Exploring Pratt-Trees

Orges Leka

March 17, 2026

Abstract

This note explores a family of structures attached to Pratt trees from several related mathematical viewpoints. To each integer $n \geq 1$ we associate its *Pratt prime forest* and, for every prime p, the vertex count $m_p(n)$ recording how often the label p occurs in that forest. These exponents lead to the finite product identity

$$n \;=\; \prod_{p \text{ prime}} \Big(1 - \frac{1}{p}\Big)^{-m_p(n)},$$

which may be read as a multiplicative reformulation of n and also as a product formula for the divisibility probability $1/n$. We discuss this identity in several ways: through divisibility in finite residue rings, through an independent Bernoulli sieve indexed by the Pratt forest, through covariance kernels and Gram matrices, through Dirichlet series naturally attached to the exponents, and through sparse feature embeddings in Hilbert space. In particular, if $\phi(n) = (m_p(n))_p$, then a fixed weight vector recovers $\log n$ by an inner product. We also show that the passage from the usual valuation coordinates $(v_p(n))_p$ to the Pratt coordinates $(m_p(n))_p$ is given by an invertible triangular change of basis on finitely supported prime-indexed sequences. The purpose of the paper is exploratory: rather than pursuing a single maximal theorem, it records how one recursive combinatorial structure can be read multiplicatively, probabilistically, and linearly.

Contents

1 Pratt trees and Pratt prime forests

1.1 Pratt trees

Let p be prime. The *Pratt tree* T_p is a rooted tree with vertex labels in the primes defined recursively:

- For $p = 2$, T_2 is the single vertex labeled 2.
- For $p \geq 3$, the root is labeled p, and its children are the primes r dividing $p-1$, with multiplicity $v_r(p-1)$; each child labeled r is the root of a copy of T_r.

Thus, the subtree multiset below the root of T_p is precisely the Pratt prime forest of $p-1$ (defined below).

1.2 Pratt prime forests and vertex counts

Definition 1 (Pratt prime forest)**.** Let $n \geq 1$ with prime factorization $n = \prod_p p^{v_p(n)}$. The *Pratt prime forest* of n is the multiset

$$F(n) := \biguplus_{p|n} v_p(n) \cdot T_p,$$

i.e. the disjoint union of $v_p(n)$ copies of T_p for each prime divisor p of n.

Definition 2 (Vertex counts)**.** For a prime q, let

$$m_q(n) \ := \ \#\{\text{vertices labeled } q \text{ in } F(n)\},$$

counted with multiplicity.

Lemma 1 (Complete additivity)**.** *For every prime q and all $n, m \geq 1$,*

$$m_q(nm) = m_q(n) + m_q(m).$$

Equivalently,

$$m_q(n) = \sum_{p|n} v_p(n)\, m_q(p).$$

Proof. For each prime p, the multiplicity of T_p in $F(nm)$ is

$$v_p(nm) = v_p(n) + v_p(m).$$

Hence $F(nm)$ is the multiset sum of $F(n)$ and $F(m)$. Counting vertices labeled q therefore gives the first identity, and the second is just the same statement written after expanding the forest of n prime by prime. □

Lemma 2 (Prime recursion)**.** *If $p \geq 3$ is prime, then for every prime q,*

$$m_q(p) = m_q(p-1) + \mathbf{1}_{\{q=p\}}.$$

Proof. In T_p, the root contributes one vertex labeled p. All other vertices lie in the children subtrees, which (with multiplicity) form the Pratt prime forest of $p-1$. Counting q-labels gives the recursion. □

2 The Pratt product formula

Define the multiplicative functional

$$W(n) \ := \prod_{p \text{ prime}} \left(\frac{p}{p-1}\right)^{m_p(n)} \ = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}. \tag{1}$$

The product is *finite* for each fixed n, because $F(n)$ has finitely many vertices.

Theorem 1 (Pratt product identity)**.** *For every $n \geq 1$,*

$$W(n) = n, \qquad \textit{i.e.} \qquad n \ = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}.$$

Equivalently,

$$\frac{1}{n} \ = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{m_p(n)}.$$

Proof. First note from Lemma 1 that $m_p(nm) = m_p(n) + m_p(m)$ for each prime p, so W is completely multiplicative:

$$W(nm) = W(n)W(m).$$

It therefore suffices to prove $W(p) = p$ for primes p.

We argue by strong induction on the prime p. The base case $p = 2$ holds because $m_2(2) = 1$ and $m_q(2) = 0$ for $q \neq 2$, hence $W(2) = 2$.

For an odd prime $p \geq 3$, Lemma 2 gives $m_q(p) = m_q(p-1)$ for $q \neq p$, and $m_p(p) = m_p(p-1) + 1$. Plugging into (1) yields the recursion

$$W(p) = \left(\frac{p}{p-1}\right) W(p-1).$$

By induction on the integer $p - 1$ (or by complete multiplicativity and the already established prime cases $\leq p - 1$), we have $W(p-1) = p - 1$, so $W(p) = p$. Finally, complete multiplicativity implies $W(n) = n$ for all n. □

Remark 1 (Uniqueness (finite support)). For each fixed n, the family $(m_p(n))_p$ has finite support. With this finiteness, the representation in Theorem 1 is unique: if $\prod_p (1 - \frac{1}{p})^{-a_p} = \prod_p (1 - \frac{1}{p})^{-b_p}$ with only finitely many nonzero a_p, b_p, then $a_p = b_p$ for all primes p. Indeed, let P be the largest prime with $a_P \neq b_P$. The P-adic valuation of the left-hand side minus the right-hand side isolates the exponent at P, because every factor with larger prime index is absent and every smaller prime contributes no P in the denominator. Thus the exponents agree term by term.

3 Two probability interpretations of $1/n$

3.1 Divisibility as a probability

Fix $N \geq 1$ and put $L := \mathrm{lcm}(1, 2, \ldots, N)$. Consider the finite probability space

$$\Omega_N := \mathbb{Z}/L\mathbb{Z}, \qquad U \sim \text{uniform on } \Omega_N.$$

For each $1 \leq n \leq N$ define the indicator random variable

$$X_n := \mathbf{1}_{\{n|U\}}.$$

Lemma 3. *For* $1 \leq n \leq N$,

$$\mathbb{P}(X_n = 1) = \frac{1}{n}.$$

More generally, for $1 \leq a, b \leq N$,

$$\mathbb{P}(X_a = 1,\ X_b = 1) = \frac{1}{\mathrm{lcm}(a, b)}.$$

Proof. Because $n \mid L$, the subset $\{u \in \Omega_N : n \mid u\}$ is the subgroup $n\mathbb{Z}/L\mathbb{Z}$ of size L/n, hence has probability $(L/n)/L = 1/n$. The joint event $\{a \mid U\} \cap \{b \mid U\}$ is $\{\operatorname{lcm}(a,b) \mid U\}$, giving the second claim. □

Remark 2. Letting $N \to \infty$ recovers the classical natural-density statement: for a random integer M (for instance uniform on $\{1, \dots, M_0\}$ with $M_0 \to \infty$), $\mathbb{P}(n \mid M) \to 1/n$. The finite model above is convenient because probabilities are exact.

3.2 A Bernoulli sieve indexed by the Pratt forest

The Pratt identity can also be read as a product of *survival probabilities*. Fix n and its Pratt forest $F(n)$. For each vertex $v \in F(n)$ with prime label $\ell(v) = p$, let B_v be an independent Bernoulli random variable with

$$\mathbb{P}(B_v = 1) = 1 - \frac{1}{p}.$$

Define the event that all tests are passed:

$$A_n := \bigcap_{v \in F(n)} \{B_v = 1\}.$$

Then independence gives

$$\mathbb{P}(A_n) = \prod_{v \in F(n)} \mathbb{P}(B_v = 1) = \prod_p (1 - 1/p)^{m_p(n)} = \frac{1}{n}$$

by Theorem 1.

Remark 3 (A "Pratt sieve" intuition). For a fixed prime p, the factor $(1 - 1/p)$ is the probability that a uniformly random residue mod p is *nonzero*. Thus, one may think of each vertex labeled p as imposing a random nonvanishing condition modulo p. The point is not that this model is identical to ordinary divisibility, but that the same product $1/n$ arises from a recursively organized family of independent local tests.

4 $\gcd$, lcm, and covariance kernels

4.1 Rewriting $1/\operatorname{lcm}$ and $1/\gcd$ in Pratt form

For any positive integer t, Theorem 1 gives the tautological identities

$$\frac{1}{t} = \prod_p (1 - 1/p)^{m_p(t)}. \tag{2}$$

In particular,

$$\frac{1}{\operatorname{lcm}(a,b)} = \prod_p (1 - 1/p)^{m_p(\operatorname{lcm}(a,b))}, \qquad \frac{1}{\gcd(a,b)} = \prod_p (1 - 1/p)^{m_p(\gcd(a,b))}. \tag{3}$$

Remark 4 (A warning about max/min). One might hope for analogues of the valuation identities $v_p(\mathrm{lcm}) = \max(v_p(a), v_p(b))$ and $v_p(\gcd) = \min(v_p(a), v_p(b))$. However, the Pratt exponents $m_p(\cdot)$ are *not* valuations, and in general

$$m_p(\mathrm{lcm}(a,b)) \neq \max(m_p(a), m_p(b)), \qquad m_p(\gcd(a,b)) \neq \min(m_p(a), m_p(b)).$$

For instance, $a = 2$, $b = 3$ gives $\mathrm{lcm}(a,b) = 6$ and $\prod_p (1 - 1/p)^{\max(m_p(2), m_p(3))} = 1/3 \neq 1/6$. So the correct way to express $1/\mathrm{lcm}$ and $1/\gcd$ in this framework is (3).

4.2 Covariance as an expectation and a natural embedding

Recall that for any pair of random variables X, Y,

$$\mathrm{Cov}(X,Y) = \mathbb{E}\Big[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\Big] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

In our setting, $X_a := \mathbf{1}_{\{a|U\}}$ and $X_b := \mathbf{1}_{\{b|U\}}$ are indicator variables, so that

$$\mathbb{E}[X_a] = \mathbb{P}(a \mid U) = \frac{1}{a}, \qquad \mathbb{E}[X_a X_b] = \mathbb{P}(a \mid U,\, b \mid U) = \frac{1}{\mathrm{lcm}(a,b)}.$$

Hence

$$\mathrm{Cov}(X_a, X_b) = \frac{1}{\mathrm{lcm}(a,b)} - \frac{1}{ab}.$$

Natural Hilbert-space embedding. Let $H := L^2(\Omega_N, \mathbb{P})$ with inner product $\langle f, g\rangle = \mathbb{E}[fg]$. Define the centered embedding

$$\Phi(a) \;:=\; X_a - \mathbb{E}[X_a] = X_a - \frac{1}{a}.$$

Then for all a, b,

$$\langle \Phi(a), \Phi(b)\rangle_H = \mathbb{E}\Big[(X_a - \mathbb{E}X_a)(X_b - \mathbb{E}X_b)\Big] = \mathrm{Cov}(X_a, X_b).$$

Thus the covariance matrix $G_N = (\mathrm{Cov}(X_a, X_b))_{1 \leq a,b \leq N}$ is precisely the Gram matrix of the centered feature vectors $\Phi(1), \ldots, \Phi(N)$ in $L^2(\Omega_N)$.

Consequently, for every $c \in \mathbb{R}^N$,

$$c^\top G_N c = \mathrm{Var}\Big(\sum_{a=1}^{N} c_a\, \Phi(a)\Big) \geq 0,$$

which restates the positive semidefiniteness of G_N in the most transparent Hilbert-space form.

Corollary 1 (Positive semidefiniteness)**.** *The matrix*

$$G_N := \Big(\mathrm{Cov}(X_a, X_b)\Big)_{1\le a,b\le N} = \left(\frac{1}{\mathrm{lcm}(a,b)} - \frac{1}{ab}\right)_{1\le a,b\le N}$$

is positive semidefinite.

Proof. G_N is a covariance matrix, hence positive semidefinite by definition: for any $c \in \mathbb{R}^N$, $c^\top G_N c = \mathrm{Var}\Big(\sum_{a=1}^N c_a X_a\Big) \ge 0$. □

5 Gram matrices and a rank formula

5.1 Centered features

Define centered random variables

$$Y_n := X_n - \mathbb{P}(X_n = 1) = X_n - \frac{1}{n} \qquad (1 \le n \le N).$$

Then

$$\langle Y_a, Y_b\rangle := \mathbb{E}[Y_a Y_b] = \mathrm{Cov}(X_a, X_b),$$

so G_N is the Gram matrix of the vectors/functions $Y_1, \dots, Y_N$ in $L^2(\Omega_N)$.

Note that $Y_1 \equiv 0$, hence the first row and column of G_N are identically 0. So $\mathrm{rank}(G_N) \le N - 1$. The next theorem shows this is sharp.

5.2 Linear independence of the centered indicators

Let $L = \mathrm{lcm}(1, \dots, N)$. For each divisor $d \mid L$, define the disjoint indicator

$$Z_d := \mathbf{1}_{\{\gcd(U,L)=d\}}.$$

The family $\{Z_d : d \mid L\}$ is linearly independent (it has disjoint support) and spans all functions that depend only on $\gcd(U, L)$. Moreover, for $n \mid L$,

$$X_n = \mathbf{1}_{\{n|U\}} = \mathbf{1}_{\{n|\gcd(U,L)\}} = \sum_{\substack{d|L\\ n|d}} Z_d. \tag{4}$$

Lemma 4. *The functions $X_1, \dots, X_N$ are linearly independent in $L^2(\Omega_N)$.*

Proof. Suppose $\sum_{n=1}^N c_n X_n \equiv 0$. Expand using (4) and collect coefficients of Z_d: the coefficient of Z_d equals $\sum_{n|d,\ 1\le n\le N} c_n$. Since the Z_d are linearly independent, every such coefficient must be 0.

In particular, for each $d \in \{1, 2, \dots, N\}$ (which is a divisor of L), we have

$$0 = \sum_{n|d} c_n.$$

Now solve recursively by increasing d: for $d = 1$ we get $c_1 = 0$; for $d = 2$ we get $c_1 + c_2 = 0$ hence $c_2 = 0$; for $d = 3$ we get $c_1 + c_3 = 0$ hence $c_3 = 0$; and in general, when d is fixed all proper divisors of d are $< d$ and already have zero coefficients, so the relation $\sum_{n|d} c_n = 0$ forces $c_d = 0$. Thus all $c_n = 0$, proving independence. □

Lemma 5. *The centered functions $Y_2, \ldots, Y_N$ are linearly independent.*

Proof. Assume $\sum_{n=2}^{N} c_n Y_n \equiv 0$. Since $Y_n = X_n - \frac{1}{n} X_1$ (and $X_1 \equiv 1$), this implies

$$\sum_{n=2}^{N} c_n X_n \; - \; \left(\sum_{n=2}^{N} \frac{c_n}{n} \right) X_1 \equiv 0.$$

By Lemma 4, all coefficients in this linear combination of $X_1, \ldots, X_N$ must vanish. In particular $c_n = 0$ for every $n = 2, \ldots, N$. □

5.3 Rank of the covariance/Gram matrix

Theorem 2 (Rank formula)**.** *For every $N \geq 1$,*

$$\operatorname{rank}(G_N) = N - 1.$$

Equivalently, the only linear relation among $Y_1, \ldots, Y_N$ is $Y_1 \equiv 0$.

Proof. We already observed $\operatorname{rank}(G_N) \leq N - 1$ because $Y_1 \equiv 0$. On the other hand, G_N is the Gram matrix of $Y_1, \ldots, Y_N$, so its rank equals $\dim \operatorname{span}\{Y_1, \ldots, Y_N\}$. By Lemma 5, the set $\{Y_2, \ldots, Y_N\}$ is linearly independent, hence spans a subspace of dimension $N - 1$. Therefore $\operatorname{rank}(G_N) = N - 1$. □

6 A simple upper bound for the Pratt exponents

Recall the finite product representation

$$n \; = \; \prod_{p \text{ prime}} \left(1 - \frac{1}{p} \right)^{-m_p(n)} \; = \; \prod_{p \text{ prime}} \left(\frac{p}{p-1} \right)^{m_p(n)}, \tag{5}$$

where $m_p(n) \in \mathbb{Z}_{\geq 0}$ and only finitely many $m_p(n)$ are nonzero.

Proposition 1 (Pointwise bound)**.** *For every integer $n \geq 2$ and every prime p,*

$$m_p(n) \; \leq \; \frac{\log n}{\log\left(\frac{p}{p-1}\right)}.$$

Proof. Taking natural logarithms in (5) yields

$$\log n = \sum_{p \text{ prime}} m_p(n) \, \log\left(\frac{p}{p-1} \right).$$

All summands are nonnegative because $m_p(n) \geq 0$ and $\log\left(\frac{p}{p-1}\right) > 0$. Hence for each fixed prime p we have

$$m_p(n) \log\left(\frac{p}{p-1}\right) \;\leq\; \log n,$$

and division by $\log\left(\frac{p}{p-1}\right)$ gives the claim. □

7 The valuation coordinates and the Pratt coordinates

Let $(p_k)_{k\geq 1} = (2, 3, 5, 7, \dots)$ be the increasing sequence of primes. For $n \in \mathbb{N}$ define the valuation vector and the Pratt vector by

$$v(n) := (v_{p_k}(n))_{k\geq 1}, \qquad \phi(n) := (m_{p_k}(n))_{k\geq 1}.$$

Both vectors have finite support, so they belong to the space c_{00} of finitely supported real sequences.

Proposition 2 (Triangular change of basis)**.** *There is a linear operator $A : c_{00} \to c_{00}$ such that*

$$\phi(n) = A\, v(n) \qquad (n \in \mathbb{N}).$$

With respect to the ordered prime basis $(e_k)_{k\geq 1}$, the matrix of A is

$$A = (a_{ij})_{i,j\geq 1}, \qquad a_{ij} := m_{p_i}(p_j).$$

Moreover, A is lower triangular with diagonal entries $a_{jj} = 1$, hence invertible on c_{00}. In particular, the valuation coordinates $(v_p(n))_p$ and the Pratt coordinates $(m_p(n))_p$ determine one another by an invertible basis change.

