Exploring Pratt-Trees

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Abstract

We study the arithmetic obtained by counting labels in Pratt trees. For each integer $n \geq 1$ and each prime p, let $m_p(n)$ denote the number of vertices labelled p in the Pratt prime forest of n. These counts define a new coordinate system on the positive integers, additive under multiplication and rich enough to recover n through the finite Euler-type product

$$n = \prod_p \Big(1 - \frac{1}{p}\Big)^{-m_p(n)}.$$

From this starting point we develop the associated Pratt partial order, its meet operation, kernel and feature-map interpretations, Dirichlet-series identities, and several explicit Möbius inversions on the Pratt poset. A central theme is that polynomial and monomial values admit canonical decompositions over Pratt lower sets; in particular, the functions C_n arise as the Pratt–Möbius transforms of the monomials $x \mapsto x^n$, and the resulting operator extends naturally to arbitrary one-variable and multivariable polynomials.

The paper is exploratory but conceptually focused: it treats Pratt forests not merely as certificates for primality or as a bookkeeping device inside iterates of arithmetic functions, but as a standalone combinatorial-arithmetic object. In this direction it is worth noting that Erdős, Granville, Pomerance, and Spiro already used essentially the same Pratt-type decomposition in their study of iterates of Euler's function [1]; the present note isolates that structure and investigates it systematically in its own right.

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1 Introduction

Pratt trees are usually introduced as recursive certificates for primality: the tree attached to a prime p records the prime factors of $p-1$, then the prime factors of those predecessors, and so on until the process terminates at 2. In this note we shift emphasis from certification to counting. Given an integer n, we form its Pratt prime forest by taking one copy of the Pratt tree T_p for each prime factor p of n, with multiplicity $v_p(n)$. The basic arithmetic data are then the vertex counts

$$m_q(n) = \#\{\text{vertices labelled } q \text{ in the Pratt forest of } n\},$$

one count for each prime q.

These counts behave like a second system of valuations. They are completely additive under multiplication, they determine a natural partial order

$$a \le_P b \iff m_q(a) \le m_q(b) \quad \text{for all primes } q,$$

and they recover the integer itself through the Euler-type product

$$n = \prod_p \Big(1 - \frac{1}{p}\Big)^{-m_p(n)}.$$

This makes it possible to reinterpret familiar arithmetic objects in Pratt coordinates. The resulting picture mixes recursive combinatorics, multiplicative number theory, incidence algebras, and kernel methods.

Part of the underlying structure is already implicit in earlier work. In particular, Erdős, Granville, Pomerance, and Spiro, in their paper on iterates of arithmetic functions [1], worked with essentially the same recursive decomposition associated with Pratt trees. However, their focus was the behaviour of iterates of Euler's function and related maps, not the systematic study of the decomposition itself. One aim of the present paper is to isolate that Pratt-poset decomposition as a mathematical object in its own right and to show that it has a surprisingly broad range of interpretations.

The paper develops three interconnected themes.

First, we study the Pratt coordinates $(m_p(n))_p$ directly. This leads to the triangular relation between ordinary prime valuations and Pratt valuations, to the Pratt partial order and meet, and to a family of kernels and feature maps. In particular, the meet kernel admits an explicit Hilbert-space realization once the Möbius transform of the identity on the Pratt poset is identified.

Second, we examine explicit Möbius inversions on the Pratt poset. Besides the basic identity for n, we obtain canonical decompositions of n^2, n^3, and more generally n^k into sums over Pratt lower sets. This culminates in the Pratt–Möbius operator

$$\mathcal{C}(P) = \mu_P *_P (x \mapsto P(x)),$$

which sends a polynomial to an arithmetic function whose cumulative sum over $\downarrow_P x$ recovers $P(x)$. Thus monomials, cyclotomic polynomials, and multivariate polynomials all admit natural Pratt-poset expansions.

Third, we record several analytic and probabilistic reformulations of the same structure: Bernoulli-sieve models for the divisibility factors, Euler products and Dirichlet series attached to the coefficients C_n, and Gram-type decompositions of kernels built from the Pratt meet. The goal is not to force these viewpoints into a single theorem, but to show that the same recursive combinatorial object appears naturally in many different settings.

In this sense the paper is both concrete and programmatic. It proves a collection of explicit identities and structural facts, while also arguing that Pratt forests deserve to be treated as a standalone source of arithmetic constructions rather than only as an auxiliary device inside primality certificates or iterated totient arguments.

2 Pratt trees and Pratt prime forests

2.1 Pratt trees

Let p be prime. The *Pratt tree* T_p is a rooted tree with vertex labels in the primes defined recursively:

- For $p = 2$, T_2 is the single vertex labeled 2.
- For $p \geq 3$, the root is labeled p, and its children are the primes r dividing $p - 1$, with multiplicity $v_r(p-1)$; each child labeled r is the root of a copy of T_r.

Thus, the subtree multiset below the root of T_p is precisely the Pratt prime forest of $p - 1$ (defined below).

2.2 Pratt prime forests and vertex counts

Definition 1 (Pratt prime forest)**.** Let $n \geq 1$ with prime factorization $n = \prod_p p^{v_p(n)}$. The *Pratt prime forest* of n is the multiset

$$F(n) \;:=\; \biguplus_{p|n} v_p(n) \cdot T_p,$$

i.e. the disjoint union of $v_p(n)$ copies of T_p for each prime divisor p of n.

Definition 2 (Vertex counts)**.** For a prime q, let

$$m_q(n) \;:=\; \#\{\text{vertices labeled } q \text{ in } F(n)\},$$

counted with multiplicity.

Lemma 1 (Complete additivity)**.** *For every prime q and all $n, m \geq 1$,*

$$m_q(nm) = m_q(n) + m_q(m).$$

Equivalently,

$$m_q(n) = \sum_{p|n} v_p(n)\, m_q(p).$$

Proof. For each prime p, the multiplicity of T_p in $F(nm)$ is

$$v_p(nm) = v_p(n) + v_p(m).$$

Hence $F(nm)$ is the multiset sum of $F(n)$ and $F(m)$. Counting vertices labeled q therefore gives the first identity, and the second is just the same statement written after expanding the forest of n prime by prime. $\square$

Lemma 2 (Prime recursion)**.** *If $p \geq 3$ is prime, then for every prime q,*

$$m_q(p) = m_q(p-1) + \mathbf{1}_{\{q=p\}}.$$

Proof. In T_p, the root contributes one vertex labeled p. All other vertices lie in the children subtrees, which (with multiplicity) form the Pratt prime forest of $p-1$. Counting q-labels gives the recursion. $\square$

3 The Pratt product formula

Define the multiplicative functional

$$W(n) := \prod_{p \text{ prime}} \left(\frac{p}{p-1}\right)^{m_p(n)} = \prod_{p \text{ prime}} \left(1-\frac{1}{p}\right)^{-m_p(n)}. \tag{1}$$

The product is *finite* for each fixed n, because $F(n)$ has finitely many vertices.

Theorem 1 (Pratt product identity)**.** *For every $n \geq 1$,*

$$W(n) = n, \qquad i.e. \qquad n = \prod_{p \text{ prime}} \left(1-\frac{1}{p}\right)^{-m_p(n)}.$$

Equivalently,

$$\frac{1}{n} = \prod_{p \text{ prime}} \left(1-\frac{1}{p}\right)^{m_p(n)}.$$

Proof. First note from Lemma 1 that $m_p(nm) = m_p(n) + m_p(m)$ for each prime p, so W is completely multiplicative:

$$W(nm) = W(n)W(m).$$

It therefore suffices to prove $W(p) = p$ for primes p.

We argue by strong induction on the prime p. The base case $p = 2$ holds because $m_2(2) = 1$ and $m_q(2) = 0$ for $q \neq 2$, hence $W(2) = 2$.

For an odd prime $p \geq 3$, Lemma 2 gives $m_q(p) = m_q(p-1)$ for $q \neq p$, and $m_p(p) = m_p(p-1) + 1$. Plugging into (1) yields the recursion

$$W(p) = \left(\frac{p}{p-1}\right) W(p-1).$$

By induction on the integer $p-1$ (or by complete multiplicativity and the already established prime cases $\leq p-1$), we have $W(p-1) = p-1$, so $W(p) = p$. Finally, complete multiplicativity implies $W(n) = n$ for all n. $\square$

Remark 1 (Uniqueness (finite support)). For each fixed n, the family $(m_p(n))_p$ has finite support. With this finiteness, the representation in Theorem 1 is unique: if $\prod_p(1-\frac{1}{p})^{-a_p} = \prod_p(1-\frac{1}{p})^{-b_p}$ with only finitely many nonzero a_p, b_p, then $a_p = b_p$ for all primes p. Indeed, let P be the largest prime with $a_P \neq b_P$. The P-adic valuation of the left-hand side minus the right-hand side isolates the exponent at P, because every factor with larger prime index is absent and every smaller prime contributes no P in the denominator. Thus the exponents agree term by term.

4 Two probability interpretations of $1/n$

4.1 Divisibility as a probability

Fix $N \geq 1$ and put $L := \mathrm{lcm}(1, 2, \ldots, N)$. Consider the finite probability space

$$\Omega_N := \mathbb{Z}/L\mathbb{Z}, \qquad U \sim \text{uniform on } \Omega_N.$$

For each $1 \leq n \leq N$ define the indicator random variable

$$X_n := \mathbf{1}_{\{n|U\}}.$$

Lemma 3. *For $1 \leq n \leq N$,*

$$\mathbf{P}(X_n = 1) = \frac{1}{n}.$$

More generally, for $1 \leq a, b \leq N$,

$$\mathbf{P}(X_a = 1,\ X_b = 1) = \frac{1}{\mathrm{lcm}(a, b)}.$$

Proof. Because $n \mid L$, the subset $\{u \in \Omega_N : \ n \mid u\}$ is the subgroup $n\mathbb{Z}/L\mathbb{Z}$ of size L/n, hence has probability $(L/n)/L = 1/n$. The joint event $\{a \mid U\} \cap \{b \mid U\}$ is $\{\mathrm{lcm}(a, b) \mid U\}$, giving the second claim. □

Remark 2. Letting $N \to \infty$ recovers the classical natural-density statement: for a random integer M (for instance uniform on $\{1, \ldots, M_0\}$ with $M_0 \to \infty$), $\mathbf{P}(n \mid M) \to 1/n$. The finite model above is convenient because probabilities are exact.

