Cayley–Truchet Puzzle

Square Permutation Tiles with Smooth Bézier Connections

Orges Leka

08.05.2026

Abstract

We describe a square Truchet-like tile associated with a permutation $\sigma \in S_n$. The tile consists of an outer square and an inner circle. On the boundary of the square there are $2n$ ports, chosen so that opposite sides have the same discrete port positions. Therefore the tiles are compatible with the square grid and can be used in rectangular tilings and in Cayley-table constructions. Along the boundary of the square the ports are read cyclically and alternately as inputs and outputs. For each $i = 1, \ldots, n$, a smooth curve connects the input port I_i with the output port $O_{\sigma(i)}$. The connections are drawn by cubic Bézier segments, or by spline-like curves. The boundary tangents are fixed: horizontal on the west and east sides, and vertical on the north and south sides. The transition pieces from the square boundary to the inner circle are chosen as monotone Bézier segments without local loops or curls.

Contents

1 Motivation

For a tiling by square cells, two conditions should be satisfied:

(1) opposite sides should have the same discrete port positions;

(2) a channel should continue into a neighbouring cell without a kink.

Therefore we use

- the same horizontal port positions on the north and south sides;
- the same vertical port positions on the west and east sides;
- horizontal tangents on the west/east sides and vertical tangents on the north/south sides;

- smooth Bézier or spline curves instead of polygonal lines.

The only delicate part is the transition from the boundary of the square into the inner circle. These transition pieces are chosen explicitly so that no small loops, curls, or local backtracking effects occur.

2 Geometric Definition

Fix $n \in \mathbb{N}$ and a permutation $\sigma \in S_n$. Write

$$a := \left\lceil \frac{n}{2} \right\rceil, \qquad b := \left\lfloor \frac{n}{2} \right\rfloor,$$

so that $a + b = n$.

Definition 1 (Smooth Bézier grid-compatible square permutation tile)**.** *Let*

$$Q = [-1, 1]^2 \subset \mathbb{R}^2$$

be the outer square, and let

$$C_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}, \qquad 0 < r < 1,$$

be a centred inner circle.

On the north and south sides of the square choose the a horizontal port positions

$$x_j = -1 + \frac{2j}{a+1}, \qquad j = 1, \dots, a,$$

and define

$$N_j = (x_j, 1), \qquad S_j = (x_j, -1), \qquad j = 1, \dots, a.$$

On the west and east sides choose the b vertical port positions

$$y_k = 1 - \frac{2k}{b+1}, \qquad k = 1, \dots, b,$$

and define

$$W_k = (-1, y_k), \qquad E_k = (1, y_k), \qquad k = 1, \dots, b.$$

The $2n = 2a + 2b$ boundary points are read cyclically around the boundary of the square:

- *on the north side from left to right: $N_1, \dots, N_a$;*
- *on the east side from top to bottom: $E_1, \dots, E_b$;*
- *on the south side from right to left: $S_a, \dots, S_1$;*
- *on the west side from bottom to top: $W_b, \dots, W_1$.*

Denote this cyclic list by

$$P_1, P_2, \dots, P_{2n}.$$

Define alternating input and output ports by

$$I_i := P_{2i-1}, \qquad O_i := P_{2i}, \qquad i = 1, \dots, n.$$

Thus the ports occur along the boundary in the cyclic order

$$I_1, O_1, I_2, O_2, \dots, I_n, O_n.$$

To every boundary point $P \in \partial Q$ assign a transition point $\pi(P)$ on the inner circle, chosen so that the tangent direction at the boundary is compatible with the side of the square:

$$\pi(x,1) = (x,r), \qquad \pi(x,-1) = (x,-r), \qquad \pi(-1,y) = (-r,y), \qquad \pi(1,y) = (r,y).$$

The tile associated with σ, denoted by T_σ, consists of the square Q and n smooth curves

$$\gamma_i : [0,1] \to Q, \qquad i = 1,\dots,n,$$

with the following properties:

- *(i)* $\gamma_i(0) = I_i$ *and* $\gamma_i(1) = O_{\sigma(i)}$*;*
- *(ii) the curve runs from I_i to $\pi(I_i)$, then through the inner circular region, and finally from $\pi(O_{\sigma(i)})$ to $O_{\sigma(i)}$;*
- *(iii) the outer transition pieces $I_i \to \pi(I_i)$ and $\pi(O_{\sigma(i)}) \to O_{\sigma(i)}$ are monotone cubic Bézier segments with prescribed tangents; in particular, they have no local loops or curls;*
- *(iv) the middle part inside the inner circular region is made of smooth cubic Bézier or spline-like segments;*
- *(v) at the transition points between consecutive segments, γ_i is tangent-continuous, that is, C^1-continuous;*
- *(vi) at boundary points on the west and east sides, γ_i has a horizontal tangent, while at boundary points on the north and south sides, it has a vertical tangent;*
- *(vii) crossings are allowed, but are minimized as far as possible.*

Remark 1 (Grid compatibility)**.** *The north and south sides have the same horizontal port positions, and the west and east sides have the same vertical port positions. Therefore neighbouring tiles are geometrically compatible along common edges.*

Remark 2 (Tangential kink-freeness)**.** *The prescribed boundary tangents ensure that a channel continues from one tile to a neighbouring tile without a kink: horizontally across west/east edges and vertically across north/south edges.*

3 Cayley–Truchet Tiles of a Finite Group

Let $G = \{g_1,\dots,g_n\}$ be a finite group with a fixed ordering of its elements. For each entry of the Cayley table, or equivalently for each pair $(g_i, g_j) \in G \times G$, we assign a permutation tile. The tile is determined by the product $g_i g_j$ and by the permutation associated with that group element under the chosen regular action and ordering.

Thus a group of order n produces a deck of

$$n^2$$

Cayley–Truchet tiles. The deck may be placed on the square grid. Whenever two tiles share an edge, their matching port positions allow the strands to continue across the common boundary.

4 Puzzle Rules

The construction can be used as a tile-laying puzzle or game. An online prototype is available at the project page.[1]

[1] https://www.orges-leka.de/cayley-truchet-puzzle.html

Material

For a finite group G of order n, the puzzle consists of the n^2 Cayley–Truchet tiles associated with the Cayley table of G.

Setup

Shuffle the tiles. The playing area is the infinite square grid. Tiles are placed into grid cells without rotation, unless a separate variant explicitly allows rotations.

Move

On each move, choose one unplaced tile and place it into an empty grid cell. Two neighbouring tiles are connected whenever a strand exits one tile through a boundary port and enters the matching port of the adjacent tile.

Goal

The goal is to form a single closed curve that passes through as many distinct tiles as possible. A curve component is called closed if it has no loose endpoint on the exterior boundary of the placed configuration.

Score

The score of a closed curve is the number of distinct tiles visited by that curve. The main score of a layout is the largest such number among all closed curve components in the layout. Equivalently, define

$$\lambda(G)$$

to be the maximum possible number of distinct tiles visited by one closed curve, using the Cayley–Truchet deck associated with G.

Proposition 1 (The Klein four group)**.** *For the Klein four group V_4, the maximum number of tiles that can be visited by a single closed curve in the Cayley–Truchet puzzle is*

$$\lambda(V_4) = 14.$$

Proof. The group V_4 has order 4. Hence its Cayley–Truchet deck contains

$$|V_4|^2 = 4^2 = 16$$

tiles.

In the Cayley–Truchet deck of V_4 there are exactly four stopper tiles. Here a stopper tile means a tile whose local strand structure closes off one end of a curve segment inside the tile. Thus, when a single curve component is followed through the grid, a stopper tile can occur only as one of the two terminal closing pieces of that component.

Consequently, a single closed curve component can contain at most two of the four stopper tiles. It cannot contain all four stopper tiles: using four stoppers would necessarily create at least two separately closed components, or else force the same component to have more than two terminal closing pieces, which is impossible for one non-branching curve.

Therefore at least two of the sixteen tiles cannot belong to one single closed curve component. This gives the upper bound

$$\lambda(V_4) \leq 16 - 2 = 14.$$

On the other hand, Figure 7 shows an explicit placement of the V_4 tiles containing one closed curve component which visits 14 distinct tiles. Hence

$$\lambda(V_4) \geq 14.$$

Combining the upper bound and the explicit construction, we obtain

$$\lambda(V_4) = 14.$$

□

5 Short Version of the Definition

The construction can be summarized as follows:

> *A smooth Bézier grid-compatible square permutation tile associated with $\sigma \in S_n$ consists of a square with a centred inner circle. Choose $a = \lceil n/2 \rceil$ port positions on the north and south sides and $b = \lfloor n/2 \rfloor$ port positions on the west and east sides. The north and south sides have the same x-coordinates, while the west and east sides have the same y-coordinates. Reading the resulting $2n$ boundary points cyclically around the square, label them alternately as $I_1, O_1, I_2, O_2, \ldots, I_n, O_n$. Each boundary point is assigned a transition point on the inner circle with a compatible tangent direction. For each $i = 1, \ldots, n$, a smooth spline curve made of cubic Bézier segments connects the input port I_i to the output port $O_{\sigma(i)}$. The curve enters and leaves west/east sides horizontally and north/south sides vertically. The outer transition pieces are chosen to be loop-free. Crossings are allowed, but are minimized as far as possible.*

6 Question Suggested by the Puzzle

For a finite group G, what is the value of $\lambda(G)$? More generally, can $\lambda(G)$ be computed or bounded in terms of group-theoretic data of G? In particular, what are the values of $\lambda(G)$ for all groups of order at most 8?

7 Figures

This section collects the basic visual examples used in the definition and in the puzzle. All image files are expected to be located in the same directory as this `.tex` file.

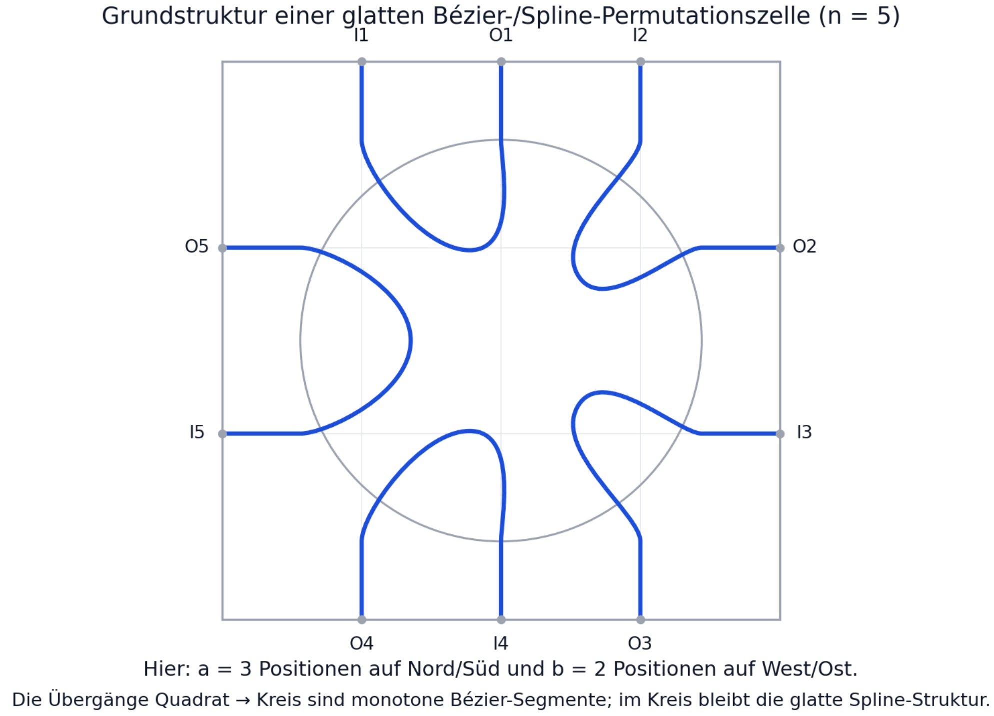

Figure 1: Basic structure of a smooth Bézier grid-compatible permutation tile. The outer square carries alternating input and output ports, and the strands pass through the inner circular region.