Proof. By Lemma 1,

$$m_{p_i}(n) = \sum_{j\geq 1} v_{p_j}(n)\, m_{p_i}(p_j),$$

where the sum is finite because $v_{p_j}(n) = 0$ for all but finitely many j. This is exactly the coordinate formula for $\phi(n) = Av(n)$.

If $i > j$, then $p_i > p_j$, and the label p_i cannot occur in the Pratt tree T_{p_j}: every label in T_{p_j} is at most p_j. Thus $a_{ij} = m_{p_i}(p_j) = 0$ whenever $i > j$, so A is lower triangular. Also, the root of T_{p_j} contributes one vertex labeled p_j, hence $a_{jj} = m_{p_j}(p_j) = 1$. Therefore A is unit lower triangular. Any finite principal truncation of a unit lower triangular matrix is invertible, and since vectors in c_{00} have finite support, this gives an inverse on all of c_{00}. □

Remark 5. Proposition 2 makes precise the idea that the Pratt exponents refine ordinary prime valuations. The two coordinate systems carry the same information, but the transition matrix is not diagonal: the recursive ancestry encoded in the Pratt trees mixes the prime coordinates in a triangular way.

8 Extension to $\mathbb{Q}_{>0}$ and a linear readout of $\log n$

8.1 Pratt exponents for integers and the weight vector

For $n \in \mathbb{N}$ the vector

$$\phi(n) = \sum_{k\geq 1} m_{p_k}(n) e_k$$

is finitely supported, so in particular it belongs to ℓ^2. Recall the finite product identity

$$n \;=\; \prod_{p\ \text{prime}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}. \tag{6}$$

Introduce the weight vector

$$w = (w_k)_{k\geq 1}, \qquad w_k := -\log\!\left(1 - \frac{1}{p_k}\right) > 0.$$

Since $w_k \sim 1/p_k$ as $k \to \infty$ and $\sum_k p_k^{-2} < \infty$, we have $w \in \ell^2$. Taking logarithms in (6) gives the following.

Proposition 3 (Linear readout of the logarithm)**.** *For every $n \geq 1$,*

$$\log n = \sum_{k\geq 1} m_{p_k}(n)\, w_k = \langle \phi(n), w\rangle_{\ell^2}.$$

Equivalently,

$$n = \exp\!\big(\langle \phi(n), w\rangle\big).$$

Proof. Because $\phi(n)$ has finite support, the inner product with w is just a finite sum. Taking logarithms in (6) yields

$$\log n = \sum_{p\ \text{prime}} m_p(n)\left(-\log\!\left(1 - \frac{1}{p}\right)\right) = \sum_{k\geq 1} m_{p_k}(n) w_k.$$

This is exactly $\langle \phi(n), w\rangle$. □

Remark 6. This Hilbert-space point of view is close in spirit to the embedding discussed in the companion paper on the first 100,000 numbers, where the feature vector $\phi(n)$ is used as a sparse prime-indexed coordinate system and the logarithm becomes a fixed linear observable on that space.[1]

8.2 Canonical extension to positive rationals

Let $\mathbb{Q}_{>0}$ be the multiplicative group of positive rationals. Every $q \in \mathbb{Q}_{>0}$ can be written uniquely as $q = a/b$ with $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$.

[1] See the discussion around the feature map and the Hilbert-space readout in the linked paper by the same author.

Definition 3 (Pratt exponent of a rational)**.** For $q = a/b \in \mathbb{Q}_{>0}$ in lowest terms, define for each prime p

$$m_p(q) \ := \ m_p(a) - m_p(b) \in \mathbb{Z}, \tag{7}$$

and define the feature vector

$$\phi(q) := (m_p(q))_p \in c_{00}(\mathbb{P}).$$

Lemma 6 (Additivity on $\mathbb{Q}_{>0}$)**.** *For all $q_1, q_2 \in \mathbb{Q}_{>0}$ and all primes p one has*

$$m_p(q_1 q_2) = m_p(q_1) + m_p(q_2), \qquad \text{hence} \qquad \phi(q_1 q_2) = \phi(q_1) + \phi(q_2).$$

Proof. This follows immediately from the definition (7) and the complete additivity of $m_p(\cdot)$ on $\mathbb{N}$. □

Proposition 4 (Pratt product and linear readout on $\mathbb{Q}_{>0}$)**.** *For every $q \in \mathbb{Q}_{>0}$ one has the finite product identity*

$$q \ = \prod_{p \text{ prime}} \Big(1 - \frac{1}{p}\Big)^{-m_p(q)}, \tag{8}$$

and the logarithmic identity

$$\log q \ = \ \sum_p m_p(q) \left(-\log\Big(1 - \frac{1}{p}\Big)\right). \tag{9}$$

Proof. Write $q = a/b$ in lowest terms. Apply (6) to a and b and divide:

$$\frac{a}{b} = \prod_p \Big(1 - \frac{1}{p}\Big)^{-m_p(a)} \cdot \prod_p \Big(1 - \frac{1}{p}\Big)^{m_p(b)} = \prod_p \Big(1 - \frac{1}{p}\Big)^{-(m_p(a) - m_p(b))}.$$

This is (8). Taking logarithms yields (9). □

8.3 Continued fraction convergents and a one-functional limit

Fix a real number $r > 0$ and let

$$r = [a_0; a_1, a_2, \dots]$$

be its continued fraction expansion. Denote by $r_k = p_k/q_k \in \mathbb{Q}_{>0}$ the k-th convergent. It is classical that $r_k \to r$ as $k \to \infty$.

Theorem 3 (Convergents yield a sequence with prescribed log-limit)**.** *Let $r > 0$ and $(r_k)_{k \geq 0}$ be its continued fraction convergents. Then the associated feature vectors $\phi(r_k) \in c_{00}$ satisfy*

$$\lim_{k \to \infty} \langle w, \phi(r_k) \rangle = \lim_{k \to \infty} \log(r_k) = \log(r). \tag{10}$$

Proof. By Proposition 4, $\langle w, \phi(r_k) \rangle = \log(r_k)$ holds for every k. Since $r_k \to r$ and $\log(\cdot)$ is continuous on $(0, \infty)$, we have $\log(r_k) \to \log(r)$. This proves (10). □

Remark 7. The identity $\langle w, \phi(r_k)\rangle \to \log(r)$ is not by itself a statement of weak convergence in ℓ^2; it only records convergence against one fixed test vector w. What it shows is that every positive real number can be approximated by rationals whose Pratt feature vectors reproduce the correct logarithmic limit through the same linear functional.

9 A coefficient recursion from differentiating a Dirichlet series

Fix $n \in \mathbb{N}$ and write $m_p := m_p(n)$ for the Pratt exponents. Consider the Dirichlet series

$$D_n(s) \;:=\; \prod_p \Big(1 - p^{-s}\Big)^{-m_p} \;=\; \sum_{r\geq 1} \frac{\chi_n(r)}{r^s}, \qquad \Re(s) > 1, \tag{11}$$

where the coefficients $\chi_n(r)$ are given multiplicatively by the Euler factors

$$(1 - p^{-s})^{-m_p} = \sum_{e\geq 0} \binom{m_p + e - 1}{e} p^{-es}.$$

Equivalently,

$$\chi_n(r) = \prod_{p\mid r} \binom{m_p(n) + v_p(r) - 1}{v_p(r)}. \tag{12}$$

Proposition 5 (Derivative identity and coefficient comparison)**.** *For every $r \geq 2$ we have the recursion*

$$\chi_n(r) \;=\; \frac{1}{\log r} \sum_{p\mid r} \sum_{k=1}^{v_p(r)} m_p(n)\, \log(p)\, \chi_n\!\left(\frac{r}{p^k}\right). \tag{13}$$

(And $\chi_n(1) = 1$ by (11)*.)*

Proof. Differentiate $D_n(s)$ in two ways.

(1) Termwise differentiation of the Dirichlet series. From (11) we get

$$D_n'(s) = \frac{d}{ds} \sum_{r\geq 1} \chi_n(r)\, r^{-s} = \sum_{r\geq 1} \chi_n(r)\, \frac{d}{ds}(r^{-s}) = -\sum_{r\geq 1} \chi_n(r)\, \log(r)\, r^{-s}.$$

Hence

$$-\,D_n'(s) = \sum_{r\geq 1} \chi_n(r)\, \log(r)\, r^{-s}. \tag{14}$$

(2) Logarithmic differentiation of the Euler product. We compute

$$\log D_n(s) = -\sum_p m_p \log(1 - p^{-s}),$$

so

$$\frac{D_n'(s)}{D_n(s)} = -\sum_p m_p \cdot \frac{d}{ds}\log(1-p^{-s}) = -\sum_p m_p \cdot \frac{\log(p)\,p^{-s}}{1-p^{-s}}.$$

Using the geometric series expansion

$$\frac{p^{-s}}{1-p^{-s}} = \sum_{k\geq 1} p^{-ks},$$

we obtain

$$-D_n'(s) = D_n(s)\sum_p m_p\,\log(p)\sum_{k\geq 1} p^{-ks}. \tag{15}$$

(3) Coefficient comparison. Insert $D_n(s) = \sum_{t\geq 1}\chi_n(t)t^{-s}$ into (15):

$$-D_n'(s) = \left(\sum_{t\geq 1}\frac{\chi_n(t)}{t^s}\right)\left(\sum_p\sum_{k\geq 1}\frac{m_p\log(p)}{p^{ks}}\right).$$

The coefficient of r^{-s} on the right-hand side equals

$$\sum_{p^k\mid r} m_p\log(p)\,\chi_n\!\left(\frac{r}{p^k}\right) = \sum_{p\mid r}\sum_{k=1}^{v_p(r)} m_p\log(p)\,\chi_n\!\left(\frac{r}{p^k}\right).$$

Comparing with (14), we get for every $r\geq 1$ the identity

$$\chi_n(r)\log(r) = \sum_{p\mid r}\sum_{k=1}^{v_p(r)} m_p\log(p)\,\chi_n\!\left(\frac{r}{p^k}\right).$$

For $r\geq 2$ we can divide by $\log r\neq 0$ and obtain (13). □

10 Bounds for the coefficients $\chi_n(r)$

Fix $n\in\mathbb{N}$. For $r\geq 1$ write $v_p(r)$ for the p-adic valuation and $m_p(n)\in\mathbb{Z}_{\geq 0}$ for the Pratt exponents of n. We consider the multiplicative function

$$\chi_n(r) \;:=\; \prod_{p\mid r}\binom{m_p(n)+v_p(r)-1}{v_p(r)}.$$

If $m_p(n)=0$ and $v_p(r)\geq 1$, then the local binomial coefficient is 0, hence $\chi_n(r)=0$. So effectively χ_n is supported on integers whose prime factors lie in the finite set $\{p:\ m_p(n)>0\}$.

Local bounds from binomial inequalities. Let $m \ge 1$ and $v \ge 1$ and set $N = m+v-1$ and $K = v$. A standard inequality valid for all $1 \le K \le N$ is

$$\frac{N^K}{K^K} \le \binom{N}{K} \le \frac{N^K}{K!} < \left(\frac{eN}{K}\right)^K.$$

Applying this with $N = m + v - 1$ and $K = v$ yields, for $m \ge 1$ and $v \ge 1$,

$$\left(\frac{m+v-1}{v}\right)^v \le \binom{m+v-1}{v} \le \left(\frac{e(m+v-1)}{v}\right)^v. \tag{16}$$

Global bounds for $\chi_n(r)$. Write $r = \prod_p p^{v_p(r)}$. For every $r \ge 1$ we obtain from (16):

$$\chi_n(r) = \prod_{p|r} \binom{m_p(n) + v_p(r) - 1}{v_p(r)} \tag{17}$$

$$\le \prod_{p|r} \left(\frac{e\,(m_p(n) + v_p(r) - 1)}{v_p(r)}\right)^{v_p(r)}. \tag{18}$$

Equivalently,

$$\log \chi_n(r) \le \sum_{p|r} v_p(r)\Big(1 + \log(m_p(n) + v_p(r) - 1) - \log v_p(r)\Big).$$

A polynomial-in-v bound when m is fixed. For fixed $m \ge 1$ the binomial coefficient is actually polynomial in v:

$$\binom{m+v-1}{v} = \binom{m+v-1}{m-1} = \frac{1}{(m-1)!}\prod_{j=1}^{m-1}(v+j).$$

Hence for all $v \ge 0$,

$$\binom{m+v-1}{v} \le \frac{(v+m-1)^{m-1}}{(m-1)!}. \tag{19}$$

Consequently, letting $S(n) := \{p : \ m_p(n) > 0\}$ (a finite set), we get for all $r \ge 1$:

$$\chi_n(r) \le \prod_{p\in S(n)} \frac{\Big(v_p(r) + m_p(n) - 1\Big)^{m_p(n)-1}}{(m_p(n)-1)!}. \tag{20}$$

Since $v_p(r) \le \log r / \log p$, this shows that for fixed n the growth of $\chi_n(r)$ is at most polylogarithmic in r (with constants depending on n).

Remark. The bounds (18) and (20) are useful because they make the coefficients in later Dirichlet-series constructions completely explicit.

11 Integrating $D_n(s)$ in two ways

Fix $n \in \mathbb{N}$. Recall the Dirichlet series

$$D_n(s) \;:=\; \sum_{r\geq 1} \frac{\chi_n(r)}{r^s} \qquad (s \in \mathbb{C}),$$

where χ_n is defined multiplicatively by the local rule

$$\chi_n(p^k) \;=\; \binom{m_p(n)+k-1}{k} \qquad (k \geq 0),$$

and $m_p(n)$ is the Pratt exponent of p in the Pratt forest of n. Then D_n admits the finite Euler product

$$D_n(s) = \prod_{p \text{ prime}} \left(\sum_{k\geq 0} \binom{m_p(n)+k-1}{k} p^{-ks} \right) = \prod_p \Big(1-p^{-s}\Big)^{-m_p(n)}. \tag{21}$$

In particular, $\lim_{\sigma\to+\infty} D_n(\sigma) = 1$.

A generalized von Mangoldt function. Define an arithmetic function $\Lambda_n : \mathbb{N} \to \mathbb{R}$ by

$$\Lambda_n(r) \;:=\; \begin{cases} m_p(n)\log p, & r = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Proposition 6 (Logarithmic derivative of D_n)**.** *For every s with $\Re(s) > 0$ one has*

$$-\frac{D_n'(s)}{D_n(s)} \;=\; \sum_{r\geq 2} \frac{\Lambda_n(r)}{r^s}. \tag{23}$$

Proof. Differentiate the Euler product (21) logarithmically:

$$\frac{D_n'(s)}{D_n(s)} = \sum_p \frac{d}{ds}\Big(-m_p(n)\log(1-p^{-s})\Big) = -\sum_p m_p(n)\frac{(\log p)\, p^{-s}}{1-p^{-s}}.$$

Expanding $\frac{p^{-s}}{1-p^{-s}} = \sum_{k\geq 1} p^{-ks}$ yields

$$-\frac{D_n'(s)}{D_n(s)} = \sum_p \sum_{k\geq 1} m_p(n)(\log p)\, p^{-ks} = \sum_{r\geq 2} \frac{\Lambda_n(r)}{r^s},$$

which is exactly (23). □

First integration: integrating the Dirichlet series for $-D'/D$. Fix $\sigma > 0$ real. Since $D_n(\sigma) \to 1$ as $\sigma \to +\infty$, we may write

$$\log D_n(\sigma) = -\int_\sigma^\infty \frac{D_n'(u)}{D_n(u)}\,du = \int_\sigma^\infty \left(-\frac{D_n'(u)}{D_n(u)}\right) du.$$

Insert (23) and integrate termwise:

$$\log D_n(\sigma) = \sum_{r\geq 2} \Lambda_n(r) \int_\sigma^\infty r^{-u}\,du = \sum_{r\geq 2} \frac{\Lambda_n(r)}{\log r}\, r^{-\sigma}. \tag{24}$$

Second integration: integrating the Euler product directly. Starting from (21), take logarithms and expand $\log(1-x) = -\sum_{k\geq 1} x^k/k$:

$$\log D_n(\sigma) = -\sum_p m_p(n) \log(1-p^{-\sigma}) = \sum_p m_p(n) \sum_{k\geq 1} \frac{1}{k}\, p^{-k\sigma}. \tag{25}$$

Comparing (24) with (25) gives the consistency check: for $r = p^k$ one has

$$\frac{\Lambda_n(p^k)}{\log(p^k)} = \frac{m_p(n)\log p}{k \log p} = \frac{m_p(n)}{k},$$

and for non-prime-powers both sides contribute 0.

Corollary 2 (A Dirichlet-integral identity for D_n). *For every $\sigma > 0$,*

$$\log D_n(\sigma) = \sum_{r\geq 2} \frac{\Lambda_n(r)}{\log r}\, r^{-\sigma} = \sum_p \sum_{k\geq 1} \frac{m_p(n)}{k}\, p^{-k\sigma}.$$

In particular, at $\sigma = 1$ one obtains the convergent identity

$$\log D_n(1) = \sum_p \sum_{k\geq 1} \frac{m_p(n)}{k\, p^k}.$$

Remark 8 (Recovering n at $s = 1$). Since $D_n(1) = \prod_p (1-\frac{1}{p})^{-m_p(n)}$, we get

$$D_n(1) = n \qquad \text{and hence} \qquad n = \exp\left(\sum_p \sum_{k\geq 1} \frac{m_p(n)}{k\, p^k}\right).$$

This is the same finite Pratt product viewed through a Dirichlet-series lens.