4.2 A Bernoulli sieve indexed by the Pratt forest

The Pratt identity can also be read as a product of *survival probabilities*. Fix n and its Pratt forest $F(n)$. For each vertex $v \in F(n)$ with prime label $\ell(v) = p$, let B_v be an independent Bernoulli random variable with

$$\mathbf{P}(B_v = 1) = 1 - \frac{1}{p}.$$

Define the event that all tests are passed:

$$A_n := \bigcap_{v \in F(n)} \{B_v = 1\}.$$

Then independence gives

$$\mathbf{P}(A_n) = \prod_{v \in F(n)} \mathbf{P}(B_v = 1) = \prod_p (1 - 1/p)^{m_p(n)} = \frac{1}{n}$$

by Theorem 1.

Remark 3 (A "Pratt sieve" intuition). For a fixed prime p, the factor $(1 - 1/p)$ is the probability that a uniformly random residue mod p is *nonzero*. Thus, one may think of each vertex labeled p as imposing a random nonvanishing condition modulo p. The point is not that this model is identical to ordinary divisibility, but that the same product $1/n$ arises from a recursively organized family of independent local tests.

5 gcd, lcm, and covariance kernels

5.1 Rewriting $1/\operatorname{lcm}$ and $1/\gcd$ in Pratt form

For any positive integer t, Theorem 1 gives the tautological identities

$$\frac{1}{t} = \prod_p (1 - 1/p)^{m_p(t)}. \tag{2}$$

In particular,

$$\frac{1}{\operatorname{lcm}(a,b)} = \prod_p (1 - 1/p)^{m_p(\operatorname{lcm}(a,b))}, \qquad \frac{1}{\gcd(a,b)} = \prod_p (1 - 1/p)^{m_p(\gcd(a,b))}. \tag{3}$$

Remark 4 (A warning about max/min). One might hope for analogues of the valuation identities $v_p(\operatorname{lcm}) = \max(v_p(a), v_p(b))$ and $v_p(\gcd) = \min(v_p(a), v_p(b))$. However, the Pratt exponents $m_p(\cdot)$ are *not* valuations, and in general

$$m_p(\operatorname{lcm}(a,b)) \neq \max(m_p(a), m_p(b)), \qquad m_p(\gcd(a,b)) \neq \min(m_p(a), m_p(b)).$$

For instance, $a = 2$, $b = 3$ gives $\operatorname{lcm}(a,b) = 6$ and $\prod_p (1 - 1/p)^{\max(m_p(2), m_p(3))} = 1/3 \neq 1/6$. So the correct way to express $1/\operatorname{lcm}$ and $1/\gcd$ in this framework is (3).

5.2 Covariance as an expectation and a natural embedding

Recall that for any pair of random variables X, Y,

$$\operatorname{Cov}(X,Y) = \mathbb{E}\Big[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\Big] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

In our setting, $X_a := \mathbf{1}_{\{a|U\}}$ and $X_b := \mathbf{1}_{\{b|U\}}$ are indicator variables, so that

$$\mathbb{E}[X_a] = \mathbf{P}(a \mid U) = \frac{1}{a}, \qquad \mathbb{E}[X_a X_b] = \mathbf{P}(a \mid U,\, b \mid U) = \frac{1}{\operatorname{lcm}(a,b)}.$$

Hence

$$\operatorname{Cov}(X_a, X_b) = \frac{1}{\operatorname{lcm}(a,b)} - \frac{1}{ab}.$$

Natural Hilbert-space embedding. Let $H := L^2(\Omega_N, \mathbf{P})$ with inner product $\langle f, g\rangle = \mathbb{E}[fg]$. Define the centered embedding

$$\Phi(a) \ := \ X_a - \mathbb{E}[X_a] = X_a - \frac{1}{a}.$$

Then for all a, b,

$$\langle \Phi(a), \Phi(b)\rangle_H = \mathbb{E}\Big[(X_a - \mathbb{E}X_a)(X_b - \mathbb{E}X_b)\Big] = \mathrm{Cov}(X_a, X_b).$$

Thus the covariance matrix $G_N = (\mathrm{Cov}(X_a, X_b))_{1\le a,b\le N}$ is precisely the Gram matrix of the centered feature vectors $\Phi(1), \dots, \Phi(N)$ in $L^2(\Omega_N)$.

Consequently, for every $c \in \mathbb{R}^N$,

$$c^\top G_N c = \mathrm{Var}\Big(\sum_{a=1}^{N} c_a\, \Phi(a)\Big) \ge 0,$$

which restates the positive semidefiniteness of G_N in the most transparent Hilbert-space form.

Corollary 1 (Positive semidefiniteness)**.** *The matrix*

$$G_N := \Big(\mathrm{Cov}(X_a, X_b)\Big)_{1\le a,b\le N} = \left(\frac{1}{\mathrm{lcm}(a,b)} - \frac{1}{ab}\right)_{1\le a,b\le N}$$

is positive semidefinite.

Proof. G_N is a covariance matrix, hence positive semidefinite by definition: for any $c \in \mathbb{R}^N$, $c^\top G_N c = \mathrm{Var}\Big(\sum_{a=1}^{N} c_a X_a\Big) \ge 0$. □

6 Gram matrices and a rank formula

6.1 Centered features

Define centered random variables

$$Y_n := X_n - \mathbf{P}(X_n = 1) = X_n - \frac{1}{n} \qquad (1 \le n \le N).$$

Then

$$\langle Y_a, Y_b\rangle := \mathbb{E}[Y_a Y_b] = \mathrm{Cov}(X_a, X_b),$$

so G_N is the Gram matrix of the vectors/functions $Y_1, \dots, Y_N$ in $L^2(\Omega_N)$.

Note that $Y_1 \equiv 0$, hence the first row and column of G_N are identically 0. So $\mathrm{rank}(G_N) \le N - 1$. The next theorem shows this is sharp.

6.2 Linear independence of the centered indicators

Let $L = \mathrm{lcm}(1, \dots, N)$. For each divisor $d \mid L$, define the disjoint indicator

$$Z_d := \mathbf{1}_{\{\gcd(U,L)=d\}}.$$

The family $\{Z_d : d \mid L\}$ is linearly independent (it has disjoint support) and spans all functions that depend only on $\gcd(U, L)$. Moreover, for $n \mid L$,

$$X_n = \mathbf{1}_{\{n|U\}} = \mathbf{1}_{\{n|\gcd(U,L)\}} = \sum_{\substack{d|L \\ n|d}} Z_d. \tag{4}$$

Lemma 4. *The functions $X_1, \dots, X_N$ are linearly independent in $L^2(\Omega_N)$.*

Proof. Suppose $\sum_{n=1}^N c_n X_n \equiv 0$. Expand using (4) and collect coefficients of Z_d: the coefficient of Z_d equals $\sum_{n|d,\ 1\le n\le N} c_n$. Since the Z_d are linearly independent, every such coefficient must be 0.

In particular, for each $d \in \{1, 2, \dots, N\}$ (which is a divisor of L), we have

$$0 = \sum_{n|d} c_n.$$

Now solve recursively by increasing d: for $d = 1$ we get $c_1 = 0$; for $d = 2$ we get $c_1 + c_2 = 0$ hence $c_2 = 0$; for $d = 3$ we get $c_1 + c_3 = 0$ hence $c_3 = 0$; and in general, when d is fixed all proper divisors of d are $< d$ and already have zero coefficients, so the relation $\sum_{n|d} c_n = 0$ forces $c_d = 0$. Thus all $c_n = 0$, proving independence. □

Lemma 5. *The centered functions $Y_2, \dots, Y_N$ are linearly independent.*

Proof. Assume $\sum_{n=2}^N c_n Y_n \equiv 0$. Since $Y_n = X_n - \frac{1}{n} X_1$ (and $X_1 \equiv 1$), this implies

$$\sum_{n=2}^N c_n X_n \; - \; \left(\sum_{n=2}^N \frac{c_n}{n} \right) X_1 \equiv 0.$$

By Lemma 4, all coefficients in this linear combination of $X_1, \dots, X_N$ must vanish. In particular $c_n = 0$ for every $n = 2, \dots, N$. □

6.3 Rank of the covariance/Gram matrix

Theorem 2 (Rank formula)**.** *For every $N \ge 1$,*

$$\mathrm{rank}(G_N) = N - 1.$$

Equivalently, the only linear relation among $Y_1, \dots, Y_N$ is $Y_1 \equiv 0$.

Proof. We already observed $\mathrm{rank}(G_N) \le N - 1$ because $Y_1 \equiv 0$. On the other hand, G_N is the Gram matrix of $Y_1, \dots, Y_N$, so its rank equals $\dim \mathrm{span}\{Y_1, \dots, Y_N\}$. By Lemma 5, the set $\{Y_2, \dots, Y_N\}$ is linearly independent, hence spans a subspace of dimension $N-1$. Therefore $\operatorname{rank}(G_N) = N-1$. $\square$

7 A simple upper bound for the Pratt exponents

Recall the finite product representation

$$n \;=\; \prod_{p\ \text{prime}} \left(1-\frac{1}{p}\right)^{-m_p(n)} \;=\; \prod_{p\ \text{prime}} \left(\frac{p}{p-1}\right)^{m_p(n)}, \tag{5}$$

where $m_p(n) \in \mathbb{Z}_{\ge 0}$ and only finitely many $m_p(n)$ are nonzero.

