Visualising Groups with Cayley Fibergraphs and Continuous Transformations

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May 5, 2026

Abstract

We introduce a geometric way to visualise a finite group G by means of its *Cayley fibergraph.* The construction starts from the Cayley table of G. For each group element $h \in G$ we consider the fiber

$$F_h = \{(i,j) \mid g_i g_j = h\},$$

where $g_1, \dots, g_n$ is a fixed ordering of the elements of G. Each fiber is enhanced with an adjacency rule and thus becomes a polygonal figure inside the square grid of the Cayley table. In this way every table entry value h is represented globally by a figure F_h rather than by a numeral. The left action of G permutes these figures regularly by $x \cdot F_h = F_{xh}$. We describe a continuous geometric realisation of this action by piecewise linear deformations of the polygonal figures. Finally, we provide a SageMath implementation that computes the figures, draws them, and produces an MP4 animation (via FFmpeg) visualising the action of every element of G.

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1 Introduction

Visual representations of finite groups usually proceed via multiplication tables, Cayley graphs, or permutation actions. The purpose of the present note is to develop a different representation scheme based directly on the full Cayley table but avoiding potential confusion with Cayley graphs. The key idea is to collect all table positions with equal product into a single geometric object. These objects, called *figures*, are then used as visual representatives of group elements.

This gives rise to a family of polygonal patterns encoded by the multiplication structure of the group. Since left multiplication permutes the product fibers, the group acts naturally on the family of figures. The action can be visualised continuously by moving the vertices of each figure and carrying the polygonal edges along.

2 The Cayley fibergraph

Let G be a finite group of order n, and fix an ordering

$$G = \{g_1, g_2, \dots, g_n\}.$$

We consider the vertex set

$$V := \{(i, j) \mid 1 \leq i, j \leq n\}.$$

The point (i, j) corresponds to the table position in row i and column j.

Definition 1 (Cayley fibergraph)**.** *The* Cayley fibergraph *of* $(G, (g_1, \dots, g_n))$ *is the graph* $\Gamma_G = (V, E)$ *with*

$$(i, j) \sim (k, \ell)$$

if and only if

$$g_i g_j = g_k g_\ell \quad \textit{and} \quad (|i - k| = 1 \textit{ or } |j - \ell| = 1).$$

Thus an edge joins two nearby table positions carrying the same product value. The graph depends on the chosen ordering of the group elements.

Definition 2 (Fiber and figure)**.** *For* $h \in G$ *define the* product fiber

$$F_h := \{(i, j) \in V \mid g_i g_j = h\}.$$

The induced subgraph of Γ_G *on* F_h *is denoted by* Γ_h*. Its geometric realisation in the Cayley table grid is called the* figure *of* h*.*

Each figure is drawn in the square grid via the embedding

$$\phi(i, j) := (j, n + 1 - i) \in \mathbb{R}^2.$$

Vertices are drawn at the points $\phi(i, j)$ and edges are straight line segments between the corresponding embedded vertices. In this way Γ_h becomes a polygonal 1-complex in the plane.

Remark 1. *For fixed* $h \in G$ *and each* $i \in \{1, \dots, n\}$*, there is a unique* j *such that* $g_i g_j = h$*, namely* $g_j = g_i^{-1} h$*. Hence every fiber contains exactly* n *vertices, one in each row and one in each column. In particular, the vertex pattern of* F_h *is a permutation pattern.*

2.1 A Lexicographic Ordering via the Regular Permutation Representation

For the visual construction it is useful to keep the simple local adjacency rule

$$|i - k| = 1 \quad \text{or} \quad |j - \ell| = 1,$$

because it produces sparse and visually readable polygonal figures. This rule, however, requires a linear ordering of the group elements. We therefore choose a canonical ordering induced by the regular permutation representation of the group.

Let $G = \{g_1, \dots, g_n\}$ be a finite group. By Cayley's theorem, G embeds into the symmetric group S_n via the left regular action

$$\lambda : G \longrightarrow S_G, \qquad \lambda(g)(x) = gx.$$

After identifying the underlying set of G with $\{1, \dots, n\}$, every element $g \in G$ is represented by a permutation

$$\lambda(g) = (\lambda(g)(1), \lambda(g)(2), \dots, \lambda(g)(n)).$$

We then order the elements of G lexicographically by these image vectors.