12 Zeros and poles of the Dirichlet series $D_q(s)$ (integers and rationals)

12.1 From Pratt exponents to an Euler product

Let $m_p(n) \in \mathbb{Z}_{\geq 0}$ denote the Pratt-forest vertex count of p in the Pratt prime forest of $n \in \mathbb{N}$. We recall the finite-support identity

$$n \; = \; \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}. \tag{26}$$

It is convenient to extend the exponents from $\mathbb{N}$ to $\mathbb{Q}_{>0}$ by allowing negative exponents: if $q = \frac{a}{b}$ with $a, b \in \mathbb{N}$, define

$$m_p(q) \; := \; m_p(a) - m_p(b) \in \mathbb{Z}. \tag{27}$$

Then $m_p(q) = 0$ for all but finitely many primes p, and

$$q \; = \; \prod_{p} \left(1 - \frac{1}{p}\right)^{-m_p(q)}. \tag{28}$$

12.2 A Dirichlet series attached to q

Fix $q \in \mathbb{Q}_{>0}$ and write $m_p := m_p(q)$. Define a Dirichlet series for $\Re(s) > 1$ by the Euler product

$$D_q(s) \; := \; \prod_{p} \left(1 - p^{-s}\right)^{-m_p}. \tag{29}$$

Because only finitely many $m_p \neq 0$, the product is finite. Therefore $D_q(s)$ extends meromorphically to all of $\mathbb{C}$ as a finite product of elementary factors. Expanding the Euler product gives a Dirichlet series

$$D_q(s) \; = \; \sum_{r \geq 1} \frac{\chi_q(r)}{r^s}, \qquad \Re(s) > 1, \tag{30}$$

where χ_q is multiplicative and determined on prime powers by

$$\chi_q(p^k) = \begin{cases} \binom{m_p+k-1}{k}, & m_p \geq 1, \\ (-1)^k \binom{-m_p}{k}, & m_p \leq 0, \end{cases} \qquad (k \geq 0). \tag{31}$$

Indeed, for an integer $m \geq 1$ one has the generating function $\sum_{k \geq 0} \binom{m+k-1}{k} t^k = (1-t)^{-m}$, while for $m = -r \leq 0$ one has $(1-t)^r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} t^k$.

Remark 9. For $q \in \mathbb{N}$ all $m_p \geq 0$, so the local factors are of the form $(1-p^{-s})^{-m_p}$ and $D_q(s)$ has no zeros. For general rationals q, some m_p may be negative, and then $(1-p^{-s})^{-m_p} = (1-p^{-s})^{|m_p|}$ introduces zeros.

12.3 The divisor of $D_q(s)$

For each prime p define the p-lattice

$$\Lambda_p := \left\{ \frac{2\pi i k}{\log p} \, : \, k \in \mathbb{Z} \right\}.$$

These are exactly the solutions to $p^{-s} = 1$, i.e. the zeros of $1 - p^{-s}$.

Theorem 4 (Zeros and poles of $D_q(s)$)**.** *Let $q \in \mathbb{Q}_{>0}$ and $D_q(s)$ be defined by (29). Then:*

1. *For each prime p with $m_p(q) > 0$, the function $D_q(s)$ has poles of order $m_p(q)$ at every point of Λ_p.*
2. *For each prime p with $m_p(q) < 0$, the function $D_q(s)$ has zeros of order $-m_p(q)$ at every point of Λ_p.*
3. *There are no other zeros or poles.*

Moreover, if $p \neq \ell$ are distinct primes, then $\Lambda_p \cap \Lambda_\ell = \{0\}$. Hence the only point at which different prime lattices can overlap is $s = 0$.

Proof. Write $m_p = m_p(q)$. Since $D_q(s) = \prod_p (1 - p^{-s})^{-m_p}$ is a finite product, it suffices to analyze a single factor. The function $1 - p^{-s}$ vanishes exactly when $p^{-s} = 1$, i.e. when $e^{-s \log p} = 1$, which is equivalent to $s \in \Lambda_p$.

Near any $s_0 \in \Lambda_p$, one has $1 - p^{-s} = (s - s_0) \log p + O((s - s_0)^2)$, so $(1 - p^{-s})^{-m_p}$ has a pole of order m_p if $m_p > 0$ and a zero of order $-m_p$ if $m_p < 0$. Multiplying the finitely many factors gives (1)–(3).

Finally, if $s \in \Lambda_p \cap \Lambda_\ell$ with $p \neq \ell$, then $\frac{2\pi i k}{\log p} = \frac{2\pi i m}{\log \ell}$ for some integers k, m. If $k = m = 0$ this is $s = 0$. Otherwise we get $k \log \ell = m \log p$, hence $\log(\ell^k) = \log(p^m)$ and $\ell^k = p^m$, impossible for distinct primes. □

Corollary 3 (Integer case: no zeros)**.** *If $n \in \mathbb{N}_{\geq 2}$, then $D_n(s)$ has no zeros and has poles exactly at*

$$\bigcup_{p:\, m_p(n) > 0} \Lambda_p,$$

where the pole order along Λ_p equals $m_p(n)$.

12.4 Logarithmic derivative

From (29) we get, for all s away from the divisor,

$$\frac{D_q'(s)}{D_q(s)} = \sum_p m_p \, (\log p) \, \frac{p^{-s}}{1 - p^{-s}}. \tag{32}$$

This makes the singularities transparent: each summand has simple poles at Λ_p, and the coefficient m_p records the multiplicity with sign.

12.5 A remark about the value at $s = 1$

On the half-plane $\Re(s) > 1$, the Dirichlet series (30) converges and equals the Euler product (29). Since the Euler product is finite, there is in fact no analytic difficulty at $s = 1$:

$$D_q(1) = \prod_p (1 - p^{-1})^{-m_p(q)} = q,$$

which is exactly the rational Pratt product identity (28). Thus the point $s = 1$ is not a delicate boundary phenomenon here; it is simply the place where the finite product recovers the original rational number.

13 A Pratt-zeta function built from Pratt exponents

This section records a two-variable Dirichlet series whose Euler product interpolates between two classical zeta functions.

13.1 A two-variable Pratt-Dirichlet series

Fix a complex parameter u and define

$$A_u(n) \;:=\; \prod_{p \text{ prime}} \Big(1 - p^{-u}\Big)^{m_p(n)}. \tag{33}$$

This product is finite for each fixed n, because $m_p(n) = 0$ for all but finitely many primes p.

Multiplicativity. Since each $m_p(\cdot)$ is completely additive in n,

$$m_p(nm) = m_p(n) + m_p(m) \qquad (n, m \in \mathbb{N}),$$

it follows that A_u is completely multiplicative:

$$A_u(nm) = A_u(n)\, A_u(m) \qquad (n, m \in \mathbb{N}). \tag{34}$$

Definition 4 (Pratt double zeta function)**.** For complex parameters (u, s) with $\Re(s)$ sufficiently large, define

$$\mathcal{Z}(u, s) \;:=\; \sum_{n \geq 1} \frac{A_u(n)}{n^s}. \tag{35}$$

13.2 Euler product in the s-variable

Because A_u is completely multiplicative, $\mathcal{Z}(u, s)$ admits an Euler product:

$$\mathcal{Z}(u, s) \;=\; \prod_{p \text{ prime}} \left(\sum_{k \geq 0} \frac{A_u(p^k)}{p^{ks}} \right) \;=\; \prod_{p \text{ prime}} \frac{1}{1 - A_u(p)\, p^{-s}}, \tag{36}$$

where we used $A_u(p^k) = A_u(p)^k$, a direct consequence of $m_q(p^k) = k\, m_q(p)$ for every prime q.

Moreover, by the prime-step recursion for Pratt trees,

$$m_q(p) = m_q(p-1) + \mathbf{1}_{\{q=p\}},$$

one gets a convenient recursion for the local coefficient:

$$A_u(p) \;=\; (1 - p^{-u})\, A_u(p-1) \qquad (p \geq 3 \text{ prime}), \tag{37}$$

with the base $A_u(2) = 1 - 2^{-u}$.

13.3 Two key specializations

Limit $u \to +\infty$. As $\Re(u) \to +\infty$, one has $(1-p^{-u}) \to 1$ for every prime p, hence $A_u(n) \to 1$ for each fixed n. Formally, and whenever $\Re(s)$ is large enough to justify interchanging limit and summation, this gives

$$\mathcal{Z}(u, s) \longrightarrow \sum_{n\geq 1} \frac{1}{n^s} = \zeta(s). \tag{38}$$

Value $u = 1$. At $u = 1$, we have

$$\frac{1}{n} = \prod_p \Big(1 - \frac{1}{p}\Big)^{m_p(n)} = A_1(n).$$

Therefore

$$\mathcal{Z}(1, s) \;=\; \sum_{n\geq 1} \frac{A_1(n)}{n^s} \;=\; \sum_{n\geq 1} \frac{1}{n^{s+1}} \;=\; \zeta(s+1). \tag{39}$$

14 A conditional distribution from the Pratt forest

Fix $n \geq 1$ and recall the multiplicative coefficients $\chi_n(r)$ defined by

$$\chi_n(p^k) = \binom{m_p(n) + k - 1}{k} \qquad (k \geq 0),$$

equivalently

$$\chi_n(r) = \prod_{p\mid r} \binom{m_p(n) + v_p(r) - 1}{v_p(r)}.$$

Define

$$\mu_n(r) \;:=\; \frac{\chi_n(r)}{n\, r} \qquad (r \geq 1). \tag{40}$$

Then μ_n is a probability distribution on $\mathbb{N}$. Indeed, by the Dirichlet-Euler identity

$$D_n(s) = \sum_{r \geq 1} \frac{\chi_n(r)}{r^s} = \prod_p (1 - p^{-s})^{-m_p(n)},$$

we have

$$\sum_{r \geq 1} \frac{\chi_n(r)}{r} = D_n(1) = \prod_p (1 - 1/p)^{-m_p(n)} = n,$$

hence $\sum_{r \geq 1} \mu_n(r) = 1$.

14.1 Interpretation as a conditional law

There is a natural primewise independent stochastic construction that realizes μ_n as a conditional distribution.

For each prime p, let K_p be a $\mathbb{Z}_{\geq 0}$-valued random variable, and assume that the family $(K_p)_p$ is independent with

$$\mathbb{P}(K_p = k \mid n) \;=\; \binom{m_p(n) + k - 1}{k} \Big(1 - \frac{1}{p}\Big)^{m_p(n)} \Big(\frac{1}{p}\Big)^k, \qquad k \geq 0. \tag{41}$$

For fixed p, this is a negative-binomial law: the number of failures before $m_p(n)$ successes in Bernoulli trials with success probability $1 - \frac{1}{p}$.

Now define the random integer

$$R \;:=\; \prod_p p^{K_p},$$

which is almost surely finite because $m_p(n) = 0$ for all but finitely many p (and then $K_p \equiv 0$ for such primes by (41)). By independence and the multiplicative definition of χ_n, for every $r \geq 1$ we obtain

$$\begin{aligned}
\mathbb{P}(R = r \mid n) &= \prod_p \mathbb{P}(K_p = v_p(r) \mid n) \\
&= \prod_p \binom{m_p(n) + v_p(r) - 1}{v_p(r)} \Big(1 - \frac{1}{p}\Big)^{m_p(n)} \Big(\frac{1}{p}\Big)^{v_p(r)} \\
&= \left(\prod_p \Big(1 - \frac{1}{p}\Big)^{m_p(n)}\right) \frac{\chi_n(r)}{r} \;=\; \frac{\chi_n(r)}{n\, r} \;=\; \mu_n(r),
\end{aligned}$$

where in the last step we used the Pratt product identity. Hence μ_n is precisely the conditional law of R given n.

15 Bhattacharyya coefficient and Hellinger affinity

Let μ, ν be probability distributions on a countable set Ω. The *Bhattacharyya coefficient* (also called the Hellinger affinity) is

$$\mathrm{BC}(\mu,\nu) \ := \ \sum_{x\in\Omega} \sqrt{\mu(x)\nu(x)} \in [0,1]. \tag{42}$$

Equivalently, writing $\sqrt{\mu}, \sqrt{\nu} \in \ell^2(\Omega)$,

$$\mathrm{BC}(\mu,\nu) = \langle \sqrt{\mu}, \sqrt{\nu} \rangle_{\ell^2(\Omega)}.$$

In particular, $\mathrm{BC}(\mu,\nu) = 1$ iff $\mu = \nu$, and $\mathrm{BC}(\mu,\nu) = 0$ iff μ and ν have disjoint supports.

15.1 The Bhattacharyya coefficient for Pratt measures

For $a, b \in \mathbb{N}$ define μ_a, μ_b by (40). Then

$$\mathrm{BC}(\mu_a,\mu_b) = \sum_{r\geq 1} \sqrt{\mu_a(r)\mu_b(r)} = \sum_{r\geq 1} \frac{\sqrt{\chi_a(r)\chi_b(r)}}{r\sqrt{ab}}. \tag{43}$$

If we introduce the vectors

$$\phi_\chi(n) := \sum_{r\geq 1} \sqrt{\frac{\chi_n(r)}{r}}\, e_r \qquad \text{and} \qquad \psi(n) := \frac{\phi_\chi(n)}{\sqrt{n}},$$

then $\|\phi_\chi(n)\|^2 = \sum_{r\geq 1} \chi_n(r)/r = n$, so $\|\psi(n)\| = 1$, and

$$\langle \psi(a), \psi(b) \rangle = \sum_{r\geq 1} \sqrt{\mu_a(r)\mu_b(r)} = \mathrm{BC}(\mu_a,\mu_b).$$

Thus the normalized inner product of these feature vectors is exactly the Bhattacharyya coefficient of the associated Pratt measures.

16 Representation of rational numbers via divisor convolution

For $n \geq 1$ define

$$\psi_n(r) \ := \ \prod_p (-1)^{v_p(r)} \binom{m_p(n)}{v_p(r)}.$$

Only finitely many factors differ from 1, and if $v_p(r) > m_p(n)$ then the corresponding binomial coefficient is 0. Therefore ψ_n has finite support. The Euler product expansion of

$(1-1/p)^{m_p(n)}$ gives

$$\frac{1}{n} = \prod_p \Big(1-\frac{1}{p}\Big)^{m_p(n)} = \sum_{r=1}^{\infty} \frac{\psi_n(r)}{r}.$$

Recall also that

$$n = \prod_p \Big(1-\frac{1}{p}\Big)^{-m_p(n)} = \sum_{r=1}^{\infty} \frac{\chi_n(r)}{r}.$$

It follows that

$$\frac{m}{n} = m\cdot\frac{1}{n} = \left(\sum_{r=1}^{\infty}\frac{\chi_m(r)}{r}\right)\left(\sum_{s=1}^{\infty}\frac{\psi_n(s)}{s}\right) = \sum_{t=1}^{\infty}\frac{(\chi_m*\psi_n)(t)}{t},$$

where

$$(\chi_m*\psi_n)(t) := \sum_{d|t}\chi_m(d)\,\psi_n\Big(\frac{t}{d}\Big)$$

is the divisor convolution. Thus, for any rational number $x = a/b \in \mathbb{Q}$ with $\gcd(a,b)=1$, we obtain

$$x = \frac{a}{b} = \sum_{r=1}^{\infty}\frac{\operatorname{sign}(x)\,(\chi_a*\psi_b)(r)}{r}$$

when one also allows the overall sign of a rational number. For positive rationals the sign factor is of course 1.

17 Table data

This section records the ratios of consecutive primes

$$\frac{p_r}{p_{r+1}},$$

together with the corresponding infinite series

$$\frac{p_r}{p_{r+1}} = \sum_{t\geq 1}\frac{(\chi_{p_r}*\psi_{p_{r+1}})(t)}{t}.$$

Here the arithmetic functions χ_n and ψ_n are defined from the Pratt exponents by

$$\chi_n(p^k) = \binom{m_p(n)+k-1}{k}, \qquad \psi_n(p^k) = (-1)^k\binom{m_p(n)}{k},$$

and are extended multiplicatively. The convolution is the divisor convolution

$$(\chi_a*\psi_b)(t) = \sum_{d|t}\chi_a(d)\psi_b\Big(\frac{t}{d}\Big).$$

In the table below, the final column lists the first 12 nonzero terms of the series.

