A Report on Generalized Trigonometric Functions, Product Formulas, and Group Convolutions

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Contents

1 An Analogue to the Basel Problem

This section, based on work from 12.12.2021, explores a function analogous to the sinc function used in the resolution of the Basel problem.

Let us define the function $f(x)$ as the infinite product:

$$f(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^3}{n^3}\right)$$

Using the roots of unity, where $\omega = \exp(\frac{2\pi i}{3})$, this can be factored as:

$$f(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) \left(1 + \frac{x}{n}\omega\right) \left(1 + \frac{x}{n}\omega^2\right)$$

Expanding this product into a power series gives a representation in terms of generalized elementary symmetric polynomials:

$$f(x) = \sum_{k=0}^{\infty} \zeta_k(3) x^{3k}$$

where we define:

$$\zeta_k(3) = \sum_{1 \le n_1 < n_2 < \cdots < n_k} \left(\frac{1}{n_1 n_2 \cdots n_k}\right)^3$$

with $\zeta_0(3) = 1$ and $\zeta_1(3) = \zeta(3)$.

Inspired by the Euler reflection formula for the sinc function, we conjecture a similar identity for $f(x)$.

Conjecture 1.1. *The function $f(x)$ satisfies the reflection-type formula:*

$$f(x) = \frac{1}{\Gamma(1+x)\Gamma(1+\omega x)\Gamma(1+\omega^2 x)}$$

A series expansion of the right-hand side of the conjecture using SageMath yields coefficients that relate $\zeta_k(3)$ to powers of π and values of the Riemann zeta function.

$$\begin{aligned}
\zeta_1(3)x^3 &= \zeta(3)x^3 \\
\zeta_2(3)x^6 &= \left(-\frac{\pi^6}{1890} + \frac{1}{2}\zeta(3)^2\right) x^6 = -\frac{1}{1890}\left(\pi^6 - 945\,\zeta(3)^2\right) x^6 \\
\zeta_3(3)x^9 &= \left(-\frac{\pi^6\zeta(3)}{1890} + \frac{1}{6}\zeta(3)^3 + \frac{1}{3}\zeta(9)\right) x^9 = -\frac{1}{1890}\left(\pi^6\zeta(3) - 315\,\zeta(3)^3 - 630\,\zeta(9)\right) x^9 \\
\zeta_4(3)x^{12} &= -\frac{1}{5108103000}\left(667\,\pi^{12} + 1351350\,\pi^6\zeta(3)^2 - 212837625\,\zeta(3)^4 - 1702701000\,\zeta(9)\zeta(3)\right) x^{12} \\
\zeta_5(3)x^{15} &= -\frac{1}{5108103000}\left(667\pi^{12}\zeta(3) + 450450\pi^6\zeta(3)^3 + 900900\pi^6\zeta(9)\right. \\
&\quad \left. -42567525\zeta(3)^5 - 851350500\zeta(9)\zeta(3)^2 - 1021620600\zeta(15)\right) x^{15}
\end{aligned}$$

From these identities, one could potentially solve for $\zeta(3)$. For instance, from the coefficient of x^6:

$$\zeta(3) = \sqrt{\frac{1890\,\zeta_2(3) + \pi^6}{945}}$$

This raises several questions:

1. Is the conjectured "reflection" equality true?

2. Is there a "closed formula" for the $\zeta_k(3)$ coefficients, or are they related to known numbers in a systematic way?
3. Can this procedure be generalized to products of the form $\prod(1-\frac{x^l}{n^l})$?
4. Is there established literature on the function $f(x)$?

Proof of the Reflection Formula

As noted by Terry Tao in a related context, we can use the Weierstrass factorization of the Gamma function:

$$\Gamma(z)=\frac{e^{-\gamma_E z}}{z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1}e^{z/k}$$

Applying this for $z=x,\omega x,\omega^2 x$ and multiplying the three results, the exponential terms cancel because $1+\omega+\omega^2=0$. This gives:

$$\Gamma(x)\Gamma(\omega x)\Gamma(\omega^2 x)=\frac{e^{-\gamma_E x(1+\omega+\omega^2)}}{x\cdot\omega x\cdot\omega^2 x}\prod_{k=1}^{\infty}\left(\left(1+\frac{x}{k}\right)\left(1+\frac{\omega x}{k}\right)\left(1+\frac{\omega^2 x}{k}\right)\right)^{-1}e^{x(1+\omega+\omega^2)/k}$$

$$=\frac{1}{x^3}\prod_{k=1}^{\infty}\left(1+\frac{x^3}{k^3}\right)^{-1}$$

and therefore

$$\prod_{k=1}^{\infty}\left(1+\frac{x^3}{k^3}\right)=\frac{1}{x^3\Gamma(x)\Gamma(\omega x)\Gamma(\omega^2 x)}$$