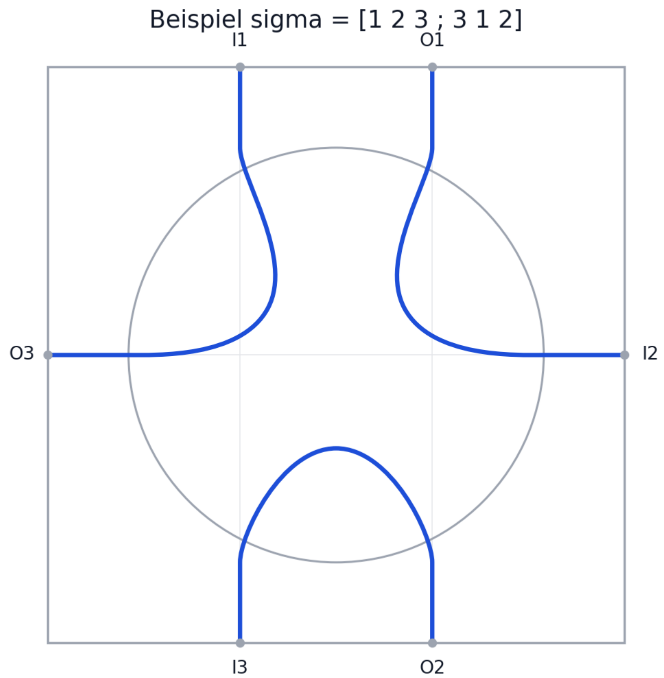

Figure 2: Example tile for the permutation $\sigma = (1\ 3\ 2)$, shown as a smooth Truchet-like permutation cell.

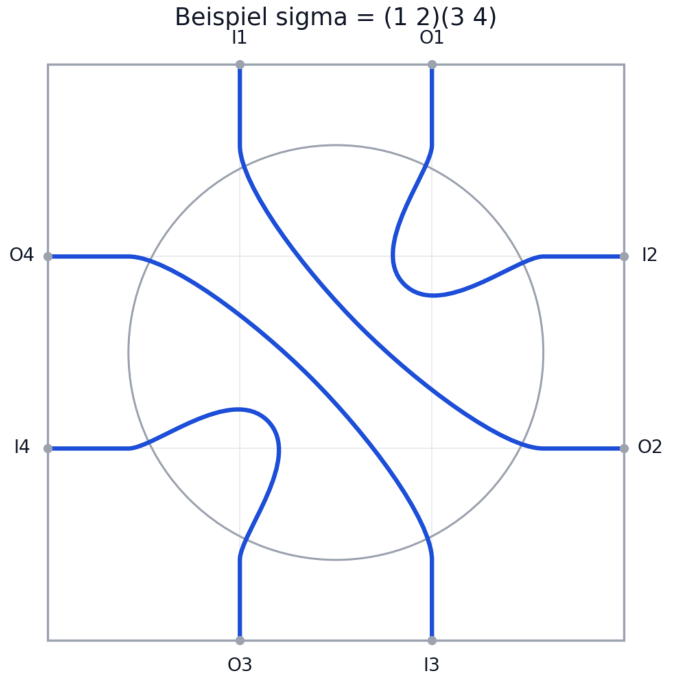

Figure 3: Example tile for an involution. The prescribed boundary tangents make the cell compatible with neighbouring cells on the square grid.

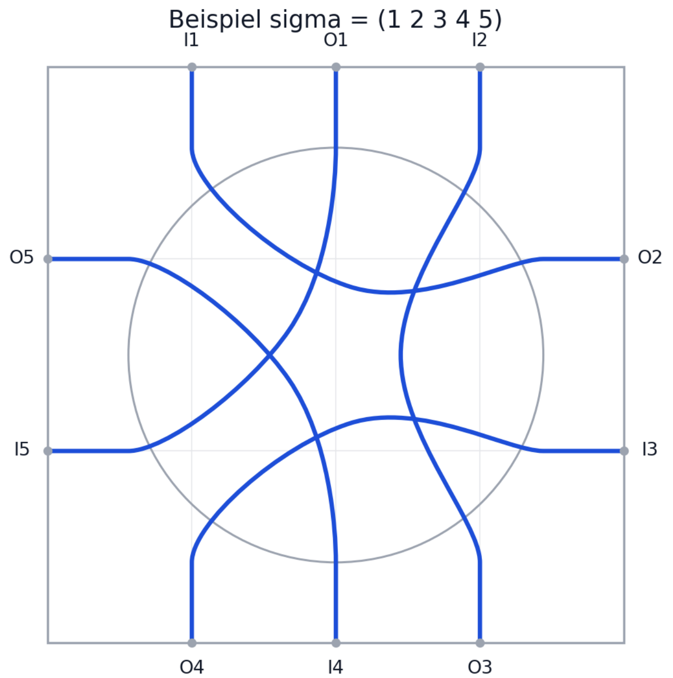

Figure 4: Example tile for a 5-cycle. Crossings are allowed, while the transition pieces at the boundary are chosen without small local loops.