r	p_r	p_{r+1}	$\frac{p_r}{p_{r+1}}$	First nonzero terms of $\sum_{t\geq 1}\frac{(\chi_{p_r} * \psi_{p_{r+1}})(t)}{t}$
1	2	3	$\frac{2}{3}$	$1-\frac{1}{3}$
2	3	5	$\frac{3}{5}$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{5}-\frac{1}{6}+\frac{1}{9}+\frac{1}{10}-\frac{1}{15}-\frac{1}{18}+\frac{1}{27}+\frac{1}{30}-\frac{1}{45}+\cdots$
3	5	7	$\frac{5}{7}$	$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{15}+\frac{1}{21}+\frac{1}{25}-\frac{1}{35}-\frac{1}{75}+\frac{1}{105}+\frac{1}{125}-\frac{1}{175}+\cdots$
4	7	11	$\frac{7}{11}$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\frac{1}{14}-\frac{1}{15}-\frac{1}{18}+\cdots$
5	11	13	$\frac{11}{13}$	$1-\frac{1}{3}+\frac{1}{5}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{25}-\frac{1}{33}+\frac{1}{39}+\frac{1}{55}-\frac{1}{65}-\frac{1}{75}+\cdots$
6	13	17	$\frac{13}{17}$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{6}+\frac{1}{9}+\frac{1}{13}-\frac{1}{17}-\frac{1}{18}-\frac{1}{26}+\frac{1}{27}+\frac{1}{34}+\frac{1}{39}+\cdots$
7	17	19	$\frac{17}{19}$	$1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}-\frac{2}{6}+\frac{1}{8}+\frac{1}{9}-\frac{2}{12}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}-\frac{1}{19}+\cdots$
8	19	23	$\frac{19}{23}$	$1-\frac{1}{2}+\frac{2}{3}-\frac{1}{5}-\frac{2}{6}+\frac{3}{9}+\frac{1}{10}-\frac{1}{11}-\frac{2}{15}-\frac{3}{18}+\frac{1}{19}+\frac{1}{22}+\cdots$
9	23	29	$\frac{23}{29}$	$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{21}+\frac{1}{23}+\frac{1}{25}-\frac{1}{29}-\frac{1}{33}-\frac{1}{35}+\cdots$
10	29	31	$\frac{29}{31}$	$1-\frac{1}{5}+\frac{1}{7}+\frac{1}{29}-\frac{1}{31}-\frac{1}{35}+\frac{1}{49}-\frac{1}{145}+\frac{1}{155}+\frac{1}{203}-\frac{1}{217}-\frac{1}{245}+\cdots$
11	31	37	$\frac{31}{37}$	$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{15}+\frac{1}{25}+\frac{1}{31}-\frac{1}{37}-\frac{1}{75}-\frac{1}{93}+\frac{1}{111}+\frac{1}{125}+\frac{1}{155}+\cdots$
12	37	41	$\frac{37}{41}$	$1-\frac{1}{2}+\frac{2}{3}-\frac{1}{5}-\frac{2}{6}+\frac{3}{9}+\frac{1}{10}-\frac{2}{15}-\frac{3}{18}+\frac{4}{27}+\frac{2}{30}+\frac{1}{37}+\cdots$
13	41	43	$\frac{41}{43}$	$1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{2}{12}-\frac{1}{14}+\cdots$
14	43	47	$\frac{43}{47}$	$1-\frac{1}{2}+\frac{2}{3}-\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{3}{9}+\frac{1}{10}-\frac{1}{11}-\frac{1}{14}-\frac{2}{15}-\frac{3}{18}+\cdots$
15	47	53	$\frac{47}{53}$	$1-\frac{1}{3}+\frac{1}{5}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{23}+\frac{1}{25}-\frac{1}{33}+\frac{1}{39}+\frac{1}{47}-\frac{1}{53}+\cdots$
16	53	59	$\frac{53}{59}$	$1-\frac{1}{7}+\frac{1}{13}-\frac{1}{29}+\frac{1}{53}-\frac{1}{59}-\frac{1}{91}+\frac{1}{169}+\frac{1}{203}-\frac{1}{371}-\frac{1}{377}$
17	59	61	$\frac{59}{61}$	$1-\frac{1}{5}+\frac{1}{7}+\frac{1}{29}-\frac{1}{35}+\frac{1}{49}+\frac{1}{59}-\frac{1}{61}-\frac{1}{145}+\frac{1}{203}-\frac{1}{245}-\frac{1}{295}+\cdots$
18	61	67	$\frac{61}{67}$	$1-\frac{1}{11}+\frac{1}{61}-\frac{1}{67}$
19	67	71	$\frac{67}{71}$	$1-\frac{1}{7}+\frac{1}{11}+\frac{1}{67}-\frac{1}{71}-\frac{1}{77}+\frac{1}{121}$
20	71	73	$\frac{71}{73}$	$1-\frac{1}{3}+\frac{1}{5}+\frac{1}{7}-\frac{1}{15}-\frac{1}{21}+\frac{1}{25}+\frac{1}{35}+\frac{1}{49}+\frac{1}{71}-\frac{1}{73}-\frac{1}{75}+\cdots$

17.1 When is the convolution series finite?

A natural question is the following: for which pairs $a, b \in \mathbb{N}$ does the convolution expansion

$$\frac{a}{b} = \sum_{t\geq 1}\frac{(\chi_a * \psi_b)(t)}{t}$$

reduce to a *finite* sum?

At first sight this is not obvious from the coefficient formula alone, since the convolution

$$(\chi_a * \psi_b)(t) = \sum_{d\mid t}\chi_a(d)\psi_b\left(\frac{t}{d}\right)$$

mixes the two arithmetic functions in a nontrivial way. The answer becomes transparent once one rewrites the associated Dirichlet series in Euler-product form.

Definition 5 (Pratt exponent order)**.** For $a, b \in \mathbb{N}$, write

$$a \preceq_{\mathrm{P}} b \qquad :\Longleftrightarrow \qquad m_p(a) \leq m_p(b) \text{ for every prime } p.$$

We call this the *Pratt exponent order.*

Thus $a \preceq_{\mathrm{P}} b$ means that the Pratt exponent vector of a is dominated componentwise by that of b.

Theorem 5 (Finiteness criterion for the convolution series)**.** *Let $a, b \in \mathbb{N}$. Then the convolution expansion*

$$\frac{a}{b} = \sum_{t \geq 1} \frac{(\chi_a * \psi_b)(t)}{t}$$

is a finite sum if and only if

$$a \preceq_{\mathrm{P}} b, \qquad \text{equivalently,} \qquad m_p(a) \leq m_p(b) \text{ for all primes } p.$$

Proof. Recall that

$$\sum_{r \geq 1} \frac{\chi_a(r)}{r^s} = \prod_p (1 - p^{-s})^{-m_p(a)}$$

and

$$\sum_{r \geq 1} \frac{\psi_b(r)}{r^s} = \prod_p (1 - p^{-s})^{m_p(b)}.$$

Multiplying these Dirichlet series gives

$$\sum_{t \geq 1} \frac{(\chi_a * \psi_b)(t)}{t^s} = \prod_p (1 - p^{-s})^{m_p(b) - m_p(a)}.$$

Set

$$d_p := m_p(b) - m_p(a) \in \mathbb{Z}.$$

Since only finitely many Pratt exponents are nonzero, only finitely many d_p are nonzero, so the Euler product above is finite.

Now observe:

- If $d_p \geq 0$, then

  $$(1 - p^{-s})^{d_p} = \sum_{k=0}^{d_p} (-1)^k \binom{d_p}{k} p^{-ks},$$

  which is a finite polynomial in p^{-s}.

- If $d_p < 0$, then

  $$(1 - p^{-s})^{d_p} = (1 - p^{-s})^{-|d_p|}$$

has an infinite binomial expansion

$$(1-p^{-s})^{-|d_p|} = \sum_{k\geq 0} \binom{|d_p|+k-1}{k} p^{-ks},$$

and hence contributes infinitely many nonzero terms.

Therefore the full Dirichlet series

$$\sum_{t\geq 1} \frac{(\chi_a * \psi_b)(t)}{t^s}$$

is a Dirichlet polynomial, equivalently the series at $s = 1$ is finite, if and only if

$$d_p \geq 0 \qquad \text{for every prime } p.$$

This is exactly the condition

$$m_p(a) \leq m_p(b) \qquad \text{for every prime } p,$$

that is, $a \preceq_{\mathrm{P}} b$. □

Remark 10 (Support in the finite case). If $a \preceq_{\mathrm{P}} b$, let

$$d_p := m_p(b) - m_p(a) \geq 0.$$

Then

$$\sum_{t\geq 1} \frac{(\chi_a * \psi_b)(t)}{t^s} = \prod_p (1-p^{-s})^{d_p}$$

is a finite product of finite polynomials. In particular, its support is contained in the divisor set of

$$M := \prod_p p^{d_p}.$$

Thus in the finite case one actually has

$$\frac{a}{b} = \sum_{t|M} \frac{(\chi_a * \psi_b)(t)}{t}.$$

Remark 11 (This order is not the sorted-sequence order). The order $\preceq_{\mathrm{P}}$ is the coordinatewise order on the Pratt exponent vectors

$$\phi(n) := (m_p(n))_p.$$

It is different from the order obtained by comparing sorted Pratt sequences term by term. For the finiteness criterion above, the relevant order is precisely the exponent order $\preceq_{\mathrm{P}}$, because the Euler product depends only on the differences $m_p(b) - m_p(a)$.

Remark 12 (Twin primes). If p and $p+2$ are both prime, then

$$1 - \frac{p}{p+2} = \frac{2}{p+2}.$$

Since the Pratt exponent vector of 2 is given by $m_2(2) = 1$ and $m_q(2) = 0$ for $q \neq 2$, while every odd prime q satisfies $m_2(q) \geq 1$, we have

$$2 \preceq_{\mathrm{P}} p+2.$$

Hence, by Theorem 5, the convolution expansion of

$$\frac{2}{p+2}$$

is always finite. Equivalently, for every twin-prime pair $p, p+2$, the normalized prime gap

$$\frac{(p+2)-p}{p+2} = \frac{2}{p+2}$$

has a finite Pratt convolution expansion.

18 Meet Kernel defined with Pratt valuation

For $n \in \mathbb{N}$, let $(m_p(n))_p$ denote the Pratt valuation vector, indexed by primes p, and define the Pratt order by

$$a \leq_P b \quad \Longleftrightarrow \quad m_p(a) \leq m_p(b) \text{ for every prime } p.$$

The Pratt product formula reads

$$n = \prod_p \left(1 - \frac{1}{p}\right)^{-m_p(n)}.$$

Since each factor $(1-1/p)^{-1} = p/(p-1)$ is strictly larger than 1, it follows immediately that

$$a \leq_P b \quad \Longrightarrow \quad a \leq b.$$

Motivated by the usual gcd-kernel, define the Pratt meet-value

$$a \wedge b := \prod_p \left(1 - \frac{1}{p}\right)^{-\min(m_p(a), m_p(b))}.$$

By construction,

$$m_p(a \wedge b) = \min(m_p(a), m_p(b)),$$

so $a \wedge b$ is the meet of a and b in the Pratt poset $(\mathbb{N}, \leq_P)$.

Proposition 7. *The function*

$$k(a,b) := a \wedge b = \prod_p \left(1 - \frac{1}{p}\right)^{-\min(m_p(a), m_p(b))}$$

is a positive semidefinite kernel on $\mathbb{N}$.

Proof. For each prime p, put

$$\alpha_p := \left(1 - \frac{1}{p}\right)^{-1} > 1.$$

Then

$$k(a,b) = \prod_p \alpha_p^{\min(m_p(a), m_p(b))}.$$

Thus it is enough to show that for every fixed $\alpha > 1$, the kernel

$$K_\alpha(u,v) := \alpha^{\min(u,v)}, \qquad u, v \in \mathbb{N}_0,$$

is positive semidefinite.

We use the telescoping identity

$$\alpha^{\min(u,v)} = 1 + \sum_{j \geq 1} (\alpha^j - \alpha^{j-1}) \, \mathbf{1}_{\{u \geq j\}} \, \mathbf{1}_{\{v \geq j\}}.$$

Indeed, if $m = \min(u,v)$, then the right-hand side equals

$$1 + \sum_{j=1}^{m} (\alpha^j - \alpha^{j-1}) = \alpha^m.$$

Each coefficient $\alpha^j - \alpha^{j-1}$ is positive, so K_α is a sum of rank-one positive semidefinite kernels. Hence K_α is positive semidefinite.

Now for each prime p, the kernel

$$(a,b) \longmapsto \alpha_p^{\min(m_p(a), m_p(b))}$$

is positive semidefinite. Since for fixed a, b only finitely many factors differ from 1, the product defining $k(a,b)$ is effectively finite. A pointwise product of positive semidefinite kernels is again positive semidefinite, so k is positive semidefinite. □

Remark 13. This is the Pratt-valuation analogue of the classical gcd-kernel

$$\gcd(a,b) = \prod_p p^{\min(v_p(a), v_p(b))},$$

with the usual p-adic exponents v_p replaced by the Pratt valuations m_p, and the local weights p replaced by $p/(p-1)$.

The kernel also admits a Möbius-expansion on the Pratt poset. Since $a \le_P b$ implies $a \le b$, every interval in $(\mathbb{N}, \le_P)$ is finite, so the Möbius function μ_P exists.

Define the Möbius transform of the identity map $\mathrm{id}(n) = n$ by

$$g(x) := \sum_{d \le_P x} \mu_P(d, x)\, d.$$

By Möbius inversion on the locally finite poset $(\mathbb{N}, \le_P)$,

$$x = \sum_{d \le_P x} g(d) \qquad (x \in \mathbb{N}).$$

Applying this with $x = a \wedge b$ yields the following.

Proposition 8. *For all $a, b \in \mathbb{N}$,*

$$a \wedge b = \sum_{\substack{d \le_P a \\ d \le_P b}} g(d),$$

where g is the Möbius transform of the identity on the Pratt poset.

Proof. Since $m_p(a \wedge b) = \min(m_p(a), m_p(b))$, one has

$$d \le_P a \wedge b \iff (d \le_P a \text{ and } d \le_P b).$$

Therefore,

$$a \wedge b = \sum_{d \le_P a \wedge b} g(d) = \sum_{\substack{d \le_P a \\ d \le_P b}} g(d),$$

as claimed. □

19 Monotonicity of $\chi_a(r)$ with respect to the Pratt order

In this section we record a simple monotonicity property of the coefficients

$$\chi_a(r),$$

viewed as functions of a with r fixed.

Recall that if

$$r = \prod_p p^{v_p(r)},$$

then

$$\chi_a(r) = \prod_{p \mid r} \binom{m_p(a) + v_p(r) - 1}{v_p(r)},$$

where $m_p(a)$ denotes the multiplicity parameter attached to a at the prime p. Recall also that

$$a \leq_P b \quad \Longleftrightarrow \quad m_p(a) \leq m_p(b) \qquad \text{for all primes } p.$$

Lemma 7. *For all positive integers a, b, r, if $a \leq_P b$, then*

$$\chi_a(r) \leq \chi_b(r).$$

Proof. Fix $r \geq 1$ and write

$$r = \prod_p p^{v_p(r)}.$$

By definition,

$$\chi_a(r) = \prod_{p \mid r} \binom{m_p(a) + v_p(r) - 1}{v_p(r)}, \qquad \chi_b(r) = \prod_{p \mid r} \binom{m_p(b) + v_p(r) - 1}{v_p(r)}.$$

Since $a \leq_P b$, we have

$$m_p(a) \leq m_p(b) \qquad \text{for every prime } p.$$

Thus it is enough to show that for each fixed integer $v \geq 0$, the function

$$m \longmapsto \binom{m + v - 1}{v}$$

is nondecreasing in $m \in \mathbb{Z}_{\geq 0}$.

If $v = 0$, then

$$\binom{m-1}{0} = 1,$$

so there is nothing to prove. Assume now that $v \geq 1$. Then Pascal's identity gives

$$\binom{m + 1 + v - 1}{v} - \binom{m + v - 1}{v} = \binom{m + v - 1}{v - 1} \geq 0.$$

Hence

$$\binom{m + v - 1}{v}$$

is nondecreasing in m.

Applying this with $m = m_p(a)$ and $m = m_p(b)$, we obtain for every prime $p \mid r$ that

$$\binom{m_p(a) + v_p(r) - 1}{v_p(r)} \leq \binom{m_p(b) + v_p(r) - 1}{v_p(r)}.$$

Since all factors are nonnegative, multiplying over all primes dividing r yields

$$\chi_a(r) \leq \chi_b(r).$$

This proves the claim. □

Remark 14. For fixed r, the quantity $\chi_a(r)$ is therefore monotone in each coordinate $m_p(a)$. Since the Pratt order is exactly the coordinatewise order on the family $(m_p(a))_p$, the lemma is an immediate structural consequence of the Euler product formula for $\chi_a(r)$.

19.1 A converse criterion via the coefficient family $\chi_a(r)$

The pointwise inequality

$$\chi_a(r) \leq \chi_b(r)$$

for a single fixed value of r does *not* imply $a \leq_P b$. Indeed, by the Euler factor formula,

$$\chi_n(r) = \prod_{p|r} \binom{m_p(n) + v_p(r) - 1}{v_p(r)},$$

so for fixed r the quantity $\chi_n(r)$ depends only on the coordinates $m_p(n)$ for primes $p \mid r$. Thus information about primes not dividing r is invisible. For example, taking $a = 3$, $b = 2$, and $r = 2$, one finds

$$\chi_3(2) = 0 \leq 1 = \chi_2(2),$$

while $3 \not\leq_P 2$.

However, if the inequality holds for *all* $r \geq 1$, then one recovers the Pratt order completely.

Proposition 9. *For $a, b \in \mathbb{N}_{\geq 1}$, the following are equivalent:*

1. *$a \leq_P b$;*
2. *$\chi_a(r) \leq \chi_b(r)$ for all $r \geq 1$.*

In particular, if $\chi_a(r) \leq \chi_b(r)$ for all $r \geq 1$, then $a \leq_P b$, and hence $a \leq b$.

Proof. The implication (1) ⇒ (2) was proved in the previous subsection.

For the converse, assume that

$$\chi_a(r) \leq \chi_b(r) \qquad \text{for all } r \geq 1.$$

Let p be any prime and specialize to $r = p$. Since $v_p(p) = 1$, the coefficient formula gives

$$\chi_a(p) = \binom{m_p(a) + 1 - 1}{1} = \binom{m_p(a)}{1} = m_p(a),$$

and similarly

$$\chi_b(p) = m_p(b).$$

Therefore the assumption for $r = p$ yields

$$m_p(a) \leq m_p(b) \qquad \text{for every prime } p.$$

By definition of the Pratt order, this is exactly the statement that

$$a \le_P b.$$

Finally, Proposition 1 shows that $a \le_P b$ implies $a \le b$. This proves the claim. □

Remark 15. The proposition shows that the full coefficient family

$$\Big(\chi_a(r)\Big)_{r\ge 1}$$

determines the Pratt-order position of a. In fact, the values at primes already suffice, since

$$\chi_a(p) = m_p(a) \qquad \text{for every prime } p.$$

Thus the family $\chi_a(\cdot)$ contains exactly the coordinate data defining the partial order $\le_P$.

20 The Mobius transform of the identity on the Pratt poset

For $a, b \in \mathbb{N}$, write

$$a \le_P b \quad \Longleftrightarrow \quad m_q(a) \le m_q(b) \text{ for every prime } q,$$

where $m_q(\cdot)$ denotes the Pratt valuation. Let g be the Mobius transform of the identity function $\mathrm{id}(n) = n$ on the locally finite poset $(\mathbb{N}, \le_P)$, that is,

$$n = \sum_{d \le_P n} g(d) \qquad (n \in \mathbb{N}).$$

We prove that g admits the explicit closed form

$$g(n) = \frac{n}{\mathrm{rad}(n)}.$$

Proposition 10. *For every $n \in \mathbb{N}$,*

$$g(n) = \frac{n}{\mathrm{rad}(n)}.$$

Equivalently,

$$\sum_{d \le_P n} \frac{d}{\mathrm{rad}(d)} = n \qquad (n \in \mathbb{N}).$$

Proof. Set

$$h(n) := \frac{n}{\mathrm{rad}(n)}.$$

It is enough to show that

$$\sum_{d \leq_P n} h(d) = n \qquad (n \in \mathbb{N}),$$

because the Mobius transform on a locally finite poset is uniquely determined by this zeta-summation identity.

