A Lorentzian Construction in Dimension 88 and Infinitely Many Further Ranks

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March 31, 2026

Abstract

We study a Conway-style Lorentzian construction attached to identities of the form

$$A^2 + (A+1)^2 + \cdots + B^2 = X^2.$$

First, we treat in detail the explicit identity

$$192^2 + 193^2 + \cdots + 279^2 = 2222^2,$$

which yields a primitive isotropic vector in the even unimodular Lorentzian lattice $II_{89,1}$. The quotient $w^{\perp}/\mathbb{Z}w$ is therefore a positive definite even unimodular lattice of rank 88. We record a compact basis, compute its theta series, and determine its modular form decomposition in weight 44. Second, using a congruence theorem of V. Pletser on sums of consecutive squares, we prove that there are infinitely many distinct integers $D \equiv 0 \pmod 8$ for which such an identity exists. Consequently, Conway's Lorentzian construction produces infinitely many pairwise non-isomorphic positive definite even unimodular lattices. We also list ten explicit examples.

Contents

1 Introduction

A classical identity

$$1^2 + 2^2 + \cdots + 24^2 = 70^2$$

underlies Conway's Lorentzian construction of the Leech lattice. The basic mechanism is to package such an identity into a primitive isotropic vector in an even unimodular Lorentzian lattice and then pass to the quotient of its orthogonal complement by the isotropic line.

The first goal of this note is to carry out this construction for the length

$$D = 88,$$

using the identity

$$192^2 + 193^2 + \cdots + 279^2 = 2222^2.$$

This yields a positive definite even unimodular lattice L_{88} of rank 88. We then determine the beginning of its theta series and its exact modular decomposition.

The second goal is conceptual. One should distinguish very carefully between the following two assertions.

- For a *fixed* length D, there may be infinitely many solutions (A, B, X). This is the Pell-type direction and stays in the same rank.
- There are infinitely many *different* lengths $D \equiv 0 \pmod 8$ for which at least one solution exists. This produces infinitely many lattices of different ranks.

For the second statement one can invoke a theorem of V. Pletser on congruence classes of admissible lengths. We then obtain infinitely many pairwise non-isomorphic even unimodular lattices from the Lorentzian construction, simply because their ranks are different.

2 The Lorentzian construction

For an integer $D \equiv 0 \pmod 8$ we consider the standard even unimodular Lorentzian lattice $II_{D+1,1}$. A convenient model is

$$II_{D+1,1} = \left\{(x_0, x_1, \ldots, x_D; y) \in \mathbb{Z}^{D+2} : x_0 + x_1 + \cdots + x_D + y \equiv 0 \pmod 2\right\},$$

equipped with the quadratic form

$$Q(x_0, \ldots, x_D; y) = x_0^2 + x_1^2 + \cdots + x_D^2 - y^2.$$

Lemma 2.1. *Let $\Lambda = II_{D+1,1}$, and let $w \in \Lambda$ be primitive and isotropic. Then*

$$L := w^{\perp}/\mathbb{Z}w$$

is a positive definite even unimodular lattice of rank D.

Proof. Because w is primitive and Λ is unimodular, there exists $u \in \Lambda$ with $(u, w) = 1$. Since Λ is even, $(u, u) \in 2\mathbb{Z}$, hence

$$u' := u - \frac{(u, u)}{2}\, w \in \Lambda$$

satisfies

$$(u', w) = 1, \qquad (u', u') = 0.$$

Thus $U := \mathbb{Z}u' \oplus \mathbb{Z}w$ is a copy of the hyperbolic plane. Let

$$K := U^{\perp} \subset \Lambda.$$

Because Λ is even unimodular and U is unimodular, K is again even and unimodular. Its rank is

$$\mathrm{rk}(K) = \mathrm{rk}(\Lambda) - 2 = (D + 2) - 2 = D.$$

Since U has signature $(1, 1)$ and Λ has signature $(D + 1, 1)$, the orthogonal complement K is positive definite.