Thus, for $g, h \in G$, we define

$$g <_{\text{lex}} h$$

if and only if

$$(\lambda(g)(1), \ldots, \lambda(g)(n))$$

is lexicographically smaller than

$$(\lambda(h)(1), \ldots, \lambda(h)(n)).$$

This gives a reproducible ordering of the elements once the regular action has been fixed. The resulting Cayley fibergraph still uses the visually simple adjacency condition

$$(i, j) \sim (k, \ell) \iff g_i g_j = g_k g_\ell \quad \text{and} \quad (|i - k| = 1 \text{ or } |j - \ell| = 1).$$

Example: C_3**.** Let

$$C_3 = \{e, a, a^2\}, \qquad a^3 = e.$$

The left regular action on the ordered set (e, a, a^2) gives

$$\lambda(e) = (1, 2, 3),$$

$$\lambda(a) = (2, 3, 1),$$

$$\lambda(a^2) = (3, 1, 2).$$

Lexicographic ordering of these image vectors yields

$$e <_{\text{lex}} a <_{\text{lex}} a^2.$$

Hence the Cayley table is indexed by the order

$$(e, a, a^2),$$

and the product fiber

$$F_h = \{(i, j) \mid g_i g_j = h\}$$

is drawn using this lexicographic regular ordering.

For example, the identity fiber is

$$F_e = \{(1, 1), (2, 3), (3, 2)\}.$$

With the local adjacency rule, the points of F_e are connected whenever their row or column indices differ by one. This produces a sparse polygonal figure while the ordering itself is obtained from a canonical permutation representation of the group.

3 The group action on figures

The set of figures is naturally indexed by the elements of G. Left multiplication acts on this set exactly as expected.

Proposition 1. *For each $x \in G$, left multiplication defines a permutation of the set of fibers by*

$$x \cdot F_h := F_{xh}.$$

This action is isomorphic to the regular left action of G on itself.

Proof. If $(i,j) \in F_h$, then $g_i g_j = h$. Let $\rho_x \in S_n$ be the permutation induced by left multiplication on the ordered group list, i.e.

$$xg_i = g_{\rho_x(i)}.$$

Then

$$g_{\rho_x(i)} g_j = (xg_i)g_j = x(g_i g_j) = xh.$$

Therefore $(\rho_x(i), j) \in F_{xh}$. Since ρ_x is a permutation, this gives a bijection from F_h to F_{xh}. $\square$

Thus the group does not primarily act on single entries of the table as isolated labels, but rather on the global figures representing those labels.

4 Continuous geometric transformations

We now describe a continuous realisation of the action. Fix $x \in G$ and $h \in G$. The action sends the figure of h to the figure of xh. In coordinates, a vertex $(i,j) \in F_h$ moves to $(\rho_x(i), j) \in F_{xh}$.

Write

$$p_{ij} := \phi(i,j), \qquad q_{ij}^{(x)} := \phi(\rho_x(i), j).$$

A continuous motion of the vertices is given by linear interpolation:

$$p_{ij}(t) := (1-t)p_{ij} + tq_{ij}^{(x)}, \qquad 0 \le t \le 1.$$

Equivalently,

$$p_{ij}(t) = \big(j,\, n+1-((1-t)i + t\rho_x(i))\big).$$

Hence each vertex moves vertically.

If an edge of the source figure connects the vertices p_{ij} and $p_{k\ell}$, we transport it by moving its endpoints:

$$e_t = [p_{ij}(t), p_{k\ell}(t)].$$

This yields a piecewise linear deformation of the source figure.

