)+\sum_{d<n,d|n}\sigma_{-s+1}(d)\phi(\frac{n}{d})\right) = U_{-s}(n) \quad (15)$$

and the inequality is shown.

□

2 Acknowledgment

I would like to thank Jean-Louis Nicolas and Kevin Broughan for useful comments and suggestions.

References

[1] Jeffrey C. Lagarias, *An Elementary Problem Equivalent to the Riemman Hypothesis*, Amer. Math. Monthly 109 (2002), 534–543

2010 *Mathematics Subject Classification*: Primary 11A99.
Keywords: Riemman hypothesis, upper bound, sum of divisors, inequality