Now every $x \in w^\perp$ can be written uniquely as

$$x = \alpha u' + \beta w + k \qquad (\alpha, \beta \in \mathbb{Z},\ k \in K),$$

and the condition $(x, w) = 0$ forces $\alpha = 0$. Hence

$$w^\perp = \mathbb{Z}w \oplus K.$$

Therefore

$$w^\perp/\mathbb{Z}w \cong K,$$

so $w^\perp/\mathbb{Z}w$ is positive definite, even, unimodular, and of rank D. □

3 The explicit rank-88 example

Take

$$A = 192, \qquad B = 279, \qquad X = 2222.$$

Then

$$B - A + 1 = 88$$

and a direct computation gives

$$192^2 + 193^2 + \cdots + 279^2 = 2222^2.$$

Define

$$w_{88} = (0, 192, 193, \ldots, 279; 2222) \in \mathbb{Z}^{90}.$$

Proposition 3.1. *The vector w_{88} is primitive, isotropic, and belongs to $II_{89,1}$. Consequently*

$$L_{88} := w_{88}^\perp/\mathbb{Z}w_{88}$$

is a positive definite even unimodular lattice of rank 88*.*

Proof. Its Lorentz norm is

$$Q(w_{88}) = 0^2 + 192^2 + 193^2 + \cdots + 279^2 - 2222^2 = 0,$$

so w_{88} is isotropic.

It is primitive because among its nonzero coordinates appear two consecutive integers, namely 192 and 193, whose greatest common divisor is 1.

Finally, modulo 2 one has $n^2 \equiv n$. Therefore

$$192 + 193 + \cdots + 279 \equiv 2222 \pmod 2,$$

hence

$$0 + 192 + 193 + \cdots + 279 + 2222$$

is even. So $w_{88} \in II_{89,1}$. The claim now follows from the lemma. □

For exact computations one should work in a fixed explicit basis of the ambient Lorentzian lattice and then compute a quotient basis by integral row reduction. For general rank D, let

$$g = \left(\tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}\right), \qquad \delta_j = e_{j+1} - e_{j+2} \quad (1 \le j \le D), \qquad h = e_{D+1} + e_{D+2}.$$

In the ordered basis

$$[g, \delta_1, \ldots, \delta_D, h]$$

of $II_{D+1,1}$, the ambient Gram matrix is

$$M_D = \begin{pmatrix} \frac{D}{4} & 0 & \cdots & 0 & 1 & \\ 0 & 2 & -1 & & 0 & \\ \vdots & -1 & \ddots & \ddots & \vdots & \\ 0 & & \ddots & 2 & -1 & \\ 0 & \cdots & 0 & -1 & 0 & 2 \\ 1 & 0 & \cdots & 0 & 2 & 0 \end{pmatrix},$$

where the tridiagonal block is indexed by $\delta_1, \dots, \delta_D$, so that

$$(\delta_j, \delta_j) = 2 \ (1 \le j \le D-1), \qquad (\delta_D, \delta_D) = 0,$$

$$(\delta_j, \delta_{j+1}) = -1 \ (1 \le j \le D-1), \qquad (\delta_{D-1}, h) = -1, \qquad (\delta_D, h) = 2, \qquad (g, h) = 1,$$

and all remaining pairings are zero.

Now let

$$w = (0, A, A+1, \dots, B; X), \qquad D = B - A + 1,$$

and write w in the basis $[g, \delta_1, \dots, \delta_D, h]$:

$$w = \sum_{i=0}^{D+1} c_i b_i, \qquad c = (c_0, \dots, c_{D+1}) \in \mathbb{Z}^{D+2}.$$

Set

$$r := c^T M_D \in \mathbb{Z}^{D+2}.$$

Because w is primitive, the row vector r is primitive. Choose

$$U \in \mathrm{GL}_{D+2}(\mathbb{Z})$$

with

$$rU = (1, 0, \dots, 0).$$

Then the last $D+1$ columns of U form a basis of $w^\perp$. Next express w in this basis,

$$w = \sum_{j=1}^{D+1} c'_j k_j, \qquad c' = (c'_1, \dots, c'_{D+1}) \in \mathbb{Z}^{D+1},$$

and choose

$$V \in \mathrm{GL}_{D+1}(\mathbb{Z})$$

with

$$c'V = (1, 0, \dots, 0).$$

Then the last D columns of the transformed basis matrix give a basis of the quotient lattice

$$L_D = w^\perp / \mathbb{Z}w.$$

This is the general algorithm used in the computer calculations.