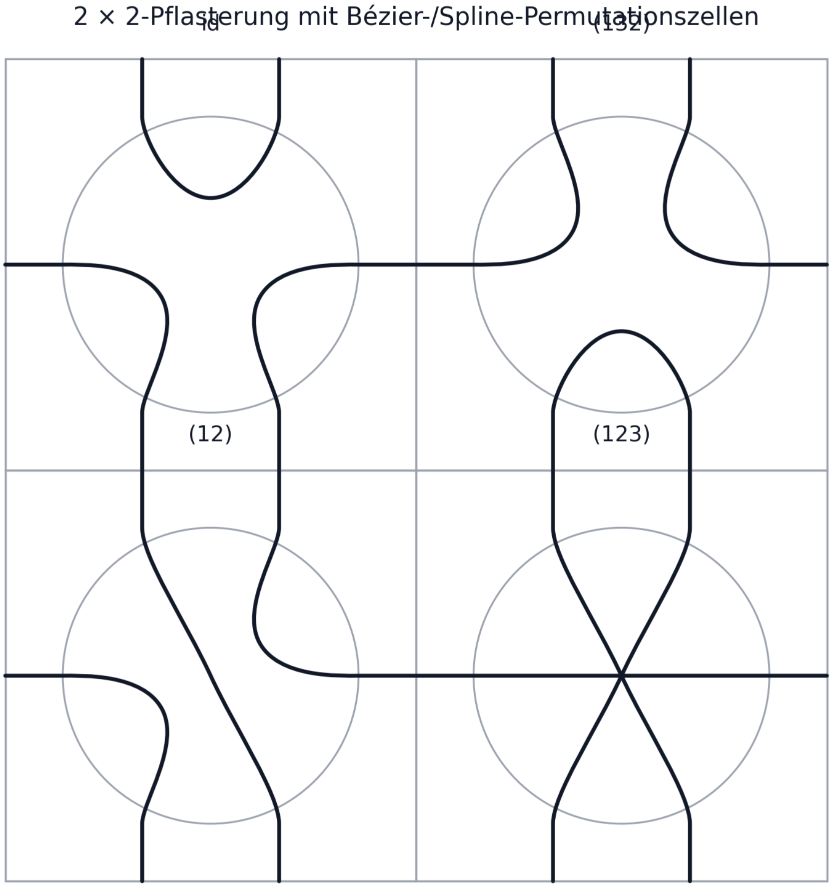

Figure 5: A small tiling by smooth Bézier permutation cells. Adjacent tiles match along common edges and the strands continue without kinks.