Proposition 1 (Pointwise bound)**.** *For every integer $n \ge 2$ and every prime p,*

$$m_p(n) \;\le\; \frac{\log n}{\log\!\left(\frac{p}{p-1}\right)}.$$

Proof. Taking natural logarithms in (5) yields

$$\log n = \sum_{p\ \text{prime}} m_p(n)\, \log\!\left(\frac{p}{p-1}\right).$$

All summands are nonnegative because $m_p(n) \ge 0$ and $\log\!\left(\frac{p}{p-1}\right) > 0$. Hence for each fixed prime p we have

$$m_p(n)\, \log\!\left(\frac{p}{p-1}\right) \;\le\; \log n,$$

and division by $\log\!\left(\frac{p}{p-1}\right)$ gives the claim. $\square$

8 The valuation coordinates and the Pratt coordinates

Let $(p_k)_{k\ge 1} = (2,3,5,7,\dots)$ be the increasing sequence of primes. For $n \in \mathbb{N}$ define the valuation vector and the Pratt vector by

$$v(n) := (v_{p_k}(n))_{k\ge 1}, \qquad \phi(n) := (m_{p_k}(n))_{k\ge 1}.$$

Both vectors have finite support, so they belong to the space c_{00} of finitely supported real sequences.

Proposition 2 (Triangular change of basis)**.** *There is a linear operator $A : c_{00} \to c_{00}$ such that*

$$\phi(n) = A\, v(n) \qquad (n \in \mathbb{N}).$$

With respect to the ordered prime basis $(e_k)_{k\ge 1}$, the matrix of A is

$$A = (a_{ij})_{i,j\ge 1}, \qquad a_{ij} := m_{p_i}(p_j).$$

Moreover, A is lower triangular with diagonal entries $a_{jj} = 1$, hence invertible on c_{00}. In particular, the valuation coordinates $(v_p(n))_p$ and the Pratt coordinates $(m_p(n))_p$ determine one another by an invertible basis change.

Proof. By Lemma 1,

$$m_{p_i}(n) = \sum_{j\geq 1} v_{p_j}(n)\, m_{p_i}(p_j),$$

where the sum is finite because $v_{p_j}(n) = 0$ for all but finitely many j. This is exactly the coordinate formula for $\phi(n) = Av(n)$.

If $i > j$, then $p_i > p_j$, and the label p_i cannot occur in the Pratt tree T_{p_j}: every label in T_{p_j} is at most p_j. Thus $a_{ij} = m_{p_i}(p_j) = 0$ whenever $i > j$, so A is lower triangular. Also, the root of T_{p_j} contributes one vertex labeled p_j, hence $a_{jj} = m_{p_j}(p_j) = 1$. Therefore A is unit lower triangular. Any finite principal truncation of a unit lower triangular matrix is invertible, and since vectors in c_{00} have finite support, this gives an inverse on all of c_{00}. □

Remark 5. Proposition 2 makes precise the idea that the Pratt exponents refine ordinary prime valuations. The two coordinate systems carry the same information, but the transition matrix is not diagonal: the recursive ancestry encoded in the Pratt trees mixes the prime coordinates in a triangular way.

9 Extension to $\mathbb{Q}_{>0}$ and a linear readout of $\log n$

9.1 Pratt exponents for integers and the weight vector

For $n \in \mathbb{N}$ the vector

$$\phi(n) = \sum_{k\geq 1} m_{p_k}(n) e_k$$

is finitely supported, so in particular it belongs to ℓ^2. Recall the finite product identity

$$n \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}. \tag{6}$$

Introduce the weight vector

$$w = (w_k)_{k\geq 1}, \qquad w_k := -\log\!\left(1 - \frac{1}{p_k}\right) > 0.$$

Since $w_k \sim 1/p_k$ as $k \to \infty$ and $\sum_k p_k^{-2} < \infty$, we have $w \in \ell^2$. Taking logarithms in (6) gives the following.

Proposition 3 (Linear readout of the logarithm)**.** *For every $n \geq 1$,*

$$\log n = \sum_{k\geq 1} m_{p_k}(n)\, w_k = \langle \phi(n), w\rangle_{\ell^2}.$$

Equivalently,

$$n = \exp\Big(\langle \phi(n), w\rangle\Big).$$

Proof. Because $\phi(n)$ has finite support, the inner product with w is just a finite sum. Taking logarithms in (6) yields

$$\log n = \sum_{p \text{ prime}} m_p(n)\Big(-\log\Big(1 - \frac{1}{p}\Big)\Big) = \sum_{k\geq 1} m_{p_k}(n) w_k.$$

This is exactly $\langle \phi(n), w\rangle$. □

Remark 6. This Hilbert-space point of view is close in spirit to the embedding discussed in the companion paper on the first 100,000 numbers, where the feature vector $\phi(n)$ is used as a sparse prime-indexed coordinate system and the logarithm becomes a fixed linear observable on that space.[1]

9.2 Canonical extension to positive rationals

Let $\mathbb{Q}_{>0}$ be the multiplicative group of positive rationals. Every $q \in \mathbb{Q}_{>0}$ can be written uniquely as $q = a/b$ with $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$.

Definition 3 (Pratt exponent of a rational)**.** For $q = a/b \in \mathbb{Q}_{>0}$ in lowest terms, define for each prime p

$$m_p(q) \;:=\; m_p(a) - m_p(b) \in \mathbb{Z}, \tag{7}$$

and define the feature vector

$$\phi(q) := (m_p(q))_p \in c_{00}(\mathbb{P}).$$

Lemma 6 (Additivity on $\mathbb{Q}_{>0}$)**.** *For all $q_1, q_2 \in \mathbb{Q}_{>0}$ and all primes p one has*

$$m_p(q_1 q_2) = m_p(q_1) + m_p(q_2), \qquad \textit{hence} \qquad \phi(q_1 q_2) = \phi(q_1) + \phi(q_2).$$

Proof. This follows immediately from the definition (7) and the complete additivity of $m_p(\cdot)$ on $\mathbb{N}$. □

Proposition 4 (Pratt product and linear readout on $\mathbb{Q}_{>0}$)**.** *For every $q \in \mathbb{Q}_{>0}$ one has the finite product identity*

$$q \;=\; \prod_{p \text{ prime}} \Big(1 - \frac{1}{p}\Big)^{-m_p(q)}, \tag{8}$$

and the logarithmic identity

$$\log q \;=\; \sum_{p} m_p(q)\,\Big(-\log\Big(1 - \frac{1}{p}\Big)\Big). \tag{9}$$

[1] See the discussion around the feature map and the Hilbert-space readout in the linked paper by the same author.

Proof. Write $q = a/b$ in lowest terms. Apply (6) to a and b and divide:

$$\frac{a}{b} = \prod_p \Big(1 - \frac{1}{p}\Big)^{-m_p(a)} \cdot \prod_p \Big(1 - \frac{1}{p}\Big)^{m_p(b)} = \prod_p \Big(1 - \frac{1}{p}\Big)^{-(m_p(a)-m_p(b))}.$$

This is (8). Taking logarithms yields (9). □

9.3 Continued fraction convergents and a one-functional limit

Fix a real number $r > 0$ and let

$$r = [a_0; a_1, a_2, \dots]$$

be its continued fraction expansion. Denote by $r_k = p_k/q_k \in \mathbb{Q}_{>0}$ the k-th convergent. It is classical that $r_k \to r$ as $k \to \infty$.

Theorem 3 (Convergents yield a sequence with prescribed log-limit)**.** *Let $r > 0$ and $(r_k)_{k\geq 0}$ be its continued fraction convergents. Then the associated feature vectors $\phi(r_k) \in c_{00}$ satisfy*

$$\lim_{k\to\infty} \langle w, \phi(r_k)\rangle = \lim_{k\to\infty} \log(r_k) = \log(r). \tag{10}$$

Proof. By Proposition 4, $\langle w, \phi(r_k)\rangle = \log(r_k)$ holds for every k. Since $r_k \to r$ and $\log(\cdot)$ is continuous on $(0, \infty)$, we have $\log(r_k) \to \log(r)$. This proves (10). □

Remark 7. The identity $\langle w, \phi(r_k)\rangle \to \log(r)$ is not by itself a statement of weak convergence in ℓ^2; it only records convergence against one fixed test vector w. What it shows is that every positive real number can be approximated by rationals whose Pratt feature vectors reproduce the correct logarithmic limit through the same linear functional.

10 A coefficient recursion from differentiating a Dirichlet series

Fix $n \in \mathbb{N}$ and write $m_p := m_p(n)$ for the Pratt exponents. Consider the Dirichlet series

$$D_n(s) \ := \ \prod_p \Big(1 - p^{-s}\Big)^{-m_p} \ = \ \sum_{r\geq 1} \frac{\chi_n(r)}{r^s}, \qquad \Re(s) > 1, \tag{11}$$

where the coefficients $\chi_n(r)$ are given multiplicatively by the Euler factors

$$(1 - p^{-s})^{-m_p} = \sum_{e\geq 0} \binom{m_p + e - 1}{e} p^{-es}.$$

Equivalently,

$$\chi_n(r) = \prod_{p\mid r} \binom{m_p(n) + v_p(r) - 1}{v_p(r)}. \tag{12}$$

Proposition 5 (Derivative identity and coefficient comparison)**.** *For every $r \geq 2$ we have the recursion*

$$\chi_n(r) \;=\; \frac{1}{\log r} \sum_{p \mid r} \sum_{k=1}^{v_p(r)} m_p(n)\, \log(p)\, \chi_n\!\left(\frac{r}{p^k}\right). \tag{13}$$

(And $\chi_n(1) = 1$ by (11).)