Using the functional equation $\Gamma(1+z)=z\Gamma(z)$ for $z=x,\omega x,\omega^2 x$, we have:

$$\Gamma(1+x)\Gamma(1+\omega x)\Gamma(1+\omega^2 x)=(x\Gamma(x))(\omega x\Gamma(\omega x))(\omega^2 x\Gamma(\omega^2 x))=x^3\Gamma(x)\Gamma(\omega x)\Gamma(\omega^2 x)$$

Combining these two results confirms the conjecture:

$$f(x)=\prod_{k=1}^{\infty}\left(1+\frac{x^3}{k^3}\right)=\frac{1}{\Gamma(1+x)\Gamma(1+\omega x)\Gamma(1+\omega^2 x)}$$

This procedure generalizes directly to $f_l(x)=\prod_{n=1}^{\infty}(1+(x/n)^l)$ for $l\geq 2$.

An Identity for $\sum 1/(n^3+1)$

Let's analyze the sum $\sum_{n=1}^{\infty}\frac{1}{n^3+1}$. Consider the function $c_0(x)=xf(\frac{x}{\pi})$. Using the product form of $f(x)$:

$$\frac{c_0(\pi x)}{\pi x}=f(x)=\prod_{n=1}^{\infty}\left(1+\frac{x^3}{n^3}\right)$$

Taking the logarithmic derivative with respect to x:

$$\frac{d}{dx}\log\left(\frac{c_0(\pi x)}{\pi x}\right)=\frac{\pi c_0'(\pi x)}{c_0(\pi x)}-\frac{1}{x}=\sum_{n=1}^{\infty}\frac{3x^2}{n^3+x^3}$$

Dividing by 3 and setting $x=1$, we find:

$$\frac{\pi c_0'(\pi)}{3c_0(\pi)}-\frac{1}{3}=\sum_{n=1}^{\infty}\frac{1}{n^3+1}$$

To evaluate the left-hand side, we use the Gamma function representation of $c_0(x)$:

$$c_0(x) = \frac{x}{\Gamma(1+\frac{x}{\pi})\Gamma(1+\omega\frac{x}{\pi})\Gamma(1+\omega^2\frac{x}{\pi})}$$

Its logarithmic derivative is:

$$\frac{c_0'(x)}{c_0(x)} = \frac{1}{x} - \frac{1}{\pi}\psi\left(1+\frac{x}{\pi}\right) - \frac{\omega}{\pi}\psi\left(1+\omega\frac{x}{\pi}\right) - \frac{\omega^2}{\pi}\psi\left(1+\omega^2\frac{x}{\pi}\right)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. Evaluating at $x = \pi$:

$$\frac{c_0'(\pi)}{c_0(\pi)} = \frac{1}{\pi} - \frac{1}{\pi}\psi(2) - \frac{\omega}{\pi}\psi(1+\omega) - \frac{\omega^2}{\pi}\psi(1+\omega^2)$$

Substituting this into our expression for the sum gives:

$$\sum_{n=1}^{\infty}\frac{1}{n^3+1} = \frac{\pi}{3}\left(\frac{1}{\pi} - \frac{\psi(2)}{\pi} - \frac{\omega\psi(1+\omega)}{\pi} - \frac{\omega^2\psi(1+\omega^2)}{\pi}\right) - \frac{1}{3}$$

$$= \frac{1}{3}\left(1 - \psi(2) - \omega\psi(1+\omega) - \omega^2\psi(1+\omega^2)\right)$$

Using $\psi(2) = 1 - \gamma_E$ and $1 = -\omega - \omega^2$, we get:

$$\sum_{n=1}^{\infty}\frac{1}{n^3+1} = \frac{1}{3}\left(\gamma_E - \omega\psi(1+\omega) - \omega^2\psi(1+\omega^2)\right)$$

This expression can be further manipulated, providing a closed-form representation for the sum in terms of special functions.

2 Functions for the Cyclic Group C_3

This section (from 20.12.2021) develops function analogues of sinh, cosh, sin, cos for the cyclic group C_3.