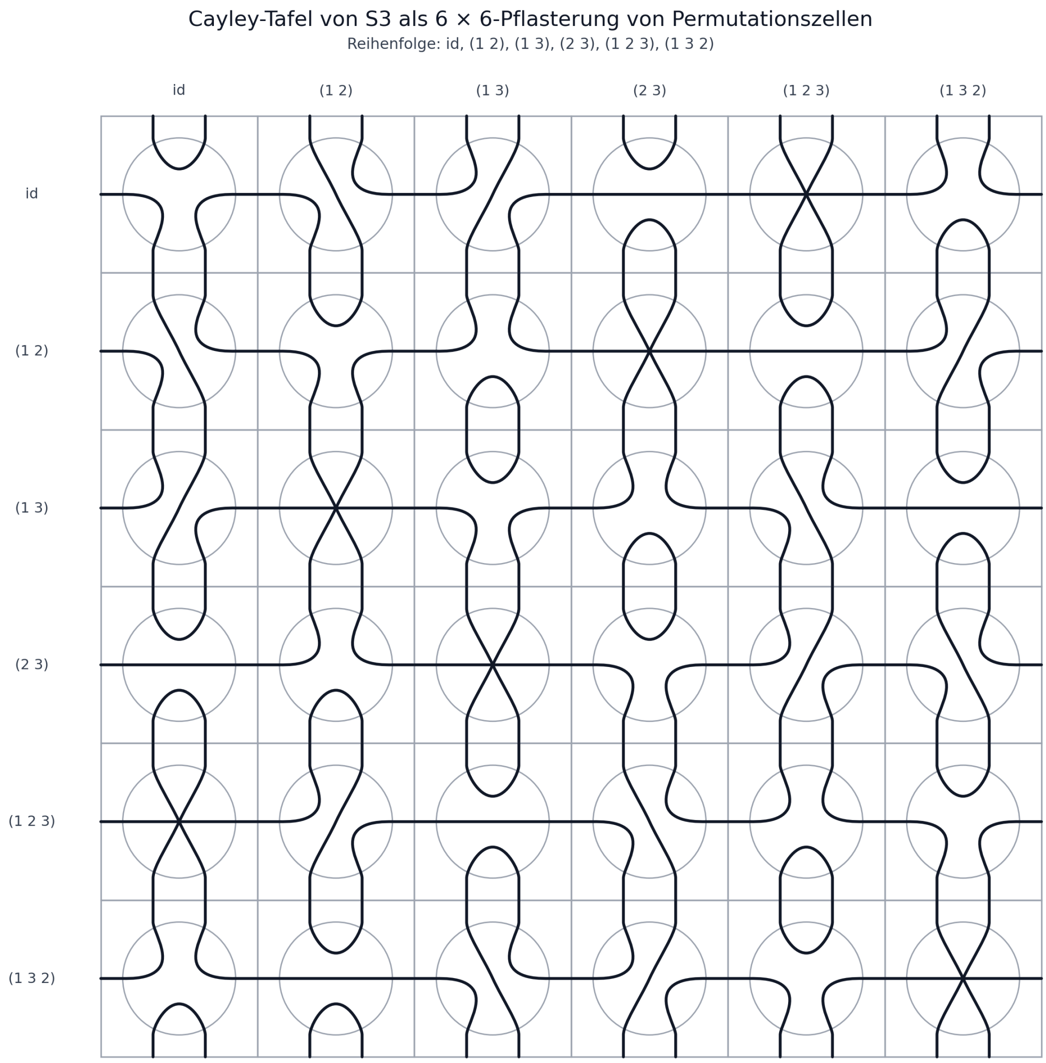

Figure 6: The full Cayley table of S_3 rendered as a 6×6 tiling of Cayley–Truchet permutation cells.

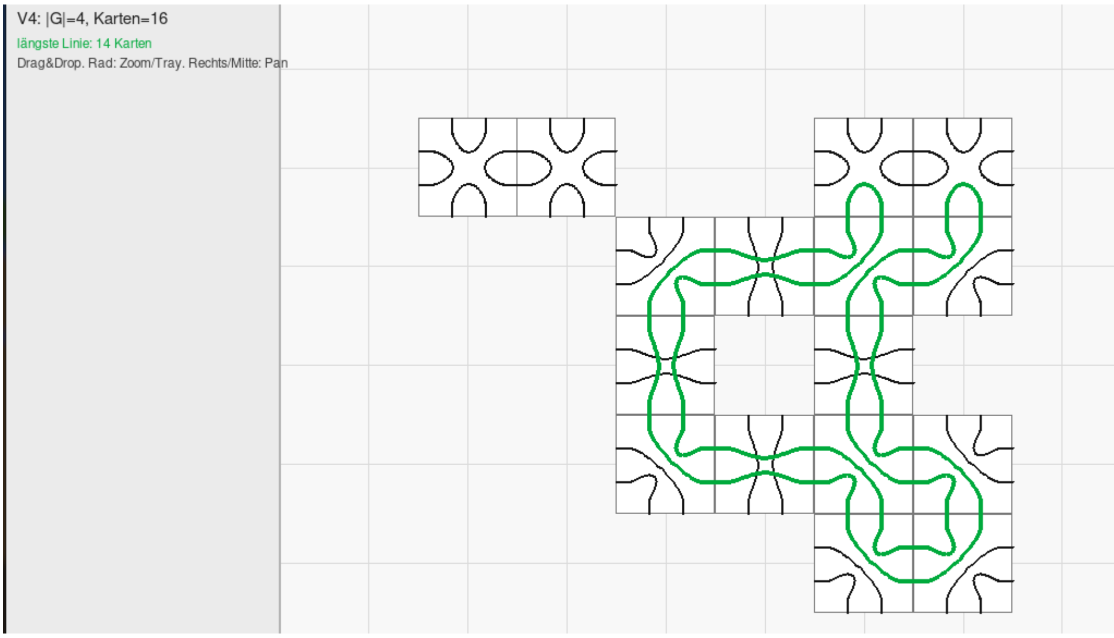

Figure 7: Example for V_4.