Remark 2 (Edge combinatorics). *The left action permutes fibers canonically, but the adjacency rule of the Cayley fibergraph depends on the chosen ordering of the group elements. Therefore left multiplication need not act by graph automorphisms of* Γ_G. *In the animation we resolve this by a two-layer interpolation: we transport the source polygonal figure continuously via its vertices and, at the same time, we fade in the true target figure. This produces a continuous and visually faithful representation of the action.*

5 Row-by-row animation of the action

Let the ordered group elements be $g_1, \dots, g_n$. For a fixed actor $x \in G$, we display in the top row the figures

$$F_{g_1}, F_{g_2}, \dots, F_{g_n}$$

and continuously transform them into the bottom row

$$F_{xg_1}, F_{xg_2}, \dots, F_{xg_n}.$$

Thus each column corresponds to one element g_j of the original indexing, while the figure in that column is acted upon by left multiplication with x.

Running through all $x \in G$ produces a complete visualisation of the regular action of G on the set of figures. This yields an animated dictionary of the group law: the animation for x shows simultaneously how x sends every figure to another one.

6 Coupled substitution fractals from Cayley fiber figures

Let $G = \{g_1, \ldots, g_n\}$ be a finite group. For the visualisation we keep the local adjacency rule

$$(i,j) \sim (k,\ell) \quad \iff \quad g_i g_j = g_k g_\ell \text{ and } \big(|i-k| = 1 \text{ or } |j-\ell| = 1\big),$$

because it produces sparse and visually appealing polygonal figures. Thus each element $g \in G$ determines a figure

$$F_g = (V_g, E_g), \qquad V_g = \{(i,j) \mid g_i g_j = g\},$$

with $|V_g| = n$.

To make the construction reproducible, we first fix an ordering $g_1, \ldots, g_n$ of the elements. For permutation groups, a convenient choice is the lexicographic ordering by image vectors:

$$g < h \quad \iff \quad \big(g(1), \ldots, g(m)\big) \text{ is lexicographically smaller than } \big(h(1), \ldots, h(m)\big),$$

where $G \leq S_m$.

Each figure F_g has exactly n vertices. We number these vertices geometrically, for example from left to right and, in case of ties, from top to bottom. Hence every figure carries a canonical list

$$v_g(1), \ldots, v_g(n).$$

Coupled substitution rule

The basic idea is not to replace each vertex of F_g by another copy of F_g, but by the whole family of figures in a coordinated way. More precisely, the i-th vertex of F_g is replaced by a scaled copy of the figure F_{g_i}. Thus every figure contains one copy of every other figure.

Let $0 < r < 1$ be a fixed scaling factor. For each $g \in G$ and each vertex $v_g(i)$, let

$$\phi_{g,i} : \mathbb{R}^2 \to \mathbb{R}^2$$

be the affine contraction that places a copy of the unit square, scaled by the factor r, around the point $v_g(i)$. Starting from the initial figures

$$\mathcal{F}_g^{(0)} = F_g,$$

we define recursively

$$\mathcal{F}_g^{(k+1)} = \bigcup_{i=1}^{n} \phi_{g,i}\big(\mathcal{F}_{g_i}^{(k)}\big).$$

Thus the k-th iterate of the figure F_g is obtained by inserting, at its i-th vertex, a scaled copy of the k-th iterate of F_{g_i}.

Connector rule

The substitution above explains where the smaller figures are placed, but it remains to specify how the edges are propagated. Suppose that the outer figure F_g has an edge

$$i \sim j$$

between its i-th and j-th labelled vertices. After substitution, the points $v_g(i)$ and $v_g(j)$ have been replaced by copies of F_{g_i} and F_{g_j}. To connect these two inserted copies, we use the following nearest-neighbour rule.

Among all n vertices of the copy of F_{g_i} and all n vertices of the copy of F_{g_j}, choose a pair

$$(k, \ell) \in \{1, \ldots, n\}^2$$

for which the Euclidean distance is minimal. In other words, we choose

$$(k,\ell) = \underset{1\le a,b\le n}{\arg\min}\left\|\phi_{g,i}(v_{g_i}(a)) - \phi_{g,j}(v_{g_j}(b))\right\|.$$

Then we add an edge between these two nearest vertices. If several pairs have the same minimal distance, we break ties lexicographically.

This produces a coupled family of recursive graph-like fractals

$$\mathcal{F}^{(k)}_{g_1}, \dots, \mathcal{F}^{(k)}_{g_n},$$

in which every figure contains scaled copies of all the others, while the outer combinatorics still controls the connector edges between the inserted copies.