Define the functions $t_k(x)$ for $k = 0, 1, 2$:

$$t_k(x) = \sum_{n=0}^{\infty}\frac{x^{3n+k}}{(3n+k)!}$$

These functions are the components of the exponential function projected onto the eigenspaces of the C_3 action. They are related via the character table of C_3, where $\omega = \exp(2\pi i/3)$:

$$\begin{pmatrix}\exp(x)\\ \exp(\omega x)\\ \exp(\omega^2 x)\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}\cdot\begin{pmatrix}t_0(x)\\ t_1(x)\\ t_2(x)\end{pmatrix}$$

From this relationship, one can derive the following addition theorem, where the matrix is circulant:

$$\begin{pmatrix}t_0(x+y)\\ t_1(x+y)\\ t_2(x+y)\end{pmatrix} = \begin{pmatrix}t_0(x) & t_2(x) & t_1(x)\\ t_1(x) & t_0(x) & t_2(x)\\ t_2(x) & t_1(x) & t_0(x)\end{pmatrix}\cdot\begin{pmatrix}t_0(y)\\ t_1(y)\\ t_2(y)\end{pmatrix}$$

The determinant of this circulant matrix gives a fundamental identity, analogous to $\cosh^2(x) - \sinh^2(x) = 1$:

$$t_0(x)^3 + t_1(x)^3 + t_2(x)^3 - 3t_0(x)t_1(x)t_2(x) = 1, \quad \forall x \in \mathbb{C}$$

We can also define trigonometric analogues. Let $\gamma = \exp(\frac{2\pi i}{9})$. Set:

- $s_0(x) := \frac{1}{\gamma} t_0(\gamma x)$
- $s_2(x) := s_0'(x) = t_2(\gamma x)$
- $s_1(x) := s_2'(x) = \gamma t_1(\gamma x)$

From the derivative relations $t_0' = t_2$, $t_2' = t_1$, $t_1' = t_0$, it follows that $s_1'(x) = \gamma^2 s_0(x)$. These functions satisfy their own addition theorem and a determinant identity analogous to $\sin^2(x) + \cos^2(x) = 1$:

$$\gamma^3 s_0(x)^3 + s_2(x)^3 + \frac{s_1(x)^3}{\gamma^3} - 3 s_0(x) s_1(x) s_2(x) = 1$$

3 Generalization to Finite Groups

This section (based on notes from 23.10.2021) generalizes the previous construction from C_3 to any finite group G.

For a finite group G, we seek functions $t_g(x)$ for $g \in G$ that satisfy the convolution-style addition formula:

$$t_g(x+y) = \sum_{h \in G} t_{gh^{-1}}(x) t_h(y)$$

This can be written as $t_{x+y}(g) = (t_x * t_y)(g)$, where t_x is the function $g \mapsto t_g(x)$ and $*$ is the group convolution.

The Fourier transform of a convolution is the product of the Fourier transforms. Let ρ be an irreducible representation of G. The Fourier transform of t_x is:

$$\widehat{t_x}(\rho) = \sum_{g \in G} t_g(x) \rho(g)$$

The addition formula is satisfied if $\widehat{t_{x+y}}(\rho) = \widehat{t_x}(\rho)\widehat{t_y}(\rho)$. This is a homomorphism property, satisfied by an exponential function.

Let $S = \{s_1, \ldots, s_r\}$ be a generating set for G with $1 \notin S$. Let $x = (x_{s_i})_{s_i \in S}$ be a vector of variables. We can define the functions t_g via their Fourier transform:

$$\widehat{t_x}(\rho) := \exp\left(\frac{1}{d_\rho} \sum_{s \in S} \chi_\rho(s) x_s\right) \mathbf{1}_{d_\rho}$$

where d_ρ is the dimension of ρ, χ_ρ is its character, and $\mathbf{1}_{d_\rho}$ is the identity matrix. This definition immediately ensures that $\widehat{t_{x+y}}(\rho) = \widehat{t_x}(\rho)\widehat{t_y}(\rho)$.

Applying the inverse Fourier transform, we obtain an explicit formula for the functions:

$$t_g(x) = \frac{1}{|G|} \sum_{\rho \text{ irred.}} d_\rho \operatorname{Tr}\left(\rho(g^{-1})\widehat{t_x}(\rho)\right) = \frac{1}{|G|} \sum_{\rho \text{ irred.}} d_\rho \chi_\rho(g^{-1}) \exp\left(\frac{1}{d_\rho} \sum_{s \in S} \chi_\rho(s) x_s\right)$$

From this formula, we see that $t_g(x)$ depends only on the conjugacy class of g.