We argue by induction on n. The case $n = 1$ is immediate. Assume $n > 1$, and write

$$n = uP^e,$$

where P is the largest prime divisor of n, $e \geq 1$, and $P \nmid u$.

Since P is the largest prime dividing n, no smaller prime contributes to the P-coordinate of the Pratt valuation. Hence

$$m_P(n) = v_P(n) = e.$$

Now let $d \leq_P n$, and write

$$d = P^k c, \qquad P \nmid c.$$

Then necessarily $0 \leq k \leq e$. Using complete additivity of the Pratt valuation together with

$$m(P) = e_P + m(P-1),$$

where e_P denotes the unit vector at the prime P, we obtain

$$m(d) = ke_P + km(P-1) + m(c),$$

while

$$m(n) = ee_P + em(P-1) + m(u).$$

Therefore

$$d \leq_P n \iff m(c) \leq m(u) + (e-k)m(P-1) \iff c \leq_P u(P-1)^{e-k}.$$

Thus every $d \leq_P n$ is uniquely of the form $d = P^k c$ with $0 \leq k \leq e$ and

$$c \leq_P u(P-1)^{e-k}.$$

Now sum $h(d)$ over all such pairs (k, c). Since $P \nmid c$, we have

$$h(P^k c) = \begin{cases} h(c), & k = 0, \\ P^{k-1} h(c), & k \geq 1. \end{cases}$$

Hence, if

$$H(n) := \sum_{d \leq_P n} h(d),$$

then

$$H(n) = H\Big(u(P-1)^e\Big) + \sum_{k=1}^{e} P^{k-1} H\Big(u(P-1)^{e-k}\Big).$$

Each argument on the right-hand side is strictly smaller than n, so the induction hypothesis gives

$$H\Big(u(P-1)^j\Big) = u(P-1)^j \qquad (0 \le j \le e).$$

Therefore

$$H(n) = u(P-1)^e + \sum_{k=1}^{e} P^{k-1} u(P-1)^{e-k} = u\left((P-1)^e + \sum_{k=1}^{e} P^{k-1}(P-1)^{e-k}\right).$$

The finite sum telescopes, because

$$P^{k-1}(P-1)^{e-k} = P^k(P-1)^{e-k} - P^{k-1}(P-1)^{e-k+1}.$$

Hence

$$\sum_{k=1}^{e} P^{k-1}(P-1)^{e-k} = P^e - (P-1)^e.$$

Substituting this into the previous display yields

$$H(n) = u\Big((P-1)^e + P^e - (P-1)^e\Big) = uP^e = n.$$

Thus $H(n) = n$ for all n, i.e.

$$\sum_{d \le_P n} \frac{d}{\operatorname{rad}(d)} = n.$$

By uniqueness of Mobius inversion on $(\mathbb{N}, \le_P)$, it follows that

$$g(n) = \frac{n}{\operatorname{rad}(n)}.$$

□

21 Feature vectors

Let $m_p(n)$ denote the Pratt valuation of $n \in \mathbb{N}$, and define the Pratt partial order by

$$a \le_P b \qquad \Longleftrightarrow \qquad m_p(a) \le m_p(b) \quad \text{for every prime } p.$$

For $a, b \in \mathbb{N}$, define their Pratt meet by

$$a \wedge_P b := \prod_p \left(1 - \frac{1}{p}\right)^{-\min\{m_p(a), m_p(b)\}}.$$

Equivalently, $a \wedge_P b$ is the unique natural number whose Pratt valuation is given by

$$m_p(a \wedge_P b) = \min\{m_p(a), m_p(b)\} \qquad \text{for all primes } p.$$

We consider the kernel

$$k(a,b) := a \wedge_P b.$$

Assume that the Möbius transform g of the identity function on the Pratt poset satisfies

$$g(n) = \frac{n}{\operatorname{rad}(n)}.$$

Then, since

$$a \wedge_P b = \sum_{\substack{d \leq_P a \\ d \leq_P b}} g(d),$$

we obtain an explicit feature map for k.

Proposition 11. *Define, for each $n \in \mathbb{N}$, the vector*

$$\phi(n) = \big(\phi_d(n)\big)_{d \geq 1}, \qquad \phi_d(n) := \sqrt{\frac{d}{\operatorname{rad}(d)}}\, \mathbf{1}_{\{d \leq_P n\}}.$$

Then $\phi(n)$ has finite support and

$$\langle \phi(a), \phi(b) \rangle_{\ell^2(\mathbb{N})} = a \wedge_P b \qquad \text{for all } a, b \in \mathbb{N}.$$

In particular, $k(a,b) = a \wedge_P b$ is a positive semidefinite kernel on $\mathbb{N}$.

Proof. Since $d \leq_P n$ implies $d \leq n$, only finitely many coordinates of $\phi(n)$ are nonzero; hence $\phi(n) \in \ell^2(\mathbb{N})$. Now

$$\langle \phi(a), \phi(b) \rangle = \sum_{d \geq 1} \sqrt{\frac{d}{\operatorname{rad}(d)}}\, \mathbf{1}_{\{d \leq_P a\}} \sqrt{\frac{d}{\operatorname{rad}(d)}}\, \mathbf{1}_{\{d \leq_P b\}}.$$

Therefore

$$\langle \phi(a), \phi(b) \rangle = \sum_{d \geq 1} \frac{d}{\operatorname{rad}(d)}\, \mathbf{1}_{\{d \leq_P a\}} \mathbf{1}_{\{d \leq_P b\}} = \sum_{\substack{d \leq_P a \\ d \leq_P b}} \frac{d}{\operatorname{rad}(d)}.$$

Using $g(d) = d/\operatorname{rad}(d)$ and the Möbius inversion formula on the Pratt poset, we obtain

$$\sum_{\substack{d \leq_P a \\ d \leq_P b}} \frac{d}{\operatorname{rad}(d)} = a \wedge_P b.$$

Hence

$$\langle \phi(a), \phi(b) \rangle = a \wedge_P b = k(a,b),$$

as claimed. Positive semidefiniteness follows immediately from this Gram representation. □

Corollary 4. *The squared norm of the feature vector is*

$$\|\phi(n)\|^2 = k(n,n) = n.$$

Hence the normalized feature map

$$\widehat{\phi}(n) := \frac{1}{\sqrt{n}}\phi(n)$$

satisfies

$$\langle \widehat{\phi}(a), \widehat{\phi}(b) \rangle = \frac{a \wedge_P b}{\sqrt{ab}}.$$

Proof. Since $n \wedge_P n = n$, the first identity is immediate from the proposition. The second follows by rescaling. □

22 Properties of the Meet Kernel $a \wedge b$ and a New Kernel

22.1 Definitions and the Partial Order

Let $m_p(a)$ denote the multiplicity of the prime p appearing in all Pratt trees associated with the integer $a \in \mathbb{N}_{\geq 1}$. For example, evaluating the prime 2, we have $m_2(2) = 1$ and $m_2(6) = 2$.

We define a relation $\leq_p$ on $\mathbb{N}_{\geq 1}$ as follows:

$$a \leq_p b \iff m_p(a) \leq m_p(b) \quad \text{for all primes } p. \tag{44}$$

22.2 The Order Proposition

Proposition 1. If $a \leq_p b$, then $a \leq b$.

Proof. We take as given the representation of any integer $n \in \mathbb{N}_{\geq 1}$ in terms of its Pratt tree prime multiplicities $m_p(n)$:

$$n = \prod_p \left(1 - \frac{1}{p}\right)^{-m_p(n)} \tag{45}$$

Assume $a \leq_p b$. By our definition of the partial order $\leq_p$, this implies that $m_p(a) \leq m_p(b)$ for all primes p. For any prime $p \geq 2$, the fraction $\frac{1}{p}$ satisfies $0 < \frac{1}{p} \leq \frac{1}{2}$, which means $0 < 1 - \frac{1}{p} < 1$. We can rewrite the factors in the product by taking the reciprocal of the base:

$$\left(1 - \frac{1}{p}\right)^{-m_p(n)} = \left(\frac{p}{p-1}\right)^{m_p(n)} \tag{46}$$

Since $p \geq 2$, the new base $\frac{p}{p-1}$ is strictly greater than 1. For any base $B > 1$, the exponential function $f(x) = B^x$ is strictly monotonically increasing. Because $m_p(a) \leq m_p(b)$, it follows that:

$$\left(\frac{p}{p-1}\right)^{m_p(a)} \leq \left(\frac{p}{p-1}\right)^{m_p(b)} \implies \left(1-\frac{1}{p}\right)^{-m_p(a)} \leq \left(1-\frac{1}{p}\right)^{-m_p(b)} \tag{47}$$

This inequality holds for every individual prime factor p. Since all factors in the product are strictly positive, multiplying these inequalities together preserves the direction of the inequality over the entire product:

$$\prod_p \left(1-\frac{1}{p}\right)^{-m_p(a)} \leq \prod_p \left(1-\frac{1}{p}\right)^{-m_p(b)} \tag{48}$$

By substituting our given representation back into both sides, we immediately obtain $a \leq b$. □

22.3 Properties of the Meet Operation

Let $a \wedge b$ denote the meet (greatest lower bound) of a and b in the poset $(\mathbb{N}_{\geq 1}, \leq_p)$. By the definition of the meet in this lattice, the prime valuations of $a \wedge b$ satisfy:

$$m_p(a \wedge b) \leq m_p(a) \quad \text{and} \quad m_p(a \wedge b) \leq m_p(b) \quad \text{for all } p. \tag{49}$$

This directly implies $a \wedge b \leq_p a$ and $a \wedge b \leq_p b$. Applying Proposition 1, we obtain:

$$a \wedge b \leq a \quad \text{and} \quad a \wedge b \leq b \tag{50}$$

Since $a \wedge b$ is less than or equal to both a and b in the standard integers, it must be bounded by their minimum:

$$a \wedge b \leq \min(a, b). \tag{51}$$

22.4 Kernel Definiteness and Feature Mapping

Definition. Let $K_c(a, b)$ be defined as the number of common predecessors in the $\leq_p$ poset:

$$K_c(a, b) := \sum_{\substack{d \leq_p a \\ d \leq_p b}} 1 = \#\{d \mid d \leq_p a \wedge b\} \tag{52}$$

Proposition 2. $K_c(a, b)$ is a positive definite (p.d.) kernel.

Proof. We construct a natural feature map $\phi(a)$ mapping into an infinite-dimensional sequence space. Let the d-th component of the vector $\phi(a)$ be the indicator function:

$$\phi(a)_d = \mathbf{1}_{\{d \leq_p a\}} \tag{53}$$

The inner product of two such feature vectors $\phi(a)$ and $\phi(b)$ is:

$$\langle \phi(a), \phi(b) \rangle = \sum_{d=1}^{\infty} \mathbf{1}_{\{d \leq_p a\}} \mathbf{1}_{\{d \leq_p b\}} = \sum_{\substack{d \leq_p a \\ d \leq_p b}} 1 = K_c(a, b) \tag{54}$$

Because the kernel function $K_c(a, b)$ can be expressed exactly as an inner product $\langle \phi(a), \phi(b) \rangle$ in a feature space, it is a positive semi-definite kernel. □

22.5 The Gram Matrix and Lattice Structure

Let G_n be the $n \times n$ Gram matrix defined by $(G_n)_{i,j} = K_c(i, j)$ for $1 \leq i, j \leq n$.

Proposition 3. $\det(G_n) = 1$.

Proof. Define an $n \times n$ matrix L such that its entries are $L_{i,d} = \mathbf{1}_{\{d \leq_p i\}}$. From Proposition 1, we know that $d \leq_p i \implies d \leq i$. This structural property ensures that $L_{i,d} = 0$ whenever $d > i$. Thus, L is a lower-triangular matrix.

The diagonal elements are $L_{i,i} = \mathbf{1}_{\{i \leq_p i\}} = 1$. Since L is lower-triangular with 1s on the main diagonal, its determinant is the product of its diagonal elements, yielding $\det(L) = 1$. The Gram matrix G_n can be factorized as $G_n = LL^T$. Therefore:

$$\det(G_n) = \det(LL^T) = \det(L) \det(L^T) = 1 \cdot 1 = 1. \square \tag{55}$$

Theorem 1. The lattice $\mathcal{L}_n$ spanned by the basis vectors $\phi(1), \phi(2), \ldots, \phi(n)$ is unimodular and integral.

Proof.

1. **Integral:** The inner product of any two basis vectors is $\langle \phi(i), \phi(j) \rangle = K_c(i, j)$. Since K_c counts a finite set of integers, $K_c(i, j) \in \mathbb{Z}$. Therefore, $\mathcal{L}_n$ is an integral lattice.
2. **Unimodular:** The volume of the fundamental parallelotope of the lattice is given by $\sqrt{\det(G_n)}$. Since $\det(G_n) = 1$, the volume is 1, proving the lattice is unimodular. □

22.6 Lattice Isometry and the Theta Series

Proposition 4. For any $n \geq 1$, the lattice $\mathcal{L}_n$ associated with the Gram matrix G_n is isometric to the standard hypercubic lattice $\mathbb{Z}^n$. Consequently, for $n = 6$, the coefficients of the theta series of $\mathcal{L}_6$ are exactly the sequence OEIS A000141.

Proof. Let G_n be the Gram matrix of the lattice $\mathcal{L}_n$. From Proposition 3, we have the exact factorization:

$$G_n = LL^T \tag{56}$$

where L is an $n \times n$ matrix with integer entries defined by $L_{i,d} = \nVdash_{\{d \leq_p i\}}$. We established that L is a lower-triangular matrix with ones on its main diagonal.

Because L is an integer matrix with $\det(L) = 1$, it is unimodular. Therefore, its inverse L^{-1} is also an integer matrix, which implies that L is an element of the general linear group over the integers, $\mathrm{GL}_n(\mathbb{Z})$.

The matrix L can be viewed as a change of basis matrix. Since $G_n = LI_nL^T$ (where I_n is the $n \times n$ identity matrix), the Gram matrix G_n is integer-equivalent to the identity matrix. Geometrically, this means that the lattice $\mathcal{L}_n$ generated by G_n is exactly the standard lattice $\mathbb{Z}^n$ represented in a different basis. Thus, there exists a lattice isometry:

$$\mathcal{L}_n \cong \mathbb{Z}^n \tag{57}$$

Because the theta series $\Theta_{\mathcal{L}}(q)$ of a lattice is a geometric invariant that depends only on the lengths of its vectors, isometric lattices share the identical theta series. Therefore:

$$\Theta_{\mathcal{L}_n}(q) = \Theta_{\mathbb{Z}^n}(q) \tag{58}$$

The theta series of the standard lattice $\mathbb{Z}^n$ is given by the sum over the squared norms of all lattice vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$:

$$\Theta_{\mathbb{Z}^n}(q) = \sum_{\mathbf{x}\in\mathbb{Z}^n} q^{\|\mathbf{x}\|^2} = \sum_{(x_1,\ldots,x_n)\in\mathbb{Z}^n} q^{x_1^2+x_2^2+\cdots+x_n^2} \tag{59}$$

By grouping the terms according to their exponent (the squared length m), we can rewrite this series as:

$$\Theta_{\mathbb{Z}^n}(q) = \sum_{m=0}^{\infty} r_n(m)q^m \tag{60}$$

where the coefficient $r_n(m)$ counts the number of distinct ways to represent the integer m as a sum of n squares.

Evaluating this for $n = 6$, the theta series becomes:

$$\Theta_{\mathcal{L}_6}(q) = \sum_{m=0}^{\infty} r_6(m)q^m \tag{61}$$

The sequence of coefficients $r_6(m)$ for $m \geq 0$ is the number of ways of writing m as a sum of 6 squares, which is precisely the defining property of the integer sequence OEIS A000141. $\square$

23 Numerical factorization of the theta series for the Pratt meet kernel

Let

$$G_N = (i \wedge_P j)_{1\leq i,j\leq N}$$

be the Gram matrix of the Pratt meet kernel restricted to $\{1, 2, \ldots, N\}$. We saw earlier that, the diagonal weights are already identified via the Möbius-transform identity

$$g(n) = \frac{n}{\operatorname{rad}(n)}.$$

The purpose of this section is to record the resulting theta-factorization explicitly, to list the numerical values for $N = 1, 2, \ldots, 20$, and to point out how this family fits the numerical strategy used by Simon Plouffe in *Numbers in the base* e^{π}.

Exact triangular factorization

For $1 \le a, d \le N$, define

$$L_N(a,d) = \mathbf{1}_{\{d \le_P a\}}, \qquad D_N = \operatorname{diag}\left(\frac{1}{\operatorname{rad}(1)}, \frac{2}{\operatorname{rad}(2)}, \ldots, \frac{N}{\operatorname{rad}(N)}\right).$$

By the feature-map description given earlier,

$$\phi_d(a) = \sqrt{\frac{d}{\operatorname{rad}(d)}}\, \mathbf{1}_{\{d \le_P a\}},$$

and therefore the restricted Gram matrix satisfies

$$G_N = L_N D_N L_N^{\mathsf{T}}.$$

Since $d \le_P a$ implies $d \le a$, the matrix L_N is lower triangular. Since also $a \le_P a$, every diagonal entry of L_N equals 1, so L_N is unimodular. Hence

$$\det(G_N) = \prod_{n=1}^{N} \frac{n}{\operatorname{rad}(n)}.$$

Corollary 5 (Theta-series factorization)**.** *Let*

$$\Theta_{G_N}(q) = \sum_{x \in \mathbb{Z}^N} q^{x^{\mathsf{T}} G_N x}.$$

Then

$$\Theta_{G_N}(q) = \prod_{n=1}^{N} \theta_3\!\left(q^{n/\operatorname{rad}(n)}\right).$$

In particular, if $q = e^{-\pi}$ *and* $T_d := \theta_3(e^{-d\pi})$*, then*

$$\Theta_{G_N}(e^{-\pi}) = \prod_{n=1}^{N} T_{n/\operatorname{rad}(n)} = T_1^{e_1(N)} T_2^{e_2(N)} T_3^{e_3(N)} T_4^{e_4(N)} T_8^{e_8(N)},$$

where

$$e_d(N) = \#\left\{1 \le n \le N : \frac{n}{\operatorname{rad}(n)} = d\right\}.$$

For $N \le 20$*, only the values* $d \in \{1, 2, 3, 4, 8\}$ *occur.*

Proof. Because L_N is unimodular, the change of variables $y = L_N^{\mathsf{T}} x$ is a bijection of $\mathbb{Z}^N$.