Example: C_3

Let

$$C_3 = \{e, a, a^2\}, \qquad a^3 = e.$$

With the lexicographic ordering induced by the permutation representation, we obtain an ordered list

$$g_1 = e, \qquad g_2 = a, \qquad g_3 = a^2.$$

Each figure F_g has three vertices. In the first substitution step, every figure F_g is replaced by three smaller figures:

$$\mathcal{F}^{(1)}_g = \phi_{g,1}(F_e) \cup \phi_{g,2}(F_a) \cup \phi_{g,3}(F_{a^2}).$$

Hence each of the three figures already contains one small copy of every figure in the family. Repeating the process yields a non-trivial coupled self-similar system rather than a single repetitive self-copy. A zoom animation is obtained by repeatedly selecting one nested copy and shrinking the viewing window towards it, in the same spirit as a classical Mandelbrot-style zoom.

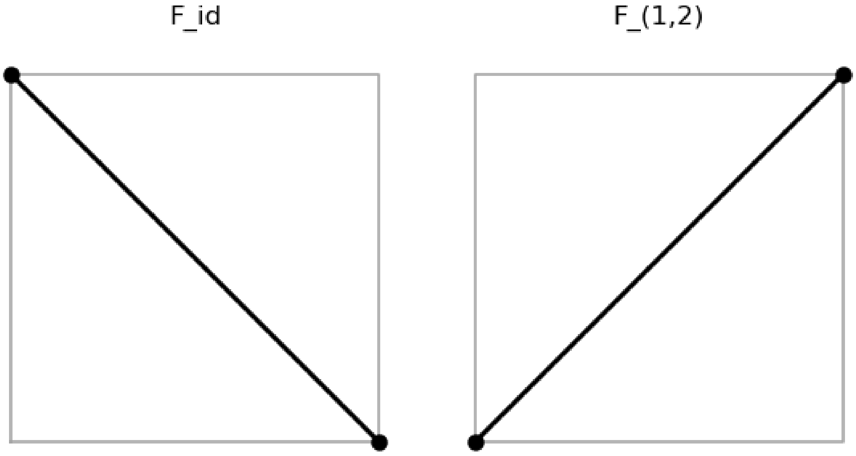

Figure 1: All figures of C2

Conclusion

The Cayley fibergraph provides a visual language for finite groups that is different from, but complementary to, the classical Cayley graph. By representing every product value as a global figure and by animating the regular action through continuous deformations, one obtains a geometric and intuitive picture of the multiplication structure of the group.