Furthermore, the determinant of the group matrix $T_G = (t_{gh^{-1}}(x))_{g,h \in G}$ can be evaluated using the Frobenius determinant formula:

$$\begin{aligned}\det(T_G) &= \prod_{\rho \text{ irred.}} \det\left(\sum_{g\in G} t_g(x)\rho(g)\right)^{d_\rho} = \prod_{\rho \text{ irred.}} \det(\widehat{t_x}(\rho))^{d_\rho} \\ &= \prod_{\rho \text{ irred.}} \det\left(\mathbf{1}_{d_\rho} \exp\left(\frac{1}{d_\rho}\sum_{s\in S}\chi_\rho(s)x_s\right)\right)^{d_\rho} \\ &= \prod_{\rho \text{ irred.}} \left(\exp\left(\frac{1}{d_\rho}\sum_{s\in S}\chi_\rho(s)x_s\right)\right)^{d_\rho^2} = \exp\left(\sum_{\rho \text{ irred.}} d_\rho \sum_{s\in S}\chi_\rho(s)x_s\right) \\ &= \exp\left(\sum_{s\in S} x_s\left(\sum_\rho d_\rho\chi_\rho(s)\right)\right)\end{aligned}$$

The inner sum $\sum_\rho d_\rho\chi_\rho(s)$ is the value of the regular character on s. For any $s \neq 1$, this sum is zero. Since we chose $1 \notin S$, we have:

$$\det(T_G) = \exp(0) = 1$$

Example: The Symmetric Group S_3

For $G = S_3$, with generating set $S = \{(1,2),(2,3)\}$ and variables x_0, x_1, the functions are:

- For $g \in \{e,(123),(132)\}$ (conjugacy class of the identity and 3-cycles):

$$t_e(x_0,x_1) = \frac{1}{6}e^{x_0+x_1} + \frac{1}{6}e^{-x_0-x_1} + \frac{2}{3}$$

$$t_{(123)}(x_0,x_1) = t_{(132)}(x_0,x_1) = \frac{1}{6}e^{x_0+x_1} + \frac{1}{6}e^{-x_0-x_1} - \frac{1}{3}$$

- For $g \in \{(12),(13),(23)\}$ (conjugacy class of transpositions):

$$t_g(x_0,x_1) = \frac{1}{6}e^{x_0+x_1} - \frac{1}{6}e^{-x_0-x_1}$$

SageMath Implementation

Listing 1: SageMath code to compute $t_g(x)$ for S3

```
G = SymmetricGroup(3)
# S = [G[3], G[5]] # Corresponds to S = {(1,2),(2,3)}
S = [list(G)[-3],list(G)[-1]] # A more robust way to select generators
X = var([("x"+str(i)) for i in range(len(S))])

# Define irreducible representations for S3
trivial = G.irreducible_characters()[0].to_representation()
sign = G.irreducible_characters()[1].to_representation()
standard = G.irreducible_characters()[2].to_representation()
irredPerms = [trivial, sign, standard]

def log_inverse_Fourier(rep, X_vars):
    m = sum([rep(S[i]).trace() * X_vars[i] for i in range(len(S))])
    return m

def tau(perm, X_vars):
```

```
    s = 0
    for rho in irredPerms:
        em = exp(log_inverse_Fourier(rho, X_vars))
        s += rho(Permutation([1,2,3])).degree() * (rho(perm.inverse())
            * em).trace()
    return 1/G.order() * s

# Verify the addition theorem
var("a0,a1,b0,b1")
X_val = [a0, a1]
Y_val = [b0, b1]
Z_val = [X_val[i] + Y_val[i] for i in range(2)]

for g in G:
    lhs = tau(g, Z_val)
    rhs = sum([tau(g*h.inverse(), Y_val) * tau(h, X_val) for h in G])
    print(f"Permutation g = {g}")
    # The symbolic verification can be slow/complex
    # print("Addition theorem satisfied:", bool(expand(lhs - rhs) == 0)
        )
    print(f"t_g(a_0,a_1) = {factor(tau(g,X_val))}\n")
```