Proof. Differentiate $D_n(s)$ in two ways.

(1) Termwise differentiation of the Dirichlet series. From (11) we get

$$D_n'(s) = \frac{d}{ds} \sum_{r \geq 1} \chi_n(r)\, r^{-s} = \sum_{r \geq 1} \chi_n(r)\, \frac{d}{ds}(r^{-s}) = -\sum_{r \geq 1} \chi_n(r)\, \log(r)\, r^{-s}.$$

Hence

$$-\,D_n'(s) = \sum_{r \geq 1} \chi_n(r)\, \log(r)\, r^{-s}. \tag{14}$$

(2) Logarithmic differentiation of the Euler product. We compute

$$\log D_n(s) = -\sum_p m_p \log(1 - p^{-s}),$$

so

$$\frac{D_n'(s)}{D_n(s)} = -\sum_p m_p \cdot \frac{d}{ds} \log(1 - p^{-s}) = -\sum_p m_p \cdot \frac{\log(p)\, p^{-s}}{1 - p^{-s}}.$$

Using the geometric series expansion

$$\frac{p^{-s}}{1 - p^{-s}} = \sum_{k \geq 1} p^{-ks},$$

we obtain

$$-\,D_n'(s) = D_n(s) \sum_p m_p\, \log(p) \sum_{k \geq 1} p^{-ks}. \tag{15}$$

(3) Coefficient comparison. Insert $D_n(s) = \sum_{t \geq 1} \chi_n(t) t^{-s}$ into (15):

$$-\,D_n'(s) = \left(\sum_{t \geq 1} \frac{\chi_n(t)}{t^s} \right) \left(\sum_p \sum_{k \geq 1} \frac{m_p \log(p)}{p^{ks}} \right).$$

The coefficient of r^{-s} on the right-hand side equals

$$\sum_{p^k \mid r} m_p \log(p)\, \chi_n\!\left(\frac{r}{p^k}\right) = \sum_{p \mid r} \sum_{k=1}^{v_p(r)} m_p \log(p)\, \chi_n\!\left(\frac{r}{p^k}\right).$$

Comparing with (14), we get for every $r \geq 1$ the identity

$$\chi_n(r)\log(r) = \sum_{p|r}\sum_{k=1}^{v_p(r)} m_p \log(p)\,\chi_n\left(\frac{r}{p^k}\right).$$

For $r \geq 2$ we can divide by $\log r \neq 0$ and obtain (13). □

11 Bounds for the coefficients $\chi_n(r)$

Fix $n \in \mathbb{N}$. For $r \geq 1$ write $v_p(r)$ for the p-adic valuation and $m_p(n) \in \mathbb{Z}_{\geq 0}$ for the Pratt exponents of n. We consider the multiplicative function

$$\chi_n(r) \;:=\; \prod_{p|r}\binom{m_p(n)+v_p(r)-1}{v_p(r)}.$$

If $m_p(n) = 0$ and $v_p(r) \geq 1$, then the local binomial coefficient is 0, hence $\chi_n(r) = 0$. So effectively χ_n is supported on integers whose prime factors lie in the finite set $\{p : \; m_p(n) > 0\}$.

Local bounds from binomial inequalities. Let $m \geq 1$ and $v \geq 1$ and set $N = m+v-1$ and $K = v$. A standard inequality valid for all $1 \leq K \leq N$ is

$$\frac{N^K}{K^K} \;\leq\; \binom{N}{K} \;\leq\; \frac{N^K}{K!} \;<\; \left(\frac{eN}{K}\right)^K.$$

Applying this with $N = m+v-1$ and $K = v$ yields, for $m \geq 1$ and $v \geq 1$,

$$\left(\frac{m+v-1}{v}\right)^v \;\leq\; \binom{m+v-1}{v} \;\leq\; \left(\frac{e(m+v-1)}{v}\right)^v. \tag{16}$$

Global bounds for $\chi_n(r)$. Write $r = \prod_p p^{v_p(r)}$. For every $r \geq 1$ we obtain from (16):

$$\chi_n(r) = \prod_{p|r}\binom{m_p(n)+v_p(r)-1}{v_p(r)} \tag{17}$$

$$\leq \prod_{p|r}\left(\frac{e\,(m_p(n)+v_p(r)-1)}{v_p(r)}\right)^{v_p(r)}. \tag{18}$$

Equivalently,

$$\log\chi_n(r) \;\leq\; \sum_{p|r} v_p(r)\Big(1+\log(m_p(n)+v_p(r)-1)-\log v_p(r)\Big).$$

A polynomial-in-v bound when m is fixed. For fixed $m \geq 1$ the binomial coefficient is actually polynomial in v:

$$\binom{m+v-1}{v} = \binom{m+v-1}{m-1} = \frac{1}{(m-1)!}\prod_{j=1}^{m-1}(v+j).$$

Hence for all $v \geq 0$,

$$\binom{m+v-1}{v} \;\leq\; \frac{(v+m-1)^{m-1}}{(m-1)!}. \tag{19}$$

Consequently, letting $S(n) := \{p: \ m_p(n) > 0\}$ (a finite set), we get for all $r \geq 1$:

$$\chi_n(r) \;\leq\; \prod_{p\in S(n)} \frac{\big(v_p(r)+m_p(n)-1\big)^{m_p(n)-1}}{(m_p(n)-1)!}. \tag{20}$$

Since $v_p(r) \leq \log r/\log p$, this shows that for fixed n the growth of $\chi_n(r)$ is at most polylogarithmic in r (with constants depending on n).

Remark. The bounds (18) and (20) are useful because they make the coefficients in later Dirichlet-series constructions completely explicit.

12 Integrating $D_n(s)$ in two ways

Fix $n \in \mathbb{N}$. Recall the Dirichlet series

$$D_n(s) \;:=\; \sum_{r\geq 1}\frac{\chi_n(r)}{r^s} \qquad (s \in \mathbb{C}),$$

where χ_n is defined multiplicatively by the local rule

$$\chi_n(p^k) \;=\; \binom{m_p(n)+k-1}{k} \qquad (k \geq 0),$$

and $m_p(n)$ is the Pratt exponent of p in the Pratt forest of n. Then D_n admits the finite Euler product

$$D_n(s) = \prod_{p\ \text{prime}}\left(\sum_{k\geq 0}\binom{m_p(n)+k-1}{k}p^{-ks}\right) = \prod_p\big(1-p^{-s}\big)^{-m_p(n)}. \tag{21}$$

In particular, $\lim_{\sigma\to+\infty} D_n(\sigma) = 1$.

A generalized von Mangoldt function. Define an arithmetic function $\Lambda_n : \mathbb{N} \to \mathbb{R}$ by

$$\Lambda_n(r) \ := \ \begin{cases} m_p(n)\log p, & r = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Proposition 6 (Logarithmic derivative of D_n)**.** *For every s with $\Re(s) > 0$ one has*

$$-\frac{D_n'(s)}{D_n(s)} \ = \ \sum_{r\geq 2} \frac{\Lambda_n(r)}{r^s}. \tag{23}$$

Proof. Differentiate the Euler product (21) logarithmically:

$$\frac{D_n'(s)}{D_n(s)} = \sum_p \frac{d}{ds}\Big(-m_p(n)\log(1-p^{-s})\Big) = -\sum_p m_p(n)\frac{(\log p)\,p^{-s}}{1-p^{-s}}.$$

Expanding $\frac{p^{-s}}{1-p^{-s}} = \sum_{k\geq 1} p^{-ks}$ yields

$$-\frac{D_n'(s)}{D_n(s)} = \sum_p \sum_{k\geq 1} m_p(n)(\log p)\,p^{-ks} = \sum_{r\geq 2} \frac{\Lambda_n(r)}{r^s},$$

which is exactly (23). □

First integration: integrating the Dirichlet series for $-D'/D$. Fix $\sigma > 0$ real. Since $D_n(\sigma) \to 1$ as $\sigma \to +\infty$, we may write

$$\log D_n(\sigma) = -\int_\sigma^\infty \frac{D_n'(u)}{D_n(u)}\,du = \int_\sigma^\infty \left(-\frac{D_n'(u)}{D_n(u)}\right) du.$$

Insert (23) and integrate termwise:

$$\log D_n(\sigma) = \sum_{r\geq 2} \Lambda_n(r) \int_\sigma^\infty r^{-u}\,du = \sum_{r\geq 2} \frac{\Lambda_n(r)}{\log r}\, r^{-\sigma}. \tag{24}$$

Second integration: integrating the Euler product directly. Starting from (21), take logarithms and expand $\log(1-x) = -\sum_{k\geq 1} x^k/k$:

$$\log D_n(\sigma) = -\sum_p m_p(n)\log(1-p^{-\sigma}) = \sum_p m_p(n) \sum_{k\geq 1} \frac{1}{k}\,p^{-k\sigma}. \tag{25}$$

Comparing (24) with (25) gives the consistency check: for $r = p^k$ one has

$$\frac{\Lambda_n(p^k)}{\log(p^k)} = \frac{m_p(n)\log p}{k\log p} = \frac{m_p(n)}{k},$$

and for non-prime-powers both sides contribute 0.