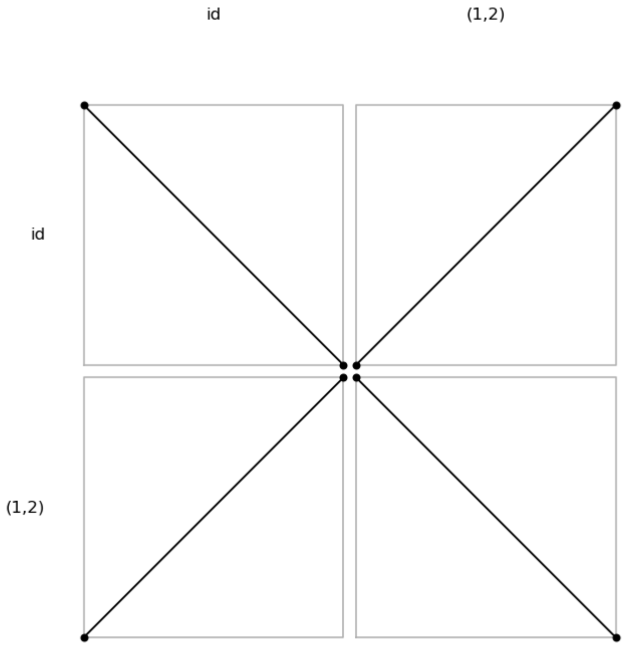

Figure 2: Cayley table rendered with the figures $F_{g_i g_j}$ of C2

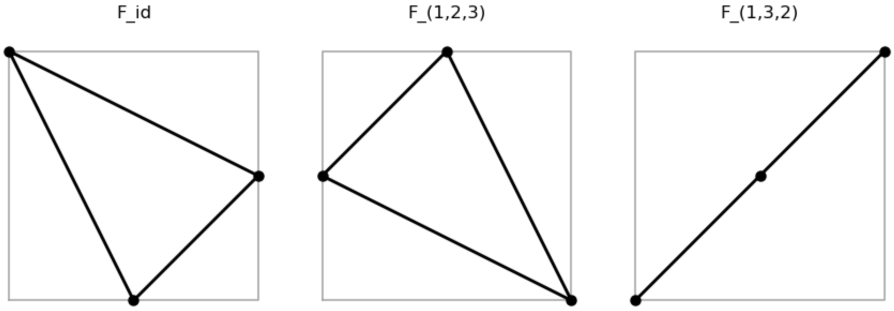

Figure 3: All figures of C3

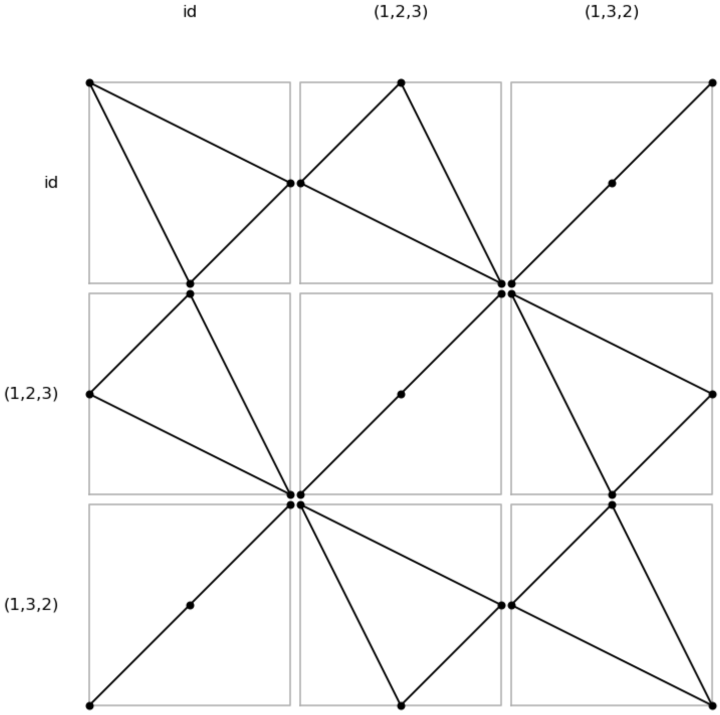

Figure 4: Cayley table rendered with the figures $F_{g_i g_j}$ of C3