Hence

$$\Theta_{G_N}(q) = \sum_{x\in\mathbb{Z}^N} q^{x^\mathsf{T} L_N D_N L_N^\mathsf{T} x} = \sum_{y\in\mathbb{Z}^N} q^{y^\mathsf{T} D_N y} = \prod_{n=1}^{N} \left(\sum_{m\in\mathbb{Z}} q^{(n/\operatorname{rad}(n))m^2} \right) = \prod_{n=1}^{N} \theta_3\!\left(q^{n/\operatorname{rad}(n)}\right).$$

The specialization $q = e^{-\pi}$ is immediate. □

Thus the exact factorization with theta values is not a conjectural output of numerical fitting: it follows directly from the proved diagonal-weight formula.

Theta values used for $N \leq 20$

For convenience, here are the theta constants needed in the range $N \leq 20$:

d	symbol	decimal value
1	T_1	1.08643481121330801458
2	T_2	1.00373488548773909105
3	T_3	1.00016139903514069402
4	T_4	1.00000697468471241799
8	T_8	1.00000000002432311342

A few exact examples are:

$$\begin{aligned} \Theta_{G_5}(e^{-\pi}) &= T_1^4 T_2, \\ \Theta_{G_9}(e^{-\pi}) &= T_1^6 T_2 T_3 T_4, \\ \Theta_{G_{20}}(e^{-\pi}) &= T_1^{13} T_2^3 T_3^2 T_4 T_8. \end{aligned}$$

All Gram matrices, determinants, and LDL^T decompositions were computed in exact arithmetic in SymPy. The decimal values below were then evaluated from the rapidly convergent series

$$T_d = 1 + 2\sum_{n\geq 1} e^{-d\pi n^2}.$$

Computed values for $N = 1, 2, \ldots, 20$

N	$\det(G_N)$	$(e_1, e_2, e_3, e_4, e_8)$	$\Theta_{G_N}(e^{-\pi})$
1	1	$(1,0,0,0,0)$	1.086434811213308
2	1	$(2,0,0,0,0)$	1.180340599016097
3	1	$(3,0,0,0,0)$	1.282363115859456
4	2	$(3,1,0,0,0)$	1.287152595250892

N	$\det(G_N)$	$(e_1, e_2, e_3, e_4, e_8)$	$\Theta_{G_N}(e^{-\pi})$
5	2	$(4, 1, 0, 0, 0)$	1.398407386824122
6	2	$(5, 1, 0, 0, 0)$	1.519278465303561
7	2	$(6, 1, 0, 0, 0)$	1.650597012632519
8	8	$(6, 1, 0, 1, 0)$	1.650608525026269
9	24	$(6, 1, 1, 1, 0)$	1.650874931649603
10	24	$(7, 1, 1, 1, 0)$	1.793567994703520
11	24	$(8, 1, 1, 1, 0)$	1.948594705723950
12	48	$(8, 2, 1, 1, 0)$	1.955872483811844
13	48	$(9, 2, 1, 1, 0)$	2.124927952707425
14	48	$(10, 2, 1, 1, 0)$	2.308595699141573
15	48	$(11, 2, 1, 1, 0)$	2.508138732564730
16	384	$(11, 2, 1, 1, 1)$	2.508138732625736
17	384	$(12, 2, 1, 1, 1)$	2.724929230477028
18	1152	$(12, 2, 2, 1, 1)$	2.725369031425653
19	1152	$(13, 2, 2, 1, 1)$	2.960935789143525
20	2304	$(13, 3, 2, 1, 1)$	2.971994545252526

24 Explicit theta constants, the constants C_N, and the coefficient sequences

Let

$$G_N := (\, i \wedge_P j \,)_{1 \le i,j \le N}$$

be the $N \times N$ Gram matrix attached to the Pratt-meet kernel $k(a, b) = a \wedge_P b$, and let

$$\Theta_N(q) := \sum_{x \in \mathbb{Z}^N} q^{x^\intercal G_N x}.$$

Write

$$d_n := \frac{n}{\mathrm{rad}(n)}, \qquad u_r(N) := \#\{1 \le n \le N : d_n = r\}.$$

For $N \le 100$ the only values of d_n are

$$r \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 27, 32\}.$$

The diagonal decomposition proved in the Pratt-meet paper gives

$$G_N = L_N \; \mathrm{diag}(d_1, \ldots, d_N) \; L_N^\intercal$$

with $L_N \in \mathrm{GL}_N(\mathbb{Z})$ lower unitriangular. Hence the theta series is invariant under the unimodular change of variables $x \mapsto L_N^{\mathsf{T}} x$, and therefore

$$\Theta_N(q) = \prod_{n=1}^{N} \theta_3(q^{d_n}) = \prod_r \theta_3(q^r)^{u_r(N)}.$$

At $q = e^{-\pi}$ we obtain the constants

$$C_N := \Theta_N(e^{-\pi}) = \prod_r T_r^{u_r(N)}, \qquad T_r := \theta_3(e^{-r\pi}).$$

Following the explicit evaluations collected on the Wikipedia page "Theta function - Explicit values" (which summarizes classical evaluations for Jacobi theta constants and Ramanujan's $\varphi(q) = \theta_3(q)$), set

$$A := \frac{\pi^{1/4}}{\Gamma(3/4)} = T_1$$

and write $T_r = A\beta_r$ for the explicitly known cases $r \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\}$. A convenient list is

$$\beta_2 = \frac{\sqrt{2+\sqrt{2}}}{2},$$
$$\beta_3 = \frac{\sqrt{1+\sqrt{3}}}{108^{1/8}},$$
$$\beta_4 = \frac{2+\sqrt[4]{8}}{4},$$
$$\beta_5 = \sqrt{\frac{2+\sqrt{5}}{5}},$$
$$\beta_6 = \frac{\sqrt{1+\sqrt[4]{3}+\sqrt{2}+\sqrt{3}}}{12^{3/8}},$$
$$\beta_7 = \frac{\sqrt{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}}{14^{3/8}7^{1/16}},$$
$$\beta_8 = \frac{\sqrt{2+\sqrt{2}}+2^{7/8}}{4},$$
$$\beta_9 = \frac{1+\sqrt[3]{2+2\sqrt{3}}}{3},$$
$$\beta_{10} = \frac{\sqrt{\sqrt[4]{64}+\sqrt[4]{80}+\sqrt[4]{81}+\sqrt[4]{100}}}{200^{1/4}},$$
$$\beta_{12} = \frac{\sqrt{1+\sqrt[4]{2}+\sqrt[4]{3}+\sqrt{2}+\sqrt{3}+\sqrt[4]{18}+\sqrt[4]{24}}}{2\cdot 108^{1/8}},$$
$$\beta_{16} = \frac{1}{4} + \frac{2^{3/4}}{8} + \frac{2^{9/16}\sqrt[4]{1+\sqrt{2}}}{4}.$$

We leave T_{27} and T_{32} as symbolic theta constants.

If

$$R_0 := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\}, \qquad \sigma_N := \sum_{r \in R_0} u_r(N),$$

then

$$C_N = A^{\sigma_N} \Big(\prod_{r \in \{2,3,4,5,6,7,8,9,10,12,16\}} \beta_r^{u_r(N)} \Big) T_{27}^{u_{27}(N)} T_{32}^{u_{32}(N)}.$$

In the catalogue below, each entry is written in a Plouffe-style format: first the exact constant C_N, then a decimal approximation, and then the first twenty coefficients

$$\Theta_N(q) = \sum_{m \geq 0} a_N(m) q^m, \qquad (a_N(0), a_N(1), \dots, a_N(19)).$$

All exact factorizations below were simplified by collecting powers with SymPy.

Catalogue entries $N = 1, \dots, 10$

$N = 1$.

$$C_1 = A.$$

$$C_1 \approx 1.0864348112133080145753\text{16}.$$

Coefficients: 1, 2, 0, 0, 2, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0.

$N = 2$.

$$C_2 = A^2.$$

$$C_2 \approx 1.180340599016096226045338.$$

Coefficients: 1, 4, 4, 0, 4, 8, 0, 0, 4, 4, 8, 0, 0, 8, 0, 0, 4, 8, 4, 0.

$N = 3$.

$$C_3 = A^3.$$

$$C_3 \approx 1.282363115859455399000143.$$

Coefficients: 1, 6, 12, 8, 6, 24, 24, 0, 12, 30, 24, 24, 8, 24, 48, 0, 6, 48, 36, 24.

$N = 4$.

$$C_4 = A^4 \, \beta_2.$$

$$C_4 \approx 1.287152595250890761600398.$$

Coefficients: 1, 6, 14, 20, 30, 40, 36, 48, 62, 42, 72, 100, 68, 120, 112, 48, 126, 108, 98, 180.

$N = 5$.

$$C_5 = A^5 \, \beta_2.$$

$$C_5 \approx 1.398407386824120966709408.$$

Coefficients: 1, 8, 26, 48, 72, 112, 144, 160, 218, 248, 240, 368, 432, 400, 576, 544, 456, 736, 650, 656.

$N = 6$.

$$C_6 = A^6\,\beta_2.$$

$$C_6 \approx 1.519278465303559255958982.$$

Coefficients: 1, 10, 42, 100, 170, 272, 420, 544, 682, 910, 1040, 1220, 1700, 1904, 2080, 2720, 2730, 2900, 3822, 3620.

$N = 7$.

$$C_7 = A^7\,\beta_2.$$

$$C_7 \approx 1.6505970126325167308960033.$$

Coefficients: 1, 12, 62, 184, 372, 632, 1048, 1584, 2110, 2820, 3720, 4472, 5704, 7464, 8512, 10160, 12660, 13552, 15686, 18984.

$N = 8$.

$$C_8 = A^8\,\beta_2\,\beta_4.$$

$$C_8 \approx 1.6506085250262671017165507.$$

Coefficients: 1, 12, 62, 184, 374, 656, 1172, 1952, 2854, 4084, 5816, 7640, 9924, 13104, 15952, 19104, 24070, 28504, 32834, 39672.

$N = 9$.

$$C_9 = A^9\,\beta_2\,\beta_3\,\beta_4.$$

$$C_9 \approx 1.6508749316496013453276888.$$

Coefficients: 1, 12, 62, 186, 398, 780, 1540, 2700, 4166, 6428, 9720, 13348, 18094, 24760, 31356, 39320, 51026, 61720, 73386, 91716.

$N = 10$.

$$C_{10} = A^{10}\,\beta_2\,\beta_3\,\beta_4.$$

$$C_{10} \approx 1.7935679947035174098187 7.$$

Coefficients: 1, 14, 86, 310, 772, 1600, 3224, 6152, 10362, 16322, 25680, 38312, 53494, 74600, 101876, 131808, 171256, 221648, 272518, 336940.

Catalogue entries $N = 11, \ldots, 20$

$N = 11$.

$$C_{11} = A^{11}\,\beta_2\,\beta_3\,\beta_4.$$

$$C_{11} \approx 1.9485947057239473661277 03.$$

Coefficients: 1, 16, 114, 482, 1394, 3172, 6596, 13220, 24210, 40248, 64800, 102148, 151462, 215776, 305636, 418632, 554166, 734112, 952382, 1197572.

$N = 12$.

$$C_{12} = A^{12}\,\beta_2^2\,\beta_3\,\beta_4.$$

$$C_{12} \approx 1.9558724838118409 61876362.$$

Coefficients: 1, 16, 116, 514, 1622, 4136, 9384, 19564, 37404, 66720, 113448, 183608, 283850, 426416, 621752, 876624, 1213858, 1651872, 2190316, 2870124.

$N = 13$.

$$C_{13} = A^{13}\,\beta_2^2\,\beta_3\,\beta_4.$$

$$C_{13} \approx 2.1249279527074212 71262599.$$

Coefficients: 1, 18, 148, 746, 2652, 7412, 17888, 39360, 79776, 149802, 265688, 449864, 726902, 1130800, 1709752, 2506112, 3573936, 5007260, 6871236, 9231928.

$N = 14$.

$$C_{14} = A^{14}\,\beta_2^2\,\beta_3\,\beta_4.$$

$$C_{14} \approx 2.3085956991415683 29849173.$$

Coefficients: 1, 20, 184, 1042, 4146, 12752, 33008, 76628, 163800, 324180, 601104, 1060256, 1787674, 2889512, 4517552, 6861120, 10118686, 14576320, 20605160, 28519492.

$N = 15$.

$$C_{15} = A^{15}\,\beta_2^2\,\beta_3\,\beta_4.$$

$$C_{15} \approx 2.5081387325647246 15724273.$$

Coefficients: 1, 22, 224, 1410, 6232, 21084, 58880, 144728, 325348, 677286, 1315520, 2416088, 4237870, 7121512, 11524288, 18082752, 27569532, 40920356, 59441632, 84656344.

$N = 16$.

$$C_{16} = A^{16}\,\beta_2^2\,\beta_3\,\beta_4\,\beta_8.$$

$$C_{16} \approx 2.5081387326257303 58586577.$$

Coefficients: 1, 22, 224, 1410, 6232, 21084, 58880, 144728, 325350, 677330, 1315968, 2418908, 4250334, 7163680, 11642048, 18372208, 28220228, 42274928, 62072672, 89488520.

$N = 17$.

$$C_{17} = A^{17}\,\beta_2^2\,\beta_3\,\beta_4\,\beta_8.$$

$$C_{17} \approx 2.7249292304770209 89203814.$$

Coefficients: 1, 24, 268, 1858, 9054, 33592, 101496, 265308, 627270, 1370200, 2788432, 5340748, 9741670, 17031472, 28643512, 46611880, 73754770, 113693488, 171261732, 253013036.

$N = 18$.

$$C_{18} = A^{18}\,\beta_2^2\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{18} \approx 2.7253690314256466 54232891.$$

Coefficients: 1, 24, 268, 1860, 9102, 34128, 105212, 283416, 694454, 1573192, 3319048, 6595288, 12482072, 22608384, 39325544, 66098936, 107835822, 171047696, 264688484, 401053192.

$N = 19$.

$$C_{19} = A^{19}\,\beta_2^2\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{19} \approx 2.9609357891435185404227\,5.$$

Coefficients: 1, 26, 316, 2396, 12824, 52380, 174004, 497560, 1279490, 3030358, 6675904, 13800752, 27065276, 50737116, 91248664, 158151024, 265564672, 433325064, 688581884, 1069269848.

$N = 20$.

$$C_{20} = A^{20}\,\beta_2^3\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{20} \approx 2.9719945452525179611140\,01.$$

Coefficients: 1, 26, 318, 2448, 13456, 57172, 199652, 602320, 1627500, 4025530, 9235516, 19866260, 40442732, 78443380, 145727224, 260620376, 450620980, 755687828, 1233063038, 1963521532.

Catalogue entries $N = 21, \ldots, 30$

$N = 21$.

$$C_{21} = A^{21}\,\beta_2^3\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{21} \approx 3.2288783326984005541334\,79.$$

Coefficients: 1, 28, 370, 3084, 18354, 84136, 314632, 1006520, 2859052, 7394876, 17685932, 39542568, 83435148, 167406816, 321199360, 592206648, 1053951838, 1817071600, 3043944838, 4969364288.

$N = 22$.

$$C_{22} = A^{22}\,\beta_2^3\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{22} \approx 3.5079658218159275524854\,54.$$

Coefficients: 1, 30, 426, 3824, 24524, 120900, 483644, 1641952, 4908800, 13281254, 33105004, 76928212, 168244556, 349103572, 691553128, 1314319768, 2407248472, 4265507068, 7335277250, 12277045292.

$N = 23$.

$$C_{23} = A^{23}\,\beta_2^3\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{23} \approx 3.8111761853673241519238\,5.$$

Coefficients: 1, 32, 486, 4676, 32174, 170008, 726296, 2616888, 8241752, 23340656, 60634860, 146422976, 331926228, 712204240, 1456212080, 2852249736, 5375661026, 9788028816, 17275961002, 29642456984.

$N = 24$.

$$C_{24} = A^{24}\,\beta_2^3\,\beta_3^2\,\beta_4^2\,\beta_8.$$

$$C_{24} \approx 3.8112027671196005649161\,26.$$

Coefficients: 1, 32, 486, 4676, 32176, 170072, 727268, 2626240, 8306100, 23680672, 62087452, 151656752, 348409732, 758885552, 1577481800, 3145095688, 6039513484, 11212437360, 20188386134, 35346965808.

$N = 25$.

$$C_{25} = A^{25}\, \beta_2^3\, \beta_3^2\, \beta_4^2\, \beta_5\, \beta_8.$$

$$C_{25} \approx 3.8112039158292825776471 99.$$

Coefficients: 1, 32, 486, 4676, 32176, 170074, 727332, 2627212, 8315452, 23745024, 62427596, 153111288, 353662212, 775497752, 1624843144, 3269270592, 6342826988, 11909256824, 21706157238, 38501929408.

$N = 26$.

$$C_{26} = A^{26}\, \beta_2^3\, \beta_3^2\, \beta_4^2\, \beta_5\, \beta_8.$$

$$C_{26} \approx 4.1406246067894068659389 42.$$

Coefficients: 1, 34, 550, 5648, 41530, 234490, 1068452, 4091228, 13634228, 40716078, 111372372, 283221876, 676525044, 1530376576, 3301033988, 6826634120, 13593947022, 26162537272, 48821848194, 88577649612.