4 Associated Partial Differential Equations

For a finite abelian group G (written additively for this section), the formulas from Section 3 simplify. The representations are 1-dimensional characters, so $d_\rho = 1$ and $\chi_\rho(g) = \rho(g)$.

$$t_g(x) = \frac{1}{|G|} \sum_{\rho \text{ irred.}} \rho(-g) \exp\left(\sum_{s\in S} \rho(s) x_s\right)$$

Let us differentiate with respect to a variable x_{s_0} for $s_0 \in S$:

$$\begin{aligned}\frac{\partial t_g(x)}{\partial x_{s_0}} &= \frac{1}{|G|} \sum_{\rho \text{ irred.}} \rho(-g)\rho(s_0) \exp\left(\sum_{s\in S} \rho(s) x_s\right) \\ &= \frac{1}{|G|} \sum_{\rho \text{ irred.}} \rho(-(g-s_0)) \exp\left(\sum_{s\in S} \rho(s) x_s\right) \\ &= t_{g-s_0}(x)\end{aligned}$$

By induction, for any $h = \sum_{i=1}^r e_i s_i \in G$, we can define a higher-order partial derivative:

$$\frac{\partial t_g(x)}{\partial h} := \frac{\partial^{e_1+\cdots+e_r} t_g(x)}{\partial x_{s_1}^{e_1} \cdots \partial x_{s_r}^{e_r}} = t_{g-h}(x)$$

This means the group G acts on the set of functions $\{t_g(x)\}$ via differentiation. Since $t_{g-h}(x) = t_{(g-h)e^{-1}}(x) = t_{gh^{-1}}(x)$ in multiplicative notation, and we proved that $\det(t_{gh^{-1}}(x)) = 1$, we arrive at a remarkable PDE satisfied by the vector of functions $T(x) = (t_g(x))_{g\in G}$:

$$\det\left(\left(\frac{\partial t_g(x)}{\partial h}\right)_{g,h\in G}\right) = 1$$

5 Monge–Ampère Equation, 04.01.2022

Let G be a finite abelian group (written additively) and let $X = (x_g)_{g\in G}$ be the corresponding vector of variables. For each $g \in G$, define

$$\tau_g(X) \;:=\; \frac{1}{|G|} \sum_{\rho \text{ irred.}} \chi_\rho(-g)\, \exp\Bigl(\sum_{s\in G} \chi_\rho(s)\, x_s\Bigr),$$

where the sum runs over all irreducible characters ρ of G. Then one checks immediately that

$$\frac{\partial}{\partial x_h}\, \tau_g(X) \;=\; \tau_{g-h}(X).$$

Set

$$u(X) \;:=\; \tau_0(X).$$

Its Hessian matrix is

$$H_u(X) \;=\; \bigl[\partial_{x_g}\partial_{x_h}\, u(X)\bigr]_{g,h\in G} \;=\; \bigl[\tau_{-(g+h)}(X)\bigr]_{g,h\in G}.$$

Hence the "discrete Monge–Ampère equation" becomes

$$\det H_u(X) \;=\; \det\bigl(\tau_{-(g+h)}(X)\bigr)_{g,h\in G} \;=\; \pm\, \det\bigl(\tau_{g-h}(X)\bigr)_{g,h\in G},$$

where the sign $\pm$ arises from the permutation $g \mapsto -g$ on the rows. The determinant $\det\bigl(\tau_{g-h}\bigr)$ is exactly the classical group determinant $\det[a_{gh^{-1}}]_{g,h\in G}$ with $a_g = \tau_g(X)$, which by Frobenius's theorem factors as

$$\det\bigl(\tau_{g-h}\bigr) \;=\; \prod_{\rho \text{ irred.}} \det\Bigl(\sum_{g\in G} \tau_g(X)\, \rho(g)\Bigr)^{\dim\rho}.$$

Thus imposing

$$\det H_u(X) \;=\; f(X)$$

amounts to finding the vector $\bigl(\tau_g(X)\bigr)$ whose group determinant matches the prescribed function $f(X)$ — in other words, to solving a Monge–Ampère–type equation in the discrete variables $\{x_g\}$.

6 Speculations on the Navier-Stokes Equations

This final section (from 30.12.2021) explores a potential connection between the functions developed above and the Navier-Stokes equations for incompressible fluid flow.