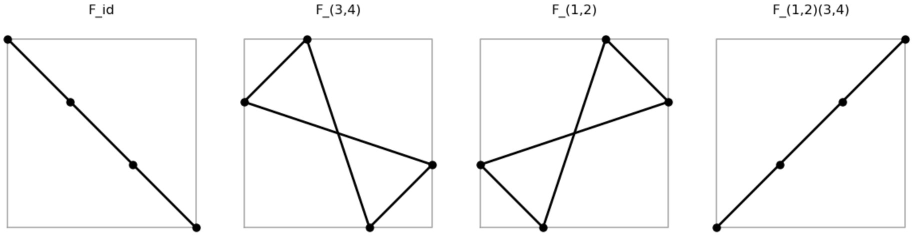

Figure 5: All figures of V4

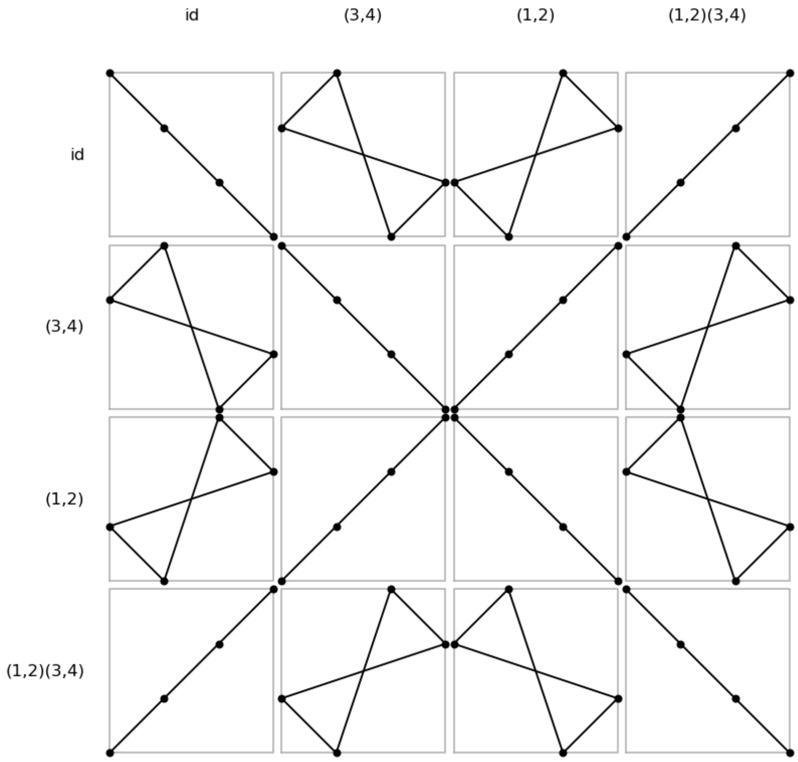

Figure 6: Cayley table rendered with the figures $F_{g_i g_j}$ of V4

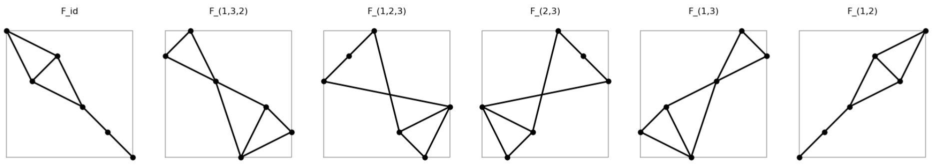