$N = 27$.

$$C_{27} = A^{27}\, \beta_2^3\, \beta_3^2\, \beta_4^2\, \beta_5\, \beta_8\, \beta_9.$$

$$C_{27} \approx 4.1406246067937590641870 2.$$

Coefficients: 1, 34, 550, 5648, 41530, 234490, 1068452, 4091228, 13634228, 40716080, 111372440, 283222976, 676536340, 1530459636, 3301502968, 6828771024, 13602129478, 26189805728, 48903280350, 88800394356.

$N = 28$.

$$C_{28} = A^{28}\, \beta_2^4\, \beta_3^2\, \beta_4^2\, \beta_5\, \beta_8\, \beta_9.$$

$$C_{28} \approx 4.1560893655478484550967 1.$$

Coefficients: 1, 34, 552, 5716, 42630, 245786, 1151512, 4560208, 15771134, 48898604, 138641996, 364666432, 899364280, 2097374568, 4656712552, 9897872752, 20232403870, 39928779936, 76330284188, 141746451832.

$N = 29$.

$$C_{29} = A^{29}\, \beta_2^4\, \beta_3^2\, \beta_4^2\, \beta_5\, \beta_8\, \beta_9.$$

$$C_{29} \approx 4.5153201652446138187322 33.$$

Coefficients: 1, 36, 620, 6820, 54064, 331114, 1644188, 6874664, 24976810, 80932446, 238742296, 651071944, 1660250844, 3993985596, 9129237252, 19942933744, 41835998352, 84619879148, 165599067476, 314480061136.

$N = 30$.

$$C_{30} = A^{30}\, \beta_2^4\, \beta_3^2\, \beta_4^2\, \beta_5\, \beta_8\, \beta_9.$$

$$C_{30} \approx 4.9056010112951747625425 59.$$

Coefficients: 1, 38, 692, 8060, 67706, 439314, 2307656, 10176680, 38834266, 131548296, 403895636, 1142307104, 3012361992, 7476460304, 17595355264, 39506840512, 85056116858, 176329800736, 353259166408, 686041561808.

Catalogue entries $N = 31, \ldots, 40$

$N = 31$.

$$C_{31} = A^{31}\,\beta_2^4\,\beta_3^2\,\beta_4^2\,\beta_5\,\beta_8\,\beta_9.$$

$$C_{31} \approx 5.3296157085942860703729 31.$$

Coefficients: 1, 40, 768, 9444, 83828, 574802, 3187668, 14808112, 59323038, 210095458, 671607616, 1970453120, 5374660852, 13764416292, 33356945772, 76986780560, 170114875228, 361472623668, 741372576384, 1472381383040.

$N = 32$.

$$C_{32} = A^{32}\,\beta_2^4\,\beta_3^2\,\beta_4^2\,\beta_5\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{32} \approx 5.3296157085942860703745 08.$$

Coefficients: 1, 40, 768, 9444, 83828, 574802, 3187668, 14808112, 59323038, 210095458, 671607616, 1970453120, 5374660852, 13764416292, 33356945772, 76986780560, 170114875230, 361472623748, 741372577920, 1472381401928.

$N = 33$.

$$C_{33} = A^{33}\,\beta_2^4\,\beta_3^2\,\beta_4^2\,\beta_5\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{33} \approx 5.7902800362061140077969 25.$$

Coefficients: 1, 42, 848, 10980, 102718, 742538, 4338808, 21202336, 89106918, 329891140, 1098173948, 3343286112, 9434232056, 24934096568, 62230143192, 147647953680, 334867374280, 729349852948, 1531451909412, 3110443353008.

$N = 34$.

$$C_{34} = A^{34}\,\beta_2^4\,\beta_3^2\,\beta_4^2\,\beta_5\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{34} \approx 6.2907617980077757674635 2.$$

Coefficients: 1, 44, 932, 12676, 124680, 948058, 5825580, 29901912, 131717026, 509590054, 1766633928, 5582040376, 16299040076, 44462548396, 114296169300, 278803489904, 649074150426, 1449131008564, 3115271685668, 6470839449048.

$N = 35$.

$$C_{35} = A^{35}\,\beta_2^4\,\beta_3^2\,\beta_4^2\,\beta_5\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{35} \approx 6.8345026064064679517538 94.$$

Coefficients: 1, 46, 1020, 14540, 150034, 1197506, 7723560, 41578424, 191770210, 774920224, 2797465284, 9175113920, 27726580232, 78080058016, 206756430064, 518571560416, 1239339014212, 2836467840348, 6243145223368, 13262523093400.

$N = 36$.

$$C_{36} = A^{36}\,\beta_2^4\,\beta_3^2\,\beta_4^2 \\ \beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{36} \approx 6.8345026954246633878162 84.$$

Coefficients: 1, 46, 1020, 14540, 150034, 1197506, 7723562, 41578516, 191772250, 774949304, 2797765352, 9177508932, 27742027352, 78163214864, 207139970484, 520121400864, 1244933944780, 2854818068188, 6298598383832, 13418683209432.

$N = 37$.

$$C_{37} = A^{37}\,\beta_2^4\,\beta_3^2\,\beta_4^2 \\ \beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{37} \approx 7.4252416456405389330514 85.$$

Coefficients: 1, 48, 1112, 16580, 179116, 1497666, 10120614, 57054720, 275229350, 1160888818, 4363111176, 14856198708, 46480618796, 135197468244, 369064325928, 952771806820, 2340743958246, 5501395932068, 12424064361824, 27061718338608.

$N = 38$.

$$C_{38} = A^{38}\,\beta_2^4\,\beta_3^2\,\beta_4^2 \\ \beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{38} \approx 8.0670410054946714428414 69.$$

Coefficients: 1, 50, 1208, 18804, 212278, 1855994, 13118170, 77329108, 389697022, 1714342852, 6705130136, 23696532724, 76743508072, 230480841704, 648188480100, 1720633097320, 4339362918920, 10453829243844, 24167306657676, 53824116931408.

$N = 39$.

$$C_{39} = A^{39}\,\beta_2^4\,\beta_3^2\,\beta_4^2 \\ \beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{39} \approx 8.7643141718546178308902 98.$$

Coefficients: 1, 52, 1308, 21220, 249888, 2280650, 16832574, 103603056, 544779794, 2497448886, 10160052280, 37261453628, 124916005172, 387396968108, 1122564135768, 3064429359308, 7934270787922, 19594296159236, 46374770793684, 105613406739280.

$N = 40$.

$$C_{40} = A^{40}\,\beta_2^4\,\beta_3^2\,\beta_4^3 \\ \beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{40} \approx 8.7643753001826870936408 63.$$

Coefficients: 1, 52, 1308, 21220, 249890, 2280754, 16835190, 103645496, 545279570, 2502010186, 10193717428, 37468659740, 126005564760, 392391865880, 1142884240328, 3138952266564, 8184102798268, 20369090095556, 48619899067836, 111742265500336.

Catalogue entries $N = 41, \ldots, 50$

$N = 41$.

$$C_{41} = A^{41}\,\beta_2^4\,\beta_3^2\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{41} \approx 9.5219224246565741232676.$$

Coefficients: 1, 54, 1412, 23836, 292332, 2780638, 21399314, 137358316, 753070342, 3597130836, 15231408284, 58063388204, 202033485820, 649407515552, 1948059968452, 5499691737080, 14714225751910, 37523169983096, 91648851762592, 215280355646432.

$N = 42$.

$$C_{42} = A^{42}\,\beta_2^4\,\beta_3^2\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{42} \approx 10.344947991819511059551554.$$

Coefficients: 1, 56, 1520, 26660, 340006, 3365410, 26963414, 180204616, 1028371638, 5108832798, 22468468692, 88800924228, 319666450584, 1060669333528, 3277343377400, 9511981249020, 26117950914344, 68251942658812, 170598505930184, 409607905510016.

$N = 43$.

$$C_{43} = A^{43}\,\beta_2^4\,\beta_3^2\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{43} \approx 11.239111618503920361077 1.$$

Coefficients: 1, 58, 1632, 29700, 393328, 4045534, 33697274, 234184764, 1389460882, 7172306896, 32740061228, 134098273884, 499325095636, 1710220580304, 5443625712660, 16244323779104, 45781606722786, 122611239897944, 313667295671244, 769873816859128.

$N = 44$.

$$C_{44} = A^{44}\,\beta_2^5\,\beta_3^2\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{44} \approx 11.281088413382950460667 4.$$

Coefficients: 1, 58, 1634, 29816, 396592, 4104934, 34483930, 242275832, 1456855432, 7640676540, 35518986256, 148442947076, 564806004748, 1978425219140, 6442343298480, 19665233309240, 56671637069870, 155114232069944, 405295989239274, 1015364493202900.

$N = 45$.

$$C_{45} = A^{45}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{45} \approx 11.282909170168207331653.$$

Coefficients: 1, 58, 1634, 29818, 396708, 4108202, 34543562, 243069016, 1465065300, 7709644400, 36003537920, 151356657940, 580087357830, 2049463191768, 6739229195900, 20794845378368, 60628488301334, 167998926876772, 444626524825614, 1128708251894304.

$N = 46$.

$$C_{46} = A^{46}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{46} \approx 12.2581452942285981242954 5.$$

Coefficients: 1, 60, 1750, 33086, 456346, 4901734, 42763234, 312215776, 1951996748, 10647991406, 51491913960, 223849875080, 885730863946, 3225057989644, 10910170871680, 34576086173172, 103378839911764, 293357759993692, 794118256263026, 2059622999437744.

$N = 47$.

$$C_{47} = A^{47}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$

$$C_{47} \approx 13.317675768560547029051 54.$$

Coefficients: 1, 62, 1870, 36586, 522520, 5814546, 52570202, 397808416, 2577340992, 14561788372, 72873423360, 327458138052, 1337334673774, 5017816613040, 17463280482356, 56844213193160, 174303098417554, 506569459790124, 1402675413980082, 3717114668204232.

$N = 48$.

$$C_{48} = A^{48}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{48} \approx 13.317675768884474367245 29.$$

Coefficients: 1, 62, 1870, 36586, 522520, 5814546, 52570202, 397808416, 2577340994, 14561788496, 72873427100, 327458211224, 1337335718814, 5017828242132, 17463385622760, 56845008809992, 174308253099538, 506598583366868, 1402821160826802, 3717769584480336.

$N = 49$.

$$C_{49} = A^{49}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{49} \approx 13.317675776380377335923 54.$$

Coefficients: 1, 62, 1870, 36586, 522520, 5814546, 52570202, 397808418, 2577341118, 14561792236, 72873500272, 327459256264, 1337347347906, 5017933382536, 17464181239592, 56850163491980, 174337376676530, 506744330221068, 1403476077249250, 3720444255917964.

$N = 50$.

$$C_{50} = A^{50}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{50} \approx 13.317679790373869945727 47.$$

Coefficients: 1, 62, 1870, 36586, 522520, 5814548, 52570326, 397812158, 2577414290, 14562837276, 72885129364, 327564396668, 1338142964742, 5023088064772, 17493304824064, 56995910492524, 174992295189058, 509419024916880, 1413511944014322, 3755372618397148.

Catalogue entries $N = 51, \ldots, 60$

$N = 51.$

$$C_{51} = A^{51}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{51} \approx 14.4687909288541228486594 6.$$

Coefficients: 1, 64, 1994, 40326, 595694, 6859712, 64203162, 503025982, 3374083646, 19729294954, 102115944692, 474130283452, 1998426659830, 7728500713848, 27685262841432, 92637754074640, 291661197727908, 869454946253244, 2467365729174502, 6696534097742740.

$N = 52.$

$$C_{52} = A^{52}\,\beta_2^6\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{52} \approx 14.5228302061194311154733.$$

Coefficients: 1, 64, 1996, 40454, 599682, 6940364, 65394550, 516745406, 3502489972, 20735347046, 108864115972, 513588954012, 2202659740602, 8676775000176, 31682244567416, 108095761554300, 347038471578064, 1054769912992432, 3050892356519704, 8436392250816260.

$N = 53.$

$$C_{53} = A^{53}\,\beta_2^6\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{53} \approx 15.7781082932682912646485 2.$$

Coefficients: 1, 66, 2124, 44446, 680592, 8139856, 79279270, 647615414, 4537180148, 27754207720, 150465599292, 732350680760, 3236842709478, 13123566374836, 49253536680440, 172487559386256, 567636347658682, 1766207411128984, 5223838142337484, 14754586215277120.

$N = 54.$

$$C_{54} = A^{54}\,\beta_2^6\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{54} \approx 15.7781082932848755874966 6.$$

Coefficients: 1, 66, 2124, 44446, 680592, 8139856, 79279270, 647615414, 4537180148, 27754207722, 150465599424, 732350685008, 3236842798370, 13123567736020, 49253552960152, 172487717944796, 567637642889510, 1766216485489280, 5223893650752924, 14754887146475704.

$N = 55.$

$$C_{55} = A^{55}\,\beta_2^6\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{55} \approx 17.1418861049180833318553 7.$$

Coefficients: 1, 68, 2256, 48694, 769486, 9501172, 95563230, 806262846, 5833772160, 36844847732, 206132573540, 1034577118932, 4710618617574, 19652763109388, 75801635910752, 272459683793656, 919088059606672, 2927747981100768, 8854889236071480, 25547950815158884.

$N = 56$.

$$C_{56} = A^{56}\,\beta_2^6\,\beta_3^3\,\beta_4^4\,\beta_5^2$$
$$\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{56} \approx 17.142005664169041314398 1.$$

Coefficients: 1, 68, 2256, 48694, 769488, 9501308, 95567742, 806360234, 5835311132, 36863850076, 206323700000, 1036189644624, 4722286161894, 19726452804852, 76213901057832, 274528838031520, 928509296841822, 2967053507319680, 9006492507897496, 26092870182843584.

$N = 57$.

$$C_{57} = A^{57}\,\beta_2^6\,\beta_3^3\,\beta_4^4\,\beta_5^2$$
$$\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{57} \approx 18.623671687568949066522 02.$$

Coefficients: 1, 70, 2392, 53206, 866878, 11040420, 114574870, 997593106, 7449570576, 48553474958, 280242535772, 1450449769604, 6806336170794, 29244754367768, 116079473070152, 429029210571916, 1487013157949120, 4863536677235428, 15093101052357184, 44655325522199004.

$N = 58$.

$$C_{58} = A^{58}\,\beta_2^6\,\beta_3^3\,\beta_4^4\,\beta_5^2$$
$$\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{58} \approx 20.233405233982600660333 33.$$

Coefficients: 1, 72, 2532, 57990, 973292, 12774316, 136660494, 1226849258, 9446490544, 63474696952, 377578635568, 2012930032144, 9722134957566, 42954535393028, 175129488958072, 664089285401168, 2358686246620754, 7896067401010496, 25052430459923044, 75700146533235160.

$N = 59$.

$$C_{59} = A^{59}\,\beta_2^6\,\beta_3^3\,\beta_4^4\,\beta_5^2$$
$$\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{59} \approx 21.982275795584245024256 36.$$

Coefficients: 1, 74, 2676, 63054, 1089274, 14721044, 162214190, 1500286226, 11902135644, 82393226674, 504801350604, 2770541006860, 13766888118922, 62525756648648, 261793742563896, 1018374396702588, 3706311541036740, 12699367858019292, 41194951189259148, 127133941181270700.

$N = 60$.

$$C_{60} = A^{60}\,\beta_2^7\,\beta_3^3\,\beta_4^4\,\beta_5^2$$
$$\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{60} \approx 22.064377078440650902931 44.$$

Coefficients: 1, 74, 2678, 63202, 1094626, 14847152, 164392738, 1529728314, 12226564026, 85393799274, 528605627244, 2935327586316, 14776492998678, 68066868104456, 289327843230120, 1143428910572336, 4229922830435820, 14736281437877816, 48608583874033838, 152538217979323152.

Catalogue entries $N = 61, \ldots, 70$

$N = 61$.

$$C_{61} = A^{61}\,\beta_2^7\,\beta_3^3\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{61} \approx 23.97150734575490920588659.$$

Coefficients: 1, 76, 2826, 68558, 1221032, 17036552, 194092398, 1858640194, 15288209906, 109876621632, 699722011416, 3995598302788, 20671601425766, 97790643889612, 426518820387824, 1727955580990684, 6546336697034478, 23332285288086568, 78659973223853614, 252043300759916392.

$N = 62$.

$$C_{62} = A^{62}\,\beta_2^7\,\beta_3^3\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{62} \approx 26.04348005768366107372015.$$

Coefficients: 1, 78, 2978, 74210, 1358150, 19478768, 228171154, 2246962106, 19007932358, 140487114550, 919863439628, 5398759611660, 28693374588270, 139353602426472, 623499586262984, 2588984806556704, 10043594779147768, 36620570546354712, 126177801194051314, 412820557813764936.

$N = 63$.

$$C_{63} = A^{63}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{63} \approx 26.04768345023667712289594.$$

Coefficients: 1, 78, 2978, 74212, 1358306, 19484724, 228319574, 2249678406, 19046889894, 140943456858, 924357363840, 5436775476376, 28974348817372, 141193329305884, 634297105492260, 2646371555881664, 10322301986717012, 37867569757838216, 131355771263507030, 432907751865984684.

$N = 64$.

$$C_{64} = A^{63}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{64} \approx 26.04768345023667712289594.$$

Coefficients: 1, 78, 2978, 74212, 1358306, 19484724, 228319574, 2249678406, 19046889894, 140943456858, 924357363840, 5436775476376, 28974348817372, 141193329305884, 634297105492260, 2646371555881664, 10322301986717012, 37867569757838216, 131355771263507030, 432907751865984684.