In the explicit rank-88 example, one convenient quotient basis obtained in this way is

$$b_1 = g + 11473\delta_1, \qquad b_2 = \delta_1 + 2\delta_2 - 3\delta_3,$$

$$b_j = -\delta_j + \delta_{j+1} \qquad (3 \le j \le 86),$$

$$b_{87} = 2501\delta_{87} + \delta_{88}, \qquad b_{88} = -1943\delta_{87} + h.$$

This basis, together with the corrected ambient matrix M_{88}, was used for the exact Gram-matrix and theta-series computations reported below.

4 The theta series of L_{88}

Write

$$\Theta_{L_{88}}(\tau) = \sum_{v \in L_{88}} q^{(v,v)/2}, \qquad q = e^{2\pi i \tau}.$$

Since L_{88} is even unimodular of rank 88, $\Theta_{L_{88}}$ is a modular form of weight 44 for $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 4.1. *The theta series of L_{88} begins*

$$\Theta_{L_{88}}(\tau) = 1+125944q^2+98241758q^3+47841024480q^4+16416521193560q^5+8153616049497120q^6+\cdots.$$

More precisely,

$$\Theta_{L_{88}} = E_4^{11} - 2640\, E_4^8\Delta + 1939624\, E_4^5\Delta^2 - 291837730\, E_4^2\Delta^3.$$

Proof. The graded dimensions

$$a_n := \#\{v \in L_{88} : (v,v) = 2n\}$$

were computed exactly for $n = 1, 2, 3$. After LLL reduction of the Gram matrix, PARI/GP returns

$$a_1 = 0, \qquad a_2 = 125944, \qquad a_3 = 98241758.$$

Hence

$$\Theta_{L_{88}}(\tau) = 1 + 125944q^2 + 98241758q^3 + \cdots.$$

Now $M_{44}(\mathrm{SL}_2(\mathbb{Z}))$ has dimension 4, because the graded ring of level-one modular forms is $\mathbb{C}[E_4, E_6]$, and the equation

$$4a + 6b = 44$$

has exactly four nonnegative solutions. A convenient basis is

$$E_4^{11}, \qquad E_4^8\Delta, \qquad E_4^5\Delta^2, \qquad E_4^2\Delta^3.$$

Writing

$$\Theta_{L_{88}} = E_4^{11} + A\, E_4^8\Delta + B\, E_4^5\Delta^2 + C\, E_4^2\Delta^3,$$

the coefficients of q^0, q^1, q^2, q^3 determine A, B, C uniquely. Solving the resulting linear system gives

$$A = -2640, \qquad B = 1939624, \qquad C = -291837730.$$

This proves the stated modular decomposition. □

Remark 4.2. The cuspidal part of $\Theta_{L_{88}}$ lies in $S_{44}(\mathrm{SL}_2(\mathbb{Z}))$, which is 3-dimensional. In Sage one finds that this cuspidal space is generated over $\mathbb{Q}$ by a single newform orbit with cubic coefficient field. The T_2-eigenvalue a_2 of that orbit has minimal polynomial

$$x^3 + 2209944x^2 - 15663522502656x - 19976984434430705664.$$

5 Infinitely many admissible ranks

The key arithmetic input is the following theorem of V. Pletser.

Theorem 5.1 (Pletser)**.** *The equation*

$$A^2 + (A+1)^2 + \cdots + B^2 = X^2$$

has solutions in integers whenever the number of terms

$$D = B - A + 1$$

satisfies

$$D \equiv 0, 9, 24, 33 \pmod{72}, \qquad D \equiv 1, 2, 16 \pmod{24}, \qquad D \equiv 11 \pmod{12}.$$

In particular, solutions exist for infinitely many

$$D \equiv 24 \pmod{72} \qquad \text{and} \qquad D \equiv 16 \pmod{24},$$

hence for infinitely many $D \equiv 0 \pmod 8$.