Figure 7: All figures of S3

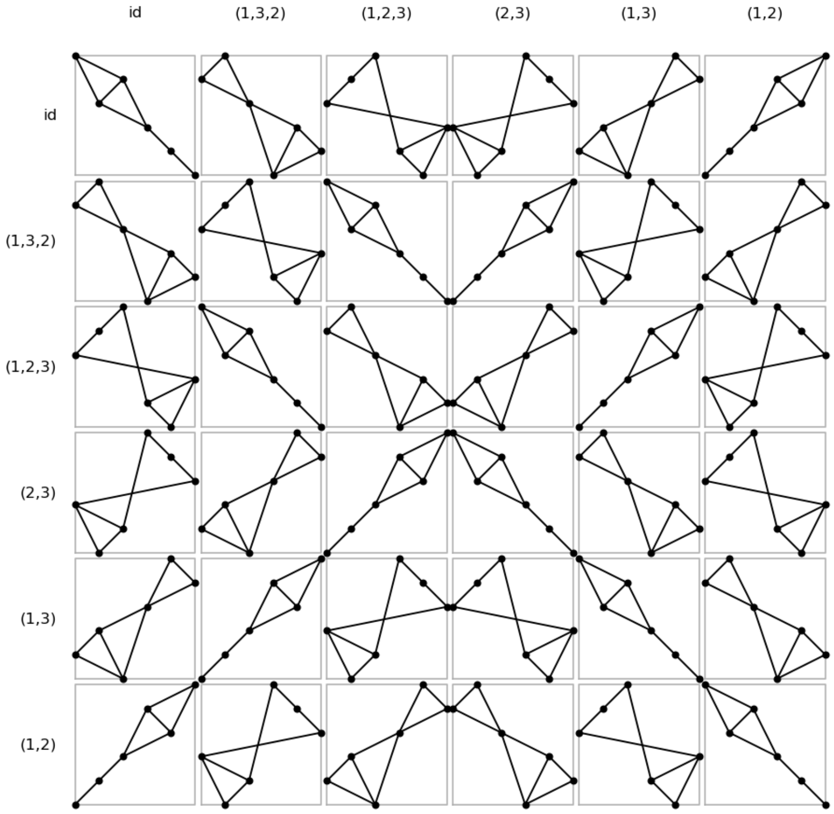

Figure 8: Cayley table rendered with the figures $F_{g_i g_j}$ of S3

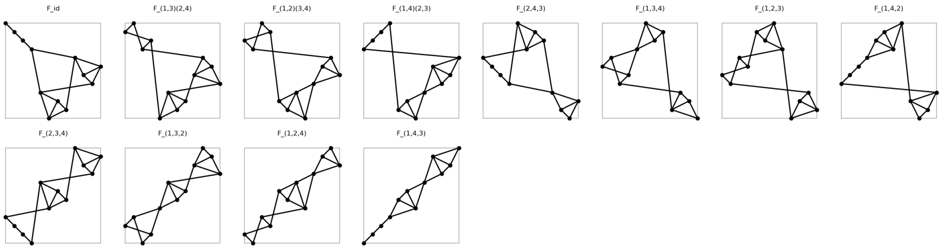

Figure 9: All figures of A4

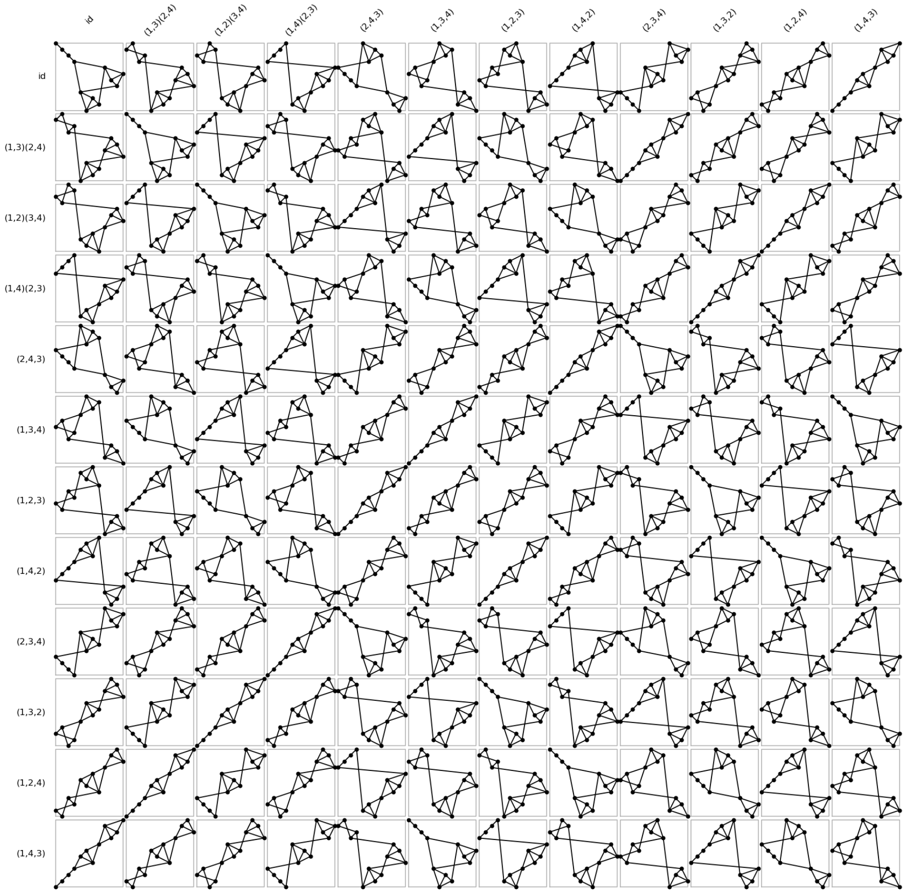