Corollary 2 (A Dirichlet-integral identity for D_n)**.** *For every $\sigma > 0$,*

$$\log D_n(\sigma) = \sum_{r\geq 2} \frac{\Lambda_n(r)}{\log r}\, r^{-\sigma} = \sum_p \sum_{k\geq 1} \frac{m_p(n)}{k}\, p^{-k\sigma}.$$

In particular, at $\sigma = 1$ one obtains the convergent identity

$$\log D_n(1) = \sum_p \sum_{k\geq 1} \frac{m_p(n)}{k\, p^k}.$$

Remark 8 (Recovering n at $s = 1$). Since $D_n(1) = \prod_p (1 - \frac{1}{p})^{-m_p(n)}$, we get

$$D_n(1) = n \qquad \text{and hence} \qquad n = \exp\left(\sum_p \sum_{k\geq 1} \frac{m_p(n)}{k\, p^k}\right).$$

This is the same finite Pratt product viewed through a Dirichlet-series lens.

13 Zeros and poles of the Dirichlet series $D_q(s)$ (integers and rationals)

13.1 From Pratt exponents to an Euler product

Let $m_p(n) \in \mathbb{Z}_{\geq 0}$ denote the Pratt-forest vertex count of p in the Pratt prime forest of $n \in \mathbb{N}$. We recall the finite-support identity

$$n \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}. \tag{26}$$

It is convenient to extend the exponents from $\mathbb{N}$ to $\mathbb{Q}_{>0}$ by allowing negative exponents: if $q = \frac{a}{b}$ with $a, b \in \mathbb{N}$, define

$$m_p(q) \;:=\; m_p(a) - m_p(b) \in \mathbb{Z}. \tag{27}$$

Then $m_p(q) = 0$ for all but finitely many primes p, and

$$q \;=\; \prod_p \left(1 - \frac{1}{p}\right)^{-m_p(q)}. \tag{28}$$

13.2 A Dirichlet series attached to q

Fix $q \in \mathbb{Q}_{>0}$ and write $m_p := m_p(q)$. Define a Dirichlet series for $\Re(s) > 1$ by the Euler product

$$D_q(s) \;:=\; \prod_p \left(1 - p^{-s}\right)^{-m_p}. \tag{29}$$

Because only finitely many $m_p \neq 0$, the product is finite. Therefore $D_q(s)$ extends meromorphically to all of $\mathbb{C}$ as a finite product of elementary factors. Expanding the Euler product gives a Dirichlet series

$$D_q(s) \;=\; \sum_{r\geq 1} \frac{\chi_q(r)}{r^s}, \qquad \Re(s) > 1, \tag{30}$$

where χ_q is multiplicative and determined on prime powers by

$$\chi_q(p^k) = \begin{cases} \binom{m_p+k-1}{k}, & m_p \geq 1, \\ (-1)^k \binom{-m_p}{k}, & m_p \leq 0, \end{cases} \qquad (k \geq 0). \tag{31}$$

Indeed, for an integer $m \geq 1$ one has the generating function $\sum_{k\geq 0} \binom{m+k-1}{k} t^k = (1-t)^{-m}$, while for $m = -r \leq 0$ one has $(1-t)^r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} t^k$.

Remark 9. For $q \in \mathbb{N}$ all $m_p \geq 0$, so the local factors are of the form $(1-p^{-s})^{-m_p}$ and $D_q(s)$ has no zeros. For general rationals q, some m_p may be negative, and then $(1-p^{-s})^{-m_p} = (1-p^{-s})^{|m_p|}$ introduces zeros.

13.3 The divisor of $D_q(s)$

For each prime p define the p-lattice

$$\Lambda_p \;:=\; \left\{ \frac{2\pi i k}{\log p} \;:\; k \in \mathbb{Z} \right\}.$$

These are exactly the solutions to $p^{-s} = 1$, i.e. the zeros of $1 - p^{-s}$.

Theorem 4 (Zeros and poles of $D_q(s)$)**.** *Let $q \in \mathbb{Q}_{>0}$ and $D_q(s)$ be defined by* (29)*. Then:*

1. *For each prime p with $m_p(q) > 0$, the function $D_q(s)$ has poles of order $m_p(q)$ at every point of Λ_p.*

2. *For each prime p with $m_p(q) < 0$, the function $D_q(s)$ has zeros of order $-m_p(q)$ at every point of Λ_p.*

3. *There are no other zeros or poles.*

Moreover, if $p \neq \ell$ are distinct primes, then $\Lambda_p \cap \Lambda_\ell = \{0\}$. Hence the only point at which different prime lattices can overlap is $s = 0$.

Proof. Write $m_p = m_p(q)$. Since $D_q(s) = \prod_p (1-p^{-s})^{-m_p}$ is a finite product, it suffices to analyze a single factor. The function $1 - p^{-s}$ vanishes exactly when $p^{-s} = 1$, i.e. when $e^{-s\log p} = 1$, which is equivalent to $s \in \Lambda_p$.

Near any $s_0 \in \Lambda_p$, one has $1 - p^{-s} = (s - s_0)\log p + O((s-s_0)^2)$, so $(1-p^{-s})^{-m_p}$ has a pole of order m_p if $m_p > 0$ and a zero of order $-m_p$ if $m_p < 0$. Multiplying the finitely many factors gives (1)–(3).

Finally, if $s \in \Lambda_p \cap \Lambda_\ell$ with $p \neq \ell$, then $\frac{2\pi i k}{\log p} = \frac{2\pi i m}{\log \ell}$ for some integers k, m. If $k = m = 0$ this is $s = 0$. Otherwise we get $k \log \ell = m \log p$, hence $\log(\ell^k) = \log(p^m)$ and $\ell^k = p^m$, impossible for distinct primes. □

Corollary 3 (Integer case: no zeros). *If $n \in \mathbb{N}_{\geq 2}$, then $D_n(s)$ has no zeros and has poles exactly at*

$$\bigcup_{p:\, m_p(n)>0} \Lambda_p,$$

where the pole order along Λ_p equals $m_p(n)$.

13.4 Logarithmic derivative

From (29) we get, for all s away from the divisor,

$$\frac{D_q'(s)}{D_q(s)} = \sum_p m_p \, (\log p) \, \frac{p^{-s}}{1 - p^{-s}}. \tag{32}$$

This makes the singularities transparent: each summand has simple poles at Λ_p, and the coefficient m_p records the multiplicity with sign.

13.5 A remark about the value at $s = 1$

On the half-plane $\Re(s) > 1$, the Dirichlet series (30) converges and equals the Euler product (29). Since the Euler product is finite, there is in fact no analytic difficulty at $s = 1$:

$$D_q(1) = \prod_p (1 - p^{-1})^{-m_p(q)} = q,$$

which is exactly the rational Pratt product identity (28). Thus the point $s = 1$ is not a delicate boundary phenomenon here; it is simply the place where the finite product recovers the original rational number.

14 A Pratt-zeta function built from Pratt exponents

This section records a two-variable Dirichlet series whose Euler product interpolates between two classical zeta functions.

14.1 A two-variable Pratt-Dirichlet series

Fix a complex parameter u and define

$$A_u(n) \;:=\; \prod_{p \text{ prime}} \Big(1 - p^{-u}\Big)^{m_p(n)}. \tag{33}$$

This product is finite for each fixed n, because $m_p(n) = 0$ for all but finitely many primes p.

Multiplicativity. Since each $m_p(\cdot)$ is completely additive in n,

$$m_p(nm) = m_p(n) + m_p(m) \qquad (n, m \in \mathbb{N}),$$

it follows that A_u is completely multiplicative:

$$A_u(nm) = A_u(n)\, A_u(m) \qquad (n, m \in \mathbb{N}). \tag{34}$$

Definition 4 (Pratt double zeta function)**.** For complex parameters (u, s) with $\Re(s)$ sufficiently large, define

$$\mathcal{Z}(u, s) \ := \ \sum_{n \geq 1} \frac{A_u(n)}{n^s}. \tag{35}$$

14.2 Euler product in the s-variable

Because A_u is completely multiplicative, $\mathcal{Z}(u, s)$ admits an Euler product:

$$\mathcal{Z}(u, s) \ = \ \prod_{p \text{ prime}} \left(\sum_{k \geq 0} \frac{A_u(p^k)}{p^{ks}} \right) \ = \ \prod_{p \text{ prime}} \frac{1}{1 - A_u(p)\, p^{-s}}, \tag{36}$$

where we used $A_u(p^k) = A_u(p)^k$, a direct consequence of $m_q(p^k) = k\, m_q(p)$ for every prime q.

Moreover, by the prime-step recursion for Pratt trees,

$$m_q(p) = m_q(p-1) + \mathbf{1}_{\{q=p\}},$$

one gets a convenient recursion for the local coefficient:

$$A_u(p) \ = \ (1 - p^{-u})\, A_u(p-1) \qquad (p \geq 3 \text{ prime}), \tag{37}$$

with the base $A_u(2) = 1 - 2^{-u}$.

14.3 Two key specializations

Limit $u \to +\infty$. As $\Re(u) \to +\infty$, one has $(1-p^{-u}) \to 1$ for every prime p, hence $A_u(n) \to 1$ for each fixed n. Formally, and whenever $\Re(s)$ is large enough to justify interchanging limit and summation, this gives

$$\mathcal{Z}(u, s) \longrightarrow \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s). \tag{38}$$

Value $u = 1$. At $u = 1$, we have

$$\frac{1}{n} = \prod_p \Big(1 - \frac{1}{p}\Big)^{m_p(n)} = A_1(n).$$

Therefore

$$\mathcal{Z}(1,s) \;=\; \sum_{n\geq 1}\frac{A_1(n)}{n^s} \;=\; \sum_{n\geq 1}\frac{1}{n^{s+1}} \;=\; \zeta(s+1). \tag{39}$$

15 A conditional distribution from the Pratt forest

Fix $n \geq 1$ and recall the multiplicative coefficients $\chi_n(r)$ defined by

$$\chi_n(p^k) = \binom{m_p(n)+k-1}{k} \qquad (k \geq 0),$$

equivalently

$$\chi_n(r) = \prod_{p\mid r}\binom{m_p(n)+v_p(r)-1}{v_p(r)}.$$

Define

$$\mu_n(r) \;:=\; \frac{\chi_n(r)}{n\,r} \qquad (r \geq 1). \tag{40}$$

Then μ_n is a probability distribution on $\mathbb{N}$. Indeed, by the Dirichlet-Euler identity

$$D_n(s) = \sum_{r\geq 1}\frac{\chi_n(r)}{r^s} = \prod_p (1-p^{-s})^{-m_p(n)},$$

we have

$$\sum_{r\geq 1}\frac{\chi_n(r)}{r} = D_n(1) = \prod_p (1-1/p)^{-m_p(n)} = n,$$

hence $\sum_{r\geq 1}\mu_n(r) = 1$.