Let G be a finite abelian group and $X = (x_g)_{g\in G}$ be a vector of variables, one for each group element. Define the functions:

$$\tau_g(X) := \frac{1}{|G|} \sum_{\rho \text{ irred.}} \rho(-g) \exp\left(\sum_{s\in G} \rho(s) x_s\right)$$

These functions satisfy the simple differentiation rule:

$$\frac{\partial \tau_g(X)}{\partial x_h} = \tau_{g-h}(X)$$

Now consider the advection term in the Navier-Stokes equations, which has the form $(u \cdot \nabla)u$. In our discrete setting, this corresponds to a sum:

$$\sum_{h\in G} u_h(X) \frac{\partial u_g(X)}{\partial x_h}$$

If we set $u_g = \tau_g$, we find a convolution identity:

$$\sum_{h\in G} \tau_h(X) \frac{\partial \tau_g(X)}{\partial x_h} = \sum_{h\in G} \tau_h(X)\tau_{g-h}(X) = (\tau_X * \tau_X)(g) = \tau_g(2X)$$

This structural similarity suggests that these functions might be useful building blocks for solutions.

Let us attempt to construct a solution to the sourceless Navier-Stokes equations in $\mathbb{R}^n \times [0, \infty)$:

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0 \quad \text{and} \quad \nabla\cdot u = 0$$

Let $G = C_n$ be a cyclic group of odd order n. Let the spatial variables be $X = (x_g)_{g\in G}$. We propose a velocity field $u(t, X) = (u_g(t, X))_{g\in G}$ and pressure $p(t, X)$.

Let $\alpha_g(X) := \tau_g(-X) - \tau_{g-1}(-X)$. This combination has zero divergence:

$$\nabla\cdot\alpha = \sum_{g\in G} \frac{\partial \alpha_g}{\partial x_g} = \sum_{g\in G} \left(\frac{\partial \tau_g(-X)}{\partial x_g} - \frac{\partial \tau_{g-1}(-X)}{\partial x_g}\right) = \sum_{g\in G} (\tau_{g-g}(-X) - \tau_{g-1-g}(-X)) = \sum_{g\in G} (\tau_0(-X) - \tau_{-1}(-X)) =$$

Since n is odd, $h \mapsto 2h$ is a permutation of G, so $\Delta\alpha_g = \sum_h \alpha_{g-2h} = \sum_k \alpha_k = 0$. So α is harmonic.

Let's propose a solution of the form $u_g(t, X) := t\alpha_g(X) + u_g^0(X)$, where u^0 is a given divergence-free initial condition at $t = 0$.

1. **Initial Condition:** $u(0, X) = u^0(X)$. This is satisfied by construction.

2. **Incompressibility:** $\nabla\cdot u = \nabla\cdot(t\alpha + u^0) = t(\nabla\cdot\alpha) + (\nabla\cdot u^0) = 0 + 0 = 0$. This holds.

Substituting into the momentum equation:

$$\frac{\partial u_g}{\partial t} + \sum_{h\in G} u_h \frac{\partial u_g}{\partial x_h} - \nu\Delta u_g + \frac{\partial p}{\partial x_g} = 0$$

$$\alpha_g + \sum_h (t\alpha_h + u_h^0)(t\alpha_{g-h} + \frac{\partial u_g^0}{\partial x_h}) - \nu\Delta(t\alpha_g + u_g^0) + \frac{\partial p}{\partial x_g} = 0$$

$$\alpha_g + t^2(\alpha * \alpha)_g + t(\alpha * \partial u_g^0)_g + t(u^0\cdot\nabla)\alpha_g + (u^0\cdot\nabla)u_g^0 - \nu\Delta u_g^0 + \frac{\partial p}{\partial x_g} = 0$$

Using $\Delta\alpha = 0$ and the initial condition $(u^0\cdot\nabla)u_g^0 - \nu\Delta u_g^0 + \nabla_g p^0 = 0$, this simplifies. The construction raises several difficult questions:

Question 6.1. *Does the vector field $\alpha_g(X)$ satisfy the Euler equations (the NS equations with $\nu = 0$) for some pressure function p_α? That is, can we find p_α such that $\nabla_g p_\alpha = -(\alpha * \alpha)_g$?*

Question 6.2. *Can one find a pressure $p(t, X)$ that absorbs all the gradient terms arising from the expansion?*

This approach recasts the non-linear advection term as a structured convolution, which may offer a new perspective, but its ultimate utility in solving this notoriously difficult problem remains an open question.