Figure 10: Cayley table rendered with the figures $F_{g_i g_j}$ of A4

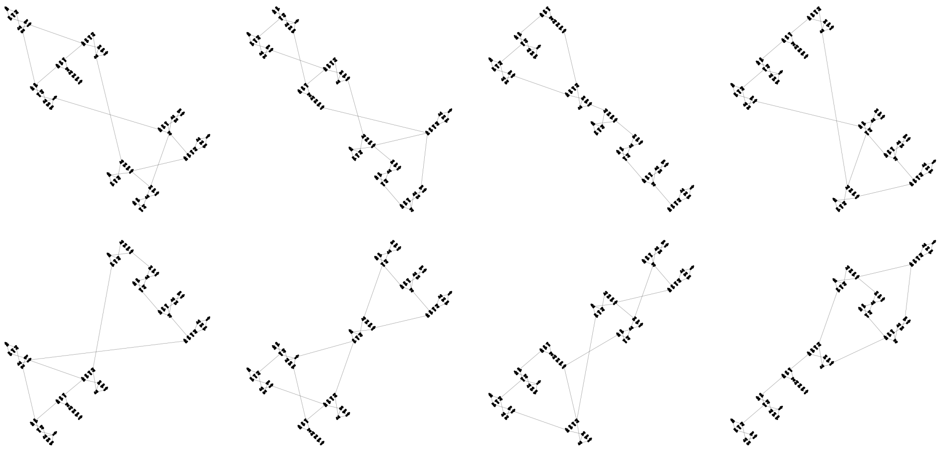

Figure 11: Fractal gallery at $k = 4$ of Quaternions-group

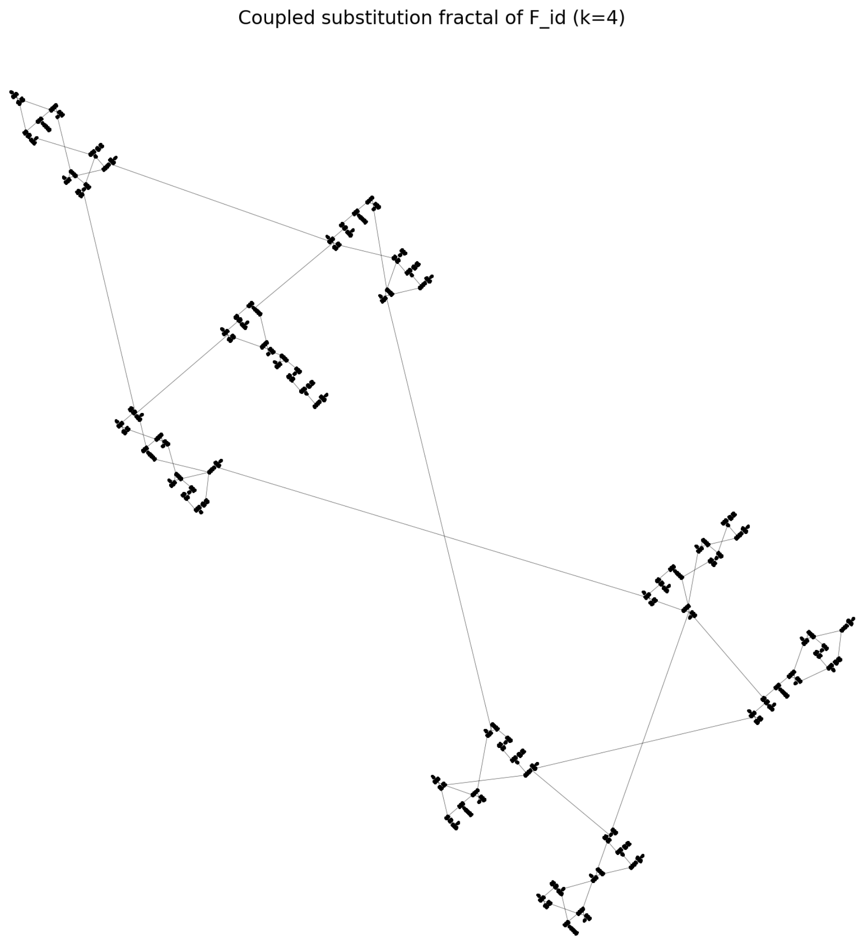

Figure 12: Fractal at $k = 4$ of neutral element in Quaternions-group