15.1 Interpretation as a conditional law

There is a natural primewise independent stochastic construction that realizes μ_n as a conditional distribution.

For each prime p, let K_p be a $\mathbb{Z}_{\geq 0}$-valued random variable, and assume that the family $(K_p)_p$ is independent with

$$\mathbf{P}(K_p = k \mid n) \;=\; \binom{m_p(n)+k-1}{k}\Big(1-\frac{1}{p}\Big)^{m_p(n)}\Big(\frac{1}{p}\Big)^k, \qquad k \geq 0. \tag{41}$$

For fixed p, this is a negative-binomial law: the number of failures before $m_p(n)$ successes in Bernoulli trials with success probability $1-\frac{1}{p}$.

Now define the random integer

$$R \;:=\; \prod_p p^{K_p},$$

which is almost surely finite because $m_p(n) = 0$ for all but finitely many p (and then $K_p \equiv 0$

for such primes by (41)). By independence and the multiplicative definition of χ_n, for every $r \ge 1$ we obtain

$$
\begin{aligned}
\mathbf{P}(R = r \mid n) &= \prod_p \mathbf{P}(K_p = v_p(r) \mid n) \\
&= \prod_p \binom{m_p(n) + v_p(r) - 1}{v_p(r)} \Big(1 - \frac{1}{p}\Big)^{m_p(n)} \Big(\frac{1}{p}\Big)^{v_p(r)} \\
&= \bigg(\prod_p \Big(1 - \frac{1}{p}\Big)^{m_p(n)}\bigg) \frac{\chi_n(r)}{r} \;=\; \frac{\chi_n(r)}{n\,r} \;=\; \mu_n(r),
\end{aligned}
$$

where in the last step we used the Pratt product identity. Hence μ_n is precisely the conditional law of R given n.

16 Bhattacharyya coefficient and Hellinger affinity

Let μ, ν be probability distributions on a countable set Ω. The *Bhattacharyya coefficient* (also called the Hellinger affinity) is

$$
\mathrm{BC}(\mu, \nu) \;:=\; \sum_{x \in \Omega} \sqrt{\mu(x)\nu(x)} \in [0, 1]. \tag{42}
$$

Equivalently, writing $\sqrt{\mu}, \sqrt{\nu} \in \ell^2(\Omega)$,

$$
\mathrm{BC}(\mu, \nu) = \langle \sqrt{\mu}, \sqrt{\nu} \rangle_{\ell^2(\Omega)}.
$$

In particular, $\mathrm{BC}(\mu, \nu) = 1$ iff $\mu = \nu$, and $\mathrm{BC}(\mu, \nu) = 0$ iff μ and ν have disjoint supports.

16.1 The Bhattacharyya coefficient for Pratt measures

For $a, b \in \mathbb{N}$ define μ_a, μ_b by (40). Then

$$
\mathrm{BC}(\mu_a, \mu_b) = \sum_{r \ge 1} \sqrt{\mu_a(r)\mu_b(r)} = \sum_{r \ge 1} \frac{\sqrt{\chi_a(r)\chi_b(r)}}{r\sqrt{ab}}. \tag{43}
$$

If we introduce the vectors

$$
\phi_\chi(n) := \sum_{r \ge 1} \sqrt{\frac{\chi_n(r)}{r}}\, e_r \qquad \text{and} \qquad \psi(n) := \frac{\phi_\chi(n)}{\sqrt{n}},
$$

then $\|\phi_\chi(n)\|^2 = \sum_{r \ge 1} \chi_n(r)/r = n$, so $\|\psi(n)\| = 1$, and

$$
\langle \psi(a), \psi(b) \rangle = \sum_{r \ge 1} \sqrt{\mu_a(r)\mu_b(r)} = \mathrm{BC}(\mu_a, \mu_b).
$$

Thus the normalized inner product of these feature vectors is exactly the Bhattacharyya coefficient of the associated Pratt measures.

17 Representation of rational numbers via divisor convolution

For $n \geq 1$ define

$$\psi_n(r) \;:=\; \prod_p (-1)^{v_p(r)} \binom{m_p(n)}{v_p(r)}.$$

Only finitely many factors differ from 1, and if $v_p(r) > m_p(n)$ then the corresponding binomial coefficient is 0. Therefore ψ_n has finite support. The Euler product expansion of $(1-1/p)^{m_p(n)}$ gives

$$\frac{1}{n} \;= \prod_p \Big(1 - \frac{1}{p}\Big)^{m_p(n)} \;= \sum_{r=1}^{\infty} \frac{\psi_n(r)}{r}.$$

Recall also that

$$n \;= \prod_p \Big(1 - \frac{1}{p}\Big)^{-m_p(n)} \;= \sum_{r=1}^{\infty} \frac{\chi_n(r)}{r}.$$

It follows that

$$\frac{m}{n} \;= m \cdot \frac{1}{n} \;= \left(\sum_{r=1}^{\infty} \frac{\chi_m(r)}{r}\right) \left(\sum_{s=1}^{\infty} \frac{\psi_n(s)}{s}\right) \;= \sum_{t=1}^{\infty} \frac{(\chi_m * \psi_n)(t)}{t},$$

where

$$(\chi_m * \psi_n)(t) \;:=\; \sum_{d \mid t} \chi_m(d)\, \psi_n\Big(\frac{t}{d}\Big)$$

is the divisor convolution. Thus, for any rational number $x = a/b \in \mathbb{Q}$ with $\gcd(a,b) = 1$, we obtain

$$x = \frac{a}{b} \;= \sum_{r=1}^{\infty} \frac{\operatorname{sign}(x)\,(\chi_a * \psi_b)(r)}{r}$$

when one also allows the overall sign of a rational number. For positive rationals the sign factor is of course 1.

18 Table data

This section records the ratios of consecutive primes

$$\frac{p_r}{p_{r+1}},$$

together with the corresponding infinite series

$$\frac{p_r}{p_{r+1}} = \sum_{t\ge 1} \frac{(\chi_{p_r} * \psi_{p_{r+1}})(t)}{t}.$$

Here the arithmetic functions χ_n and ψ_n are defined from the Pratt exponents by

$$\chi_n(p^k) = \binom{m_p(n)+k-1}{k}, \qquad \psi_n(p^k) = (-1)^k \binom{m_p(n)}{k},$$

and are extended multiplicatively. The convolution is the divisor convolution

$$(\chi_a * \psi_b)(t) = \sum_{d\mid t} \chi_a(d)\psi_b\left(\frac{t}{d}\right).$$

In the table below, the final column lists the first 12 nonzero terms of the series.

r	p_r	p_{r+1}	$\frac{p_r}{p_{r+1}}$	First nonzero terms of $\sum_{t\ge 1} \frac{(\chi_{p_r} * \psi_{p_{r+1}})(t)}{t}$
1	2	3	$\frac{2}{3}$	$1-\frac{1}{3}$
2	3	5	$\frac{3}{5}$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{5}-\frac{1}{6}+\frac{1}{9}+\frac{1}{10}-\frac{1}{15}-\frac{1}{18}+\frac{1}{27}+\frac{1}{30}-\frac{1}{45}+\cdots$
3	5	7	$\frac{5}{7}$	$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{15}+\frac{1}{21}+\frac{1}{25}-\frac{1}{35}-\frac{1}{75}+\frac{1}{105}+\frac{1}{125}-\frac{1}{175}+\cdots$
4	7	11	$\frac{7}{11}$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\frac{1}{14}-\frac{1}{15}-\frac{1}{18}+\cdots$
5	11	13	$\frac{11}{13}$	$1-\frac{1}{3}+\frac{1}{5}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{25}-\frac{1}{33}+\frac{1}{39}+\frac{1}{55}-\frac{1}{65}-\frac{1}{75}+\cdots$
6	13	17	$\frac{13}{17}$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{6}+\frac{1}{9}+\frac{1}{13}-\frac{1}{17}-\frac{1}{18}-\frac{1}{26}+\frac{1}{27}+\frac{1}{34}+\frac{1}{39}+\cdots$
7	17	19	$\frac{17}{19}$	$1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}-\frac{2}{6}+\frac{1}{8}+\frac{1}{9}-\frac{2}{12}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}-\frac{1}{19}+\cdots$
8	19	23	$\frac{19}{23}$	$1-\frac{1}{2}+\frac{2}{3}-\frac{1}{5}-\frac{2}{6}+\frac{3}{9}+\frac{1}{10}-\frac{1}{11}-\frac{2}{15}-\frac{3}{18}+\frac{1}{19}+\frac{1}{22}+\cdots$
9	23	29	$\frac{23}{29}$	$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{21}+\frac{1}{23}+\frac{1}{25}-\frac{1}{29}-\frac{1}{33}-\frac{1}{35}+\cdots$
10	29	31	$\frac{29}{31}$	$1-\frac{1}{5}+\frac{1}{7}+\frac{1}{29}-\frac{1}{31}-\frac{1}{35}+\frac{1}{49}-\frac{1}{145}+\frac{1}{155}+\frac{1}{203}-\frac{1}{217}-\frac{1}{245}+\cdots$
11	31	37	$\frac{31}{37}$	$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{15}+\frac{1}{25}+\frac{1}{31}-\frac{1}{37}-\frac{1}{75}-\frac{1}{93}+\frac{1}{111}+\frac{1}{125}+\frac{1}{155}+\cdots$
12	37	41	$\frac{37}{41}$	$1-\frac{1}{2}+\frac{2}{3}-\frac{1}{5}-\frac{2}{6}+\frac{3}{9}+\frac{1}{10}-\frac{2}{15}-\frac{3}{18}+\frac{4}{27}+\frac{2}{30}+\frac{1}{37}+\cdots$
13	41	43	$\frac{41}{43}$	$1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{2}{12}-\frac{1}{14}+\cdots$
14	43	47	$\frac{43}{47}$	$1-\frac{1}{2}+\frac{2}{3}-\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{3}{9}+\frac{1}{10}-\frac{1}{11}-\frac{1}{14}-\frac{2}{15}-\frac{3}{18}+\cdots$
15	47	53	$\frac{47}{53}$	$1-\frac{1}{3}+\frac{1}{5}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{23}+\frac{1}{25}-\frac{1}{33}+\frac{1}{39}+\frac{1}{47}-\frac{1}{53}+\cdots$
16	53	59	$\frac{53}{59}$	$1-\frac{1}{7}+\frac{1}{13}-\frac{1}{29}+\frac{1}{53}-\frac{1}{59}-\frac{1}{91}+\frac{1}{169}+\frac{1}{203}-\frac{1}{371}-\frac{1}{377}$
17	59	61	$\frac{59}{61}$	$1-\frac{1}{5}+\frac{1}{7}+\frac{1}{29}-\frac{1}{35}+\frac{1}{49}+\frac{1}{59}-\frac{1}{61}-\frac{1}{145}+\frac{1}{203}-\frac{1}{245}-\frac{1}{295}+\cdots$