Theorem 5.2. *There exist infinitely many pairwise non-isomorphic positive definite even unimodular lattices produced by the Conway Lorentzian construction.*

Proof. By Pletser's theorem there are infinitely many different integers $D \equiv 0 \pmod 8$ for which there exists at least one solution $(A, B, X) \in \mathbb{N}^3$ of

$$A^2 + (A+1)^2 + \cdots + B^2 = X^2, \qquad D = B - A + 1.$$

Fix such a D and one corresponding solution.

Define

$$w = (0, A, A+1, \ldots, B; X).$$

Exactly as in the rank-88 case, w is isotropic because its norm equals

$$A^2 + (A+1)^2 + \cdots + B^2 - X^2 = 0.$$

It is primitive because two consecutive integers occur among its coordinates. It belongs to $II_{D+1,1}$ because modulo 2 one has $n^2 \equiv n$, hence

$$A + (A+1) + \cdots + B \equiv X \pmod 2,$$

so the total coordinate sum of w is even.

Therefore

$$L_D := w^{\perp}/\mathbb{Z}w$$

is a positive definite even unimodular lattice of rank D.

If $D_1 \neq D_2$, then L_{D_1} and L_{D_2} have different ranks and therefore cannot be isomorphic. Since there are infinitely many such D, we obtain infinitely many pairwise non-isomorphic lattices. □

The point of this theorem is that it is *not* a Pell-type statement for fixed D. Such Pell families stay in one and the same rank. Here the admissible lengths D themselves vary, and hence the resulting lattices are automatically pairwise non-isomorphic.

Ten explicit examples

The following ten examples were verified directly:

D	A	B	X
24	1	24	70
88	192	279	2222
96	13	108	652
184	7	190	1518
312	15	326	3406
352	280	631	8756
376	210	585	7990
568	443	1010	17750
600	25	624	9010
856	8617	9472	264718

In each case $D = B - A + 1$ is divisible by 8, and the corresponding vector

$$w = (0, A, A + 1, \ldots, B; X)$$

produces a positive definite even unimodular lattice of rank D.

6 Conclusion

The Conway-style Lorentzian construction is remarkably flexible. The explicit identity

$$192^2 + 193^2 + \cdots + 279^2 = 2222^2$$

gives a concrete even unimodular lattice L_{88} of rank 88, and its theta series can be determined exactly. At the same time, Pletser's congruence theorem shows that the construction is not confined to isolated examples such as $D = 24$ or $D = 88$: there are infinitely many admissible lengths $D \equiv 0 \pmod 8$, and therefore infinitely many pairwise non-isomorphic positive definite even unimodular lattices arising from primitive isotropic vectors of the same general shape.

This suggests several natural directions for further work. One may try to determine more systematically the automorphism groups of these lattices, to search for frame or code structures, and to investigate whether special examples, such as the rank-88 lattice, admit additional vertex-operator-algebraic or moonshine-type interpretations.

A The theta series up to q^{20}

For reference, the coefficients a_n in

$$\Theta_{L_{88}}(\tau) = \sum_{n \geq 0} a_n q^n$$

for $0 \leq n \leq 20$ are:

n	a_n
0	1
1	0
2	125944

n	a_n
3	98241758
4	47841024480
5	16416521193560
6	8153616049497120
7	4838318500261894764
8	1484345825854165071440
9	235058267038894651353360
10	21815678349760546261624080
11	1314174916904607250037821450
12	55406546756366483504842684544
13	1731137978412282517043600228568
14	41905697387929586975300409439040
15	814126096695145239896086285108260
16	13059507256395894305220289025676240
17	177039900234213017971963929176780304
18	2067689185759009889247058766479988568
19	21143280812573974083482611994974972470
20	191892212586320237183006057631811158080

References

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