$N = 65$.

$$C_{65} = A^{64}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{65} \approx 28.29911005180189185633777.$$

Coefficients: 1, 80, 3134, 80168, 1506732, 22201492, 267294978, 2706465978, 23548963318, 179076206096, 1206700916860, 7289989566824, 39885993698336, 199423916570956, 918532517801156, 3925839774458084, 15672998295471898, 58794598483663952, 208359786877087654, 700913886219638176.

$N = 66.$

$$C_{66} = A^{65}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{66} \approx 30.745138286634015567781.$$

Coefficients: 1, 82, 3294, 86436, 1667070, 25215116, 311704230, 3241216270, 28964908738, 226218535718, 1565387919168, 9708804338768, 54513070918956, 279554059393284, 1319793797179772, 5777485323784000, 23604455244716696, 90539490005676456, 327786407032436330, 1125487552924723708.

$N = 67.$

$$C_{67} = A^{66}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{67} \approx 33.402588510166274935531865.$$

Coefficients: 1, 84, 3458, 93024, 1839944, 28549420, 362141050, 3864797602, 35450675418, 284198783428, 2018448399228, 12846062616232, 73988609586840, 389032641636772, 1882032742234908, 8436491150229540, 35268458516555150, 138307566543714056, 511505427075226810, 1792618468413175576.

$N = 68.$

$$C_{68} = A^{67}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{68} \approx 33.527343353245815461668 61.$$

Coefficients: 1, 84, 3460, 93192, 1846860, 28735468, 365820938, 3921896442, 36174957520, 291928378800, 2089349756980, 13414460369136, 78025510065184, 414724823968076, 2030010685690688, 9214564163098288, 39032594902375802, 155181117241739992, 582046381005135568, 2069259293625836320.

$N = 69.$

$$C_{69} = A^{68}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{69} \approx 36.425272946467374802304 6.$$

Coefficients: 1, 86, 3628, 100112, 2033246, 32429356, 423298794, 4653724702, 44022444124, 364335764778, 2673938156624, 17601003682900, 104926780904880, 571359704549764, 2863639090611736, 13301415186859812, 57617782092495632, 234075829044342956, 896469220716761448, 3251785362662004376.

$N = 70.$

$$C_{70} = A^{69}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{70} \approx 39.573684536988498113582 89.$$

Coefficients: 1, 88, 3800, 107368, 2233472, 36496020, 488164762, 5500522514, 53333960020, 452445511740, 3403456283940, 22958187452808, 140216833359152, 781941941955572, 4011706440883224, 19063896222046672, 84430475335474422, 350454200683322168, 1370348885658207644, 5071331982345760368.

Catalogue entries $N = 71, \ldots, 80$

$N = 71$.

$$C_{71} = A^{70}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{71} \approx 42.99422848895810553521919.$$

Coefficients: 1, 90, 3976, 114968, 2448210, 40963140, 561164402, 6477066774, 64339471992, 559186423822, 4309323637120, 29776101073316, 186239876399544, 1063280504164300, 5582397310354288, 27133226455048260, 122838712447331100, 520879141906102372, 2079281604797649508, 7850164353019051616.

$N = 72$.

$$C_{72} = A^{71}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{72} \approx 42.99422848895810918211965.$$

Coefficients: 1, 90, 3976, 114968, 2448210, 40963140, 561164402, 6477066774, 64339471992, 559186423822, 4309323637120, 29776101073316, 186239876399546, 1063280504164480, 5582397310362240, 27133226455278196, 122838712452227520, 520879141988028652, 2079281605919978312, 7850164365973185164.

$N = 73$.

$$C_{73} = A^{72}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{73} \approx 46.71042651166303245337068.$$

Coefficients: 1, 92, 4156, 122920, 2678148, 45859740, 643098634, 7599625514, 77298501960, 687947294088, 5428818813748, 38407702489056, 245920757720098, 1436878634707636, 7717577047891720, 38357574400478112, 177477658069716554, 768683256579756816, 3132205802889615692, 12063002649371202356.

$N = 74$.

$$C_{74} = A^{73}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{74} \approx 50.74783340889172429833369.$$

Coefficients: 1, 94, 4340, 131232, 2923990, 51216220, 734826426, 8886068622, 92503109284, 842636017490, 6805999599376, 49280539375892, 322890759947970, 1930096050092304, 10602192046653968, 53869545187436932, 254684663585364004, 1126512484585609300, 4685008846039509252, 18404140261589263300.

$N = 75$.

$$C_{75} = A^{74}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{75} \approx 50.74784870446405146171389.$$

Coefficients: 1, 94, 4340, 131232, 2923990, 51216222, 734826614, 8886077302, 92503371748, 842641865470, 6806102031816, 49282009028744, 322908532085214, 1930281056310872, 10603877318688948, 53883157186635684, 254783224664115788, 1127158266105505240, 4688869038139693860, 18425344645682571236.

$N = 76$.

$$C_{76} = A^{75}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{76} \approx 50.9373861081243328029794.$$

Coefficients: 1, 94, 4342, 131420, 2932670, 51478686, 740674594, 8988509746, 93973024978, 860414020262, 6991108783992, 50967293022148, 336520741996826, 2028845176800804, 11249695852512604, 57743737071412032, 275991164308237180, 1234926265762507548, 5198449099671989070, 20679759741911639392.

$N = 77$.

$$C_{77} = A^{76}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{77} \approx 55.3401494600794382902617\,9.$$

Coefficients: 1, 96, 4530, 140104, 3195512, 57344214, 843640650, 10470121774, 111955909810, 1048463027592, 8713418173892, 64967487618308, 438643274353918, 2703607494700320, 15321368526639568, 80345064843830724, 392151697912074390, 1790966472678633660, 7690802743730078582, 31192159397616272420.

$N = 78$.

$$C_{78} = A^{77}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{78} \approx 60.1234648311776539914975\,3.$$

Coefficients: 1, 98, 4722, 149164, 3475722, 63735430, 958338138, 12157683282, 132902544382, 1272489535642, 10812031510568, 82415264218700, 568802161690362, 3582990975854364, 20746010467076420, 111117738559627776, 553719135088687224, 2580677307404002892, 11303380523066689282, 46734472441600719024.

$N = 79$.

$$C_{79} = A^{78}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{79} \approx 65.3202251633504583342033\,6.$$

Coefficients: 1, 100, 4918, 158608, 3774052, 70687070, 1085818442, 14074657886, 157224862390, 1538422095268, 13358927258324, 104063642615844, 733898495514854, 4723140285257816, 27933616609277144, 152774591938894292, 777092240846690066, 3695281825338175028, 16506229703787928634, 69563490588916672604.

$N = 80$.

$$C_{80} = A^{79}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{80} \approx 65.3202251649392495793943.$$

Coefficients: 1, 100, 4918, 158608, 3774052, 70687070, 1085818442, 14074657886, 157224862392, 1538422095468, 13358927268160, 104063642933060, 733898503062958, 4723140426631956, 27933618780914028, 152774620088210064, 777092555296414846, 3695284902182365564, 16506256421642445282, 69563698716201904292.

Catalogue entries $N = 81, \ldots, 90$

$N = 81$.

$$C_{81} = A^{79}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{81} \approx 65.3202251649392495793943.$$

Coefficients: 1, 100, 4918, 158608, 3774052, 70687070, 1085818442, 14074657886, 157224862392, 1538422095468, 13358927268160, 104063642933060, 733898503062958, 4723140426631956, 27933618780914028, 152774620088210064, 777092555296414846, 3695284902182365564, 16506256421642445282, 69563698716201904292.

$N = 82$.

$$C_{82} = A^{80}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{82} \approx 70.9661664954815449848168 7.$$

Coefficients: 1, 102, 5118, 168444, 4091270, 78235374, 1227202418, 16246611986, 185381726268, 1853013194394, 16437943096180, 130809646794988, 942340238971078, 6194014284496912, 37406617630088400, 208849987107541124, 1084109620628276664, 5258916608078184152, 23952696540413205238, 102881787517518068520.

$N = 83$.

$$C_{83} = A^{81}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{83} \approx 77.1001136990506767567 4.$$

Coefficients: 1, 104, 5322, 178680, 4428160, 86418118, 1383683402, 18701353710, 217883132780, 2223933117680, 20146423890008, 163718026221556, 1204330296350478, 8082400797010396, 49827522241745332, 283924844115712736, 1503694307814525042, 7439524248667184044, 34545346697856149366, 151204913448446090492.

$N = 84$.

$$C_{84} = A^{82}\,\beta_2^{10}\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{84} \approx 77.3880737948082950188463 3.$$

Coefficients: 1, 104, 5324, 178888, 4438804, 86775478, 1392539722, 18874189946, 220650499586, 2261335825308, 20582190166212, 168165892814276, 1244623152986814, 8409837022289744, 52236185601813092, 300089683112440948, 1603349788064281266, 8007378384764844876, 37552775606332979468, 166084289381832901900.

$N = 85$.

$$C_{85} = A^{83}\,\beta_2^{10}\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{85} \approx 84.0770973434240991530111 6.$$

Coefficients: 1, 106, 5532, 189536, 4796582, 95653294, 1566101326, 21659627166, 258407757086, 2702810375438, 25107646896480, 209368021537240, 1581396239972314, 10903606008791596, 69097024200275960, 404898388886775128, 2206018438343516684, 11230898076238986276, 53672009269737956668, 241790061125104432932.

$N = 86$.

$$C_{86} = A^{84}\,\beta_2^{10}\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{86} \approx 91.34428537966588196172832.$$

Coefficients: 1, 108, 5744, 200600, 5175656, 105246670, 1757418978, 24792208890, 301736604582, 3219817196200, 30516399850220, 259626634595596, 2000649098940038, 14071804119080264, 90954451702958700, 543511176462604180, 3018978051916265902, 15664742681759117220, 76272004876237243080, 349943926657648068556.

$N = 87$.

$$C_{87} = A^{85}\,\beta_2^{10}\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{87} \approx 99.23961144187183386937622.$$

Coefficients: 1, 110, 5960, 212088, 5576858, 115598198, 1967923806, 28307448046, 351331373674, 3823500898706, 36959549080792, 320709018725304, 2520505841741594, 18079541961704052, 119159092951313008, 725939336652550728, 4110001752623772120, 21730842997303018932, 107783405582795798808, 503575019795847864716.

$N = 88$.

$$C_{88} = A^{86}\,\beta_2^{10}\,\beta_3^4\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{88} \approx 99.24030360687262379442472.$$

Coefficients: 1, 110, 5960, 212088, 5576860, 115598418, 1967935726, 28307872222, 351342527390, 3823732095102, 36963484928404, 320765633621396, 2521208504488942, 18087188963501464, 119233012049474592, 726580754690001336, 4115042764307255310, 21767002081226427256, 108021723768698436744, 505026898469153390348.

$N = 89$.

$$C_{89} = A^{87}\,\beta_2^{10}\,\beta_3^4\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{89} \approx 107.81812051388402946140 35.$$

Coefficients: 1, 112, 6180, 224008, 6001038, 126752358, 2199144482, 32244167850, 407969425554, 4526648346720, 44614884990280, 394749219234568, 3163442457210690, 23137253447823272, 155481317177531164, 965688313992064764, 5573246747311980312, 30033262690452995804, 151794201602714442564, 722523581442900547492.

$N = 90$.

$$C_{90} = A^{88}\,\beta_2^{10}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{90} \approx 107.83552225450565341282 64.$$

Coefficients: 1, 112, 6180, 224010, 6001262, 126764718, 2199592498, 32256169926, 408222930270, 4531046635684, 44679373325980, 395565158085676, 3172495753904132, 23226483217804056, 156270815616012660, 972015198906934160, 5619521254219628932, 30344225325061562848, 153725578235096861056, 733670075002012843816.

Catalogue entries $N = 91, \ldots, 100$

$N = 91$.

$$C_{91} = A^{89}\,\beta_2^{10}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6 \\ \beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{91} \approx 117.1562652626623246167965.$$

Coefficients: 1, 114, 6404, 236370, 6449284, 138767466, 2453134294, 36655802942, 472747272646, 5347746025662, 53745865782568, 484988417089848, 3964442516384044, 29580536830886212, 202813141051802168, 1285347964854315828, 7569896708053645370, 41629721616382289588, 214726579578545295800, 1043065351228767534228.

$N = 92$.

$$C_{92} = A^{90}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6 \\ \beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{92} \approx 117.5938304975895535228246.$$

Coefficients: 1, 114, 6406, 236598, 6462092, 139240206, 2466032862, 36933337874, 477653541236, 5421057631774, 54691360340668, 495683909613912, 4071934260847748, 30550513942600840, 210742030990838844, 1344509111827694136, 7975523935651794998, 44200428241582972568, 229866480486384151678, 1126325764438366293328.

$N = 93$.

$$C_{93} = A^{91}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6 \\ \beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{93} \approx 127.7580310364984490468914.$$

Coefficients: 1, 116, 6634, 249410, 6935290, 152164618, 2744526086, 41865876794, 551533141168, 6376643194660, 65538407670168, 605140496983808, 5064257387631240, 38705224592484068, 271952441875202272, 1766984546560665372, 10672686101695554516, 60212578096078846944, 318688831873647050862, 1588747853017511139488.

$N = 94$.

$$C_{94} = A^{92}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6 \\ \beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{94} \approx 138.8007723301221385384997.$$

Coefficients: 1, 118, 6866, 262678, 7434112, 166035430, 3048868590, 47355427786, 635278765336, 7480013806234, 78297183111892, 736301044091000, 6275641448380012, 48846492668006448, 349493968179839980, 2312099716794089704, 14216783793323901330, 81635361851721206680, 439657905702841551882, 2229659616934742411112.

$N = 95$.

$$C_{95} = A^{93}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6 \\ \beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{95} \approx 150.7979908827375923572496.$$

Coefficients: 1, 120, 7102, 276410, 7959470, 180903890, 3380953182, 53453690322, 730004489132, 8750903407768, 93263308461776, 892990121184088, 7749514094618040, 61412735607247164, 447343548214147520, 3012560261339688844, 18853534604519696336, 110166623694262553144, 603627632302671271402, 3113599784368380443424.

$N = 96.$

$$C_{96} = A^{94}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}^2\,T_{27}\,T_{32}.$$

$$C_{96} \approx 150.79799088273759235 72942.$$

Coefficients: 1, 120, 7102, 276410, 7959470, 180903890, 3380953182, 53453690322, 730004489132, 8750903407768, 93263308461776, 892990121184088, 7749514094618040, 61412735607247164, 447343548214147520, 3012560261339688844, 18853534604519696338, 110166623694262553384, 603627632302671285606, 3113599784368380996244.

$N = 97.$

$$C_{97} = A^{95}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}^2\,T_{27}\,T_{32}.$$

$$C_{97} \approx 163.83218675603315935 24354.$$

Coefficients: 1, 122, 7342, 290614, 8512292, 196823070, 3742775166, 60216149506, 836927788716, 10211274193814, 110771877183916, 1079623645502488, 9536954346517300, 76929265619217720, 570355546407373180, 3909033344772258424, 24894154262295690752, 147996519834525418892, 824855584289431517154, 4326880356023020421516.

$N = 98.$

$$C_{98} = A^{96}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7^2\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}^2\,T_{27}\,T_{32}.$$

$$C_{98} \approx 163.83218684824671045 41126.$$

Coefficients: 1, 122, 7342, 290614, 8512292, 196823070, 3742775166, 60216149508, 836927788960, 10211274208498, 110771877765144, 1079623662527072, 9536954740163440, 76929273104768052, 570355666839672192, 3909035018627835856, 24894174684844078380, 147996741378279786724, 824857743536722522130, 4326899429931713456116.

$N = 99.$

$$C_{99} = A^{97}\,\beta_2^{11}\,\beta_3^6\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7^2\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}^2\,T_{27}\,T_{32}.$$

$$C_{99} \approx 163.85862920512900737 3841.$$

Coefficients: 1, 122, 7342, 290616, 8512536, 196837754, 3743356394, 60233174092, 837321435100, 10218759758830, 110892310064160, 1081297518104992, 9557377288580438, 77150816860298584, 572514914164741020, 3928108928108743964, 25048033231070639068, 149137452712352777248, 832675813581463744174, 4376687779421833911892.

$N = 100.$

$$C_{100} = A^{98}\,\beta_2^{11}\,\beta_3^6\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7^2\,\beta_8^3\,\beta_9^2\,\beta_{10}\,\beta_{12}\,\beta_{16}^2\,T_{27}\,T_{32}.$$

$$C_{100} \approx 163.85862920513645016 39979.$$

Coefficients: 1, 122, 7342, 290616, 8512536, 196837754, 3743356394, 60233174092, 837321435100, 10218759758830, 110892310064162, 1081297518105236, 9557377288595122, 77150816860879816, 572514914181766092, 3928108928502419472, 25048033238557351856, 149137452832819125432, 832675815256106614374, 4376687799859353429552.

Relation to Simon Plouffe's paper

Simon Plouffe's paper *Numbers in the base* e^{π} studies high-precision evaluations of series at points of the form $e^{-m\pi}$ and then applies integer-relation detection (for instance, `lindep/algdep`) to guess exact formulas. The present family fits that framework naturally, because each value $\Theta_{G_N}(e^{-\pi})$ is already an explicit finite product of classical theta constants T_d. In that sense, the Plouffe-style numerical step is useful as a confirmation and discovery heuristic, but in the present setting the exact factorization follows directly from the proved triangular Gram decomposition.

25 Closing remarks

The Pratt exponents $m_p(n)$ provide a finite Euler-type expansion of every integer n, and hence also of the divisibility probability $1/n$. This gives a bridge between a recursive certificate-like structure and several other points of view: divisibility in finite residue rings, a Bernoulli sieve, covariance kernels, Dirichlet series, and sparse Hilbert-space embeddings.

The paper has deliberately kept a broad exploratory scope. Some sections are closer to structural facts, others to analytic reformulations, and others to possible feature-space interpretations. The common thread is that the Pratt forest supplies a recursively defined family of exponents, and those exponents can be used in many of the same places where one would ordinarily use prime valuations.

References

[1] Simon Plouffe, *Numbers in the base* e^{π}, 2025.