Continued on next page

r	p_r	p_{r+1}	$\dfrac{p_r}{p_{r+1}}$	First nonzero terms of $\sum_{t\ge1}\dfrac{(\chi_{p_r}*\psi_{p_{r+1}})(t)}{t}$
18	61	67	$\frac{61}{67}$	$1-\frac{1}{11}+\frac{1}{61}-\frac{1}{67}$
19	67	71	$\frac{67}{71}$	$1-\frac{1}{7}+\frac{1}{11}+\frac{1}{67}-\frac{1}{71}-\frac{1}{77}+\frac{1}{121}$
20	71	73	$\frac{71}{73}$	$1-\frac{1}{3}+\frac{1}{5}+\frac{1}{7}-\frac{1}{15}-\frac{1}{21}+\frac{1}{25}+\frac{1}{35}+\frac{1}{49}+\frac{1}{71}-\frac{1}{73}-\frac{1}{75}+\cdots$

18.1 When is the convolution series finite?

A natural question is the following: for which pairs $a, b \in \mathbb{N}$ does the convolution expansion

$$\frac{a}{b} = \sum_{t\ge1} \frac{(\chi_a * \psi_b)(t)}{t}$$

reduce to a *finite* sum?

At first sight this is not obvious from the coefficient formula alone, since the convolution

$$(\chi_a * \psi_b)(t) = \sum_{d\mid t} \chi_a(d)\psi_b\left(\frac{t}{d}\right)$$

mixes the two arithmetic functions in a nontrivial way. The answer becomes transparent once one rewrites the associated Dirichlet series in Euler-product form.

Definition 5 (Pratt exponent order)**.** For $a, b \in \mathbb{N}$, write

$$a \preceq_{\mathrm{P}} b \qquad :\Longleftrightarrow \qquad m_p(a) \le m_p(b) \ \text{ for every prime } p.$$

We call this the *Pratt exponent order*.

Thus $a \preceq_{\mathrm{P}} b$ means that the Pratt exponent vector of a is dominated componentwise by that of b.

Theorem 5 (Finiteness criterion for the convolution series)**.** *Let $a, b \in \mathbb{N}$. Then the convolution expansion*

$$\frac{a}{b} = \sum_{t\ge1} \frac{(\chi_a * \psi_b)(t)}{t}$$

is a finite sum if and only if

$$a \preceq_{\mathrm{P}} b, \qquad \text{equivalently,} \qquad m_p(a) \le m_p(b) \ \text{ for all primes } p.$$

Proof. Recall that

$$\sum_{r\ge1} \frac{\chi_a(r)}{r^s} = \prod_p (1-p^{-s})^{-m_p(a)}$$

and

$$\sum_{r\geq 1}\frac{\psi_b(r)}{r^s}=\prod_p(1-p^{-s})^{m_p(b)}.$$

Multiplying these Dirichlet series gives

$$\sum_{t\geq 1}\frac{(\chi_a*\psi_b)(t)}{t^s}=\prod_p(1-p^{-s})^{\,m_p(b)-m_p(a)}.$$

Set

$$d_p:=m_p(b)-m_p(a)\in\mathbb{Z}.$$

Since only finitely many Pratt exponents are nonzero, only finitely many d_p are nonzero, so the Euler product above is finite.

Now observe:

- If $d_p\geq 0$, then

  $$(1-p^{-s})^{d_p}=\sum_{k=0}^{d_p}(-1)^k\binom{d_p}{k}p^{-ks},$$

  which is a finite polynomial in p^{-s}.

- If $d_p<0$, then

  $$(1-p^{-s})^{d_p}=(1-p^{-s})^{-|d_p|}$$

  has an infinite binomial expansion

  $$(1-p^{-s})^{-|d_p|}=\sum_{k\geq 0}\binom{|d_p|+k-1}{k}p^{-ks},$$

  and hence contributes infinitely many nonzero terms.

Therefore the full Dirichlet series

$$\sum_{t\geq 1}\frac{(\chi_a*\psi_b)(t)}{t^s}$$

is a Dirichlet polynomial, equivalently the series at $s=1$ is finite, if and only if

$$d_p\geq 0\qquad\text{for every prime } p.$$

This is exactly the condition

$$m_p(a)\leq m_p(b)\qquad\text{for every prime } p,$$

that is, $a\preceq_{\mathrm{P}} b$. □

Remark 10 (Support in the finite case). If $a\preceq_{\mathrm{P}} b$, let

$$d_p:=m_p(b)-m_p(a)\geq 0.$$

Then

$$\sum_{t\geq 1} \frac{(\chi_a * \psi_b)(t)}{t^s} = \prod_p (1-p^{-s})^{d_p}$$

is a finite product of finite polynomials. In particular, its support is contained in the divisor set of

$$M := \prod_p p^{d_p}.$$

Thus in the finite case one actually has

$$\frac{a}{b} = \sum_{t|M} \frac{(\chi_a * \psi_b)(t)}{t}.$$

Remark 11 (This order is not the sorted-sequence order)*.* The order $\preceq_{\mathrm{P}}$ is the coordinatewise order on the Pratt exponent vectors

$$\phi(n) := (m_p(n))_p.$$

It is different from the order obtained by comparing sorted Pratt sequences term by term. For the finiteness criterion above, the relevant order is precisely the exponent order $\preceq_{\mathrm{P}}$, because the Euler product depends only on the differences $m_p(b) - m_p(a)$.

Remark 12 (Twin primes)*.* If p and $p+2$ are both prime, then

$$1 - \frac{p}{p+2} = \frac{2}{p+2}.$$

Since the Pratt exponent vector of 2 is given by $m_2(2) = 1$ and $m_q(2) = 0$ for $q \neq 2$, while every odd prime q satisfies $m_2(q) \geq 1$, we have

$$2 \preceq_{\mathrm{P}} p+2.$$

Hence, by Theorem 5, the convolution expansion of

$$\frac{2}{p+2}$$

is always finite. Equivalently, for every twin-prime pair $p, p+2$, the normalized prime gap

$$\frac{(p+2)-p}{p+2} = \frac{2}{p+2}$$

has a finite Pratt convolution expansion.

Proposition 7. *For all $x, y \in \mathbb{Q}_{>0}$ one has*

$$\overset{*}{\gcd}(x,y) \preceq_P x+y.$$

Equivalently, for every prime p,

$$m_p(x+y) \geq m_p(\overset{*}{\gcd}(x,y)).$$

In particular, for $x, y \in \mathbb{N}$,

$$\gcd(x,y) \leq_P x+y.$$

Proof. For each prime q we have the standard valuation inequality

$$v_q(x+y) \geq \min(v_q(x), v_q(y)).$$

Multiplying by $m_p(q) \geq 0$ and summing over all primes q gives

$$\sum_q v_q(x+y) m_p(q) \geq \sum_q \min(v_q(x), v_q(y))\, m_p(q).$$

The left-hand side is $m_p(x+y)$ by definition. The right-hand side is

$$\sum_q v_q(\overset{*}{\gcd}(x,y))\, m_p(q) = m_p(\overset{*}{\gcd}(x,y)),$$

since

$$v_q(\overset{*}{\gcd}(x,y)) = \min(v_q(x), v_q(y)).$$

Hence $m_p(x+y) \geq m_p(\gcd^*(x,y))$ for every prime p, which is exactly

$$\overset{*}{\gcd}(x,y) \leq_P x+y.$$

□

19 Meet Kernel defined with Pratt valuation

For $n \in \mathbb{N}$, let $(m_p(n))_p$ denote the Pratt valuation vector, indexed by primes p, and define the Pratt order by

$$a \leq_P b \quad \Longleftrightarrow \quad m_p(a) \leq m_p(b) \text{ for every prime } p.$$

The Pratt product formula reads

$$n = \prod_p \left(1 - \frac{1}{p}\right)^{-m_p(n)}.$$

Since each factor $(1-1/p)^{-1} = p/(p-1)$ is strictly larger than 1, it follows immediately that

$$a \leq_P b \quad \Longrightarrow \quad a \leq b.$$

Motivated by the usual gcd-kernel, define the Pratt meet-value

$$a \wedge b := \prod_p \left(1 - \frac{1}{p}\right)^{-\min(m_p(a), m_p(b))}.$$

By construction,

$$m_p(a \wedge b) = \min(m_p(a), m_p(b)),$$

so $a \wedge b$ is the meet of a and b in the Pratt poset $(\mathbb{N}, \leq_P)$.

Proposition 8. *The function*

$$k(a,b) := a \wedge b = \prod_p \left(1 - \frac{1}{p}\right)^{-\min(m_p(a), m_p(b))}$$

is a positive semidefinite kernel on $\mathbb{N}$.

Proof. For each prime p, put

$$\alpha_p := \left(1 - \frac{1}{p}\right)^{-1} > 1.$$

Then

$$k(a,b) = \prod_p \alpha_p^{\min(m_p(a), m_p(b))}.$$

Thus it is enough to show that for every fixed $\alpha > 1$, the kernel

$$K_\alpha(u,v) := \alpha^{\min(u,v)}, \qquad u, v \in \mathbb{N}_0,$$

is positive semidefinite.

We use the telescoping identity

$$\alpha^{\min(u,v)} = 1 + \sum_{j \geq 1} (\alpha^j - \alpha^{j-1}) \, \mathbf{1}_{\{u \geq j\}} \, \mathbf{1}_{\{v \geq j\}}.$$

Indeed, if $m = \min(u,v)$, then the right-hand side equals

$$1 + \sum_{j=1}^{m} (\alpha^j - \alpha^{j-1}) = \alpha^m.$$

Each coefficient $\alpha^j - \alpha^{j-1}$ is positive, so K_α is a sum of rank-one positive semidefinite kernels. Hence K_α is positive semidefinite.

Now for each prime p, the kernel

$$(a,b) \longmapsto \alpha_p^{\min(m_p(a), m_p(b))}$$

is positive semidefinite. Since for fixed a, b only finitely many factors differ from 1, the product defining $k(a,b)$ is effectively finite. A pointwise product of positive semidefinite kernels is again positive semidefinite, so k is positive semidefinite. □

Remark 13. This is the Pratt-valuation analogue of the classical gcd-kernel

$$\gcd(a, b) = \prod_p p^{\min(v_p(a), v_p(b))},$$

with the usual p-adic exponents v_p replaced by the Pratt valuations m_p, and the local weights p replaced by $p/(p-1)$.

The kernel also admits a Möbius-expansion on the Pratt poset. Since $a \leq_P b$ implies $a \leq b$, every interval in $(\mathbb{N}, \leq_P)$ is finite, so the Möbius function μ_P exists.

Define the Möbius transform of the identity map $\mathrm{id}(n) = n$ by

$$g(x) := \sum_{d \leq_P x} \mu_P(d, x)\, d.$$

By Möbius inversion on the locally finite poset $(\mathbb{N}, \leq_P)$,

$$x = \sum_{d \leq_P x} g(d) \qquad (x \in \mathbb{N}).$$

Applying this with $x = a \wedge b$ yields the following.

Proposition 9. *For all $a, b \in \mathbb{N}$,*

$$a \wedge b = \sum_{\substack{d \leq_P a \\ d \leq_P b}} g(d),$$

where g is the Möbius transform of the identity on the Pratt poset.

Proof. Since $m_p(a \wedge b) = \min(m_p(a), m_p(b))$, one has

$$d \leq_P a \wedge b \quad \Longleftrightarrow \quad (d \leq_P a \text{ and } d \leq_P b).$$

Therefore,

$$a \wedge b = \sum_{d \leq_P a \wedge b} g(d) = \sum_{\substack{d \leq_P a \\ d \leq_P b}} g(d),$$

as claimed. □

20 Monotonicity of $\chi_a(r)$ with respect to the Pratt order

In this section we record a simple monotonicity property of the coefficients

$$\chi_a(r),$$

viewed as functions of a with r fixed.

Recall that if

$$r = \prod_p p^{v_p(r)},$$

then

$$\chi_a(r) = \prod_{p \mid r} \binom{m_p(a) + v_p(r) - 1}{v_p(r)},$$

where $m_p(a)$ denotes the multiplicity parameter attached to a at the prime p. Recall also that

$$a \leq_P b \quad \Longleftrightarrow \quad m_p(a) \leq m_p(b) \qquad \text{for all primes } p.$$

Lemma 7. *For all positive integers a, b, r, if $a \leq_P b$, then*

$$\chi_a(r) \leq \chi_b(r).$$

Proof. Fix $r \geq 1$ and write

$$r = \prod_p p^{v_p(r)}.$$

By definition,

$$\chi_a(r) = \prod_{p \mid r} \binom{m_p(a) + v_p(r) - 1}{v_p(r)}, \qquad \chi_b(r) = \prod_{p \mid r} \binom{m_p(b) + v_p(r) - 1}{v_p(r)}.$$

Since $a \leq_P b$, we have

$$m_p(a) \leq m_p(b) \qquad \text{for every prime } p.$$

Thus it is enough to show that for each fixed integer $v \geq 0$, the function

$$m \longmapsto \binom{m + v - 1}{v}$$

is nondecreasing in $m \in \mathbb{Z}_{\geq 0}$.

If $v = 0$, then

$$\binom{m-1}{0} = 1,$$

so there is nothing to prove. Assume now that $v \geq 1$. Then Pascal's identity gives

$$\binom{m + 1 + v - 1}{v} - \binom{m + v - 1}{v} = \binom{m + v - 1}{v - 1} \geq 0.$$

Hence

$$\binom{m + v - 1}{v}$$

is nondecreasing in m.

Applying this with $m = m_p(a)$ and $m = m_p(b)$, we obtain for every prime $p \mid r$ that

$$\binom{m_p(a) + v_p(r) - 1}{v_p(r)} \le \binom{m_p(b) + v_p(r) - 1}{v_p(r)}.$$

Since all factors are nonnegative, multiplying over all primes dividing r yields

$$\chi_a(r) \le \chi_b(r).$$

This proves the claim. $\square$

Remark 14. For fixed r, the quantity $\chi_a(r)$ is therefore monotone in each coordinate $m_p(a)$. Since the Pratt order is exactly the coordinatewise order on the family $(m_p(a))_p$, the lemma is an immediate structural consequence of the Euler product formula for $\chi_a(r)$.

20.1 A converse criterion via the coefficient family $\chi_a(r)$

The pointwise inequality

$$\chi_a(r) \le \chi_b(r)$$

for a single fixed value of r does *not* imply $a \le_P b$. Indeed, by the Euler factor formula,

$$\chi_n(r) = \prod_{p \mid r} \binom{m_p(n) + v_p(r) - 1}{v_p(r)},$$

so for fixed r the quantity $\chi_n(r)$ depends only on the coordinates $m_p(n)$ for primes $p \mid r$. Thus information about primes not dividing r is invisible. For example, taking $a = 3$, $b = 2$, and $r = 2$, one finds

$$\chi_3(2) = 0 \le 1 = \chi_2(2),$$

while $3 \not\le_P 2$.

However, if the inequality holds for *all* $r \ge 1$, then one recovers the Pratt order completely.

Proposition 10. *For $a, b \in \mathbb{N}_{\ge 1}$, the following are equivalent:*

1. *$a \le_P b$;*
2. *$\chi_a(r) \le \chi_b(r)$ for all $r \ge 1$.*

In particular, if $\chi_a(r) \le \chi_b(r)$ for all $r \ge 1$, then $a \le_P b$, and hence $a \le b$.

Proof. The implication (1) ⇒ (2) was proved in the previous subsection.

For the converse, assume that

$$\chi_a(r) \le \chi_b(r) \qquad \text{for all } r \ge 1.$$

Let p be any prime and specialize to $r = p$. Since $v_p(p) = 1$, the coefficient formula gives

$$\chi_a(p) = \binom{m_p(a) + 1 - 1}{1} = \binom{m_p(a)}{1} = m_p(a),$$

and similarly

$$\chi_b(p) = m_p(b).$$

Therefore the assumption for $r = p$ yields

$$m_p(a) \le m_p(b) \qquad \text{for every prime } p.$$

By definition of the Pratt order, this is exactly the statement that

$$a \le_P b.$$

Finally, Proposition 1 shows that $a \le_P b$ implies $a \le b$. This proves the claim. □

Remark 15. The proposition shows that the full coefficient family

$$\Big(\chi_a(r)\Big)_{r \ge 1}$$

determines the Pratt-order position of a. In fact, the values at primes already suffice, since

$$\chi_a(p) = m_p(a) \qquad \text{for every prime } p.$$

Thus the family $\chi_a(\cdot)$ contains exactly the coordinate data defining the partial order $\le_P$.

21 The Mobius transform of the identity on the Pratt poset

For $a, b \in \mathbb{N}$, write

$$a \le_P b \quad \Longleftrightarrow \quad m_q(a) \le m_q(b) \text{ for every prime } q,$$

where $m_q(\cdot)$ denotes the Pratt valuation. Let g be the Mobius transform of the identity function $\mathrm{id}(n) = n$ on the locally finite poset $(\mathbb{N}, \le_P)$, that is,

$$n = \sum_{d \le_P n} g(d) \qquad (n \in \mathbb{N}).$$

We prove that g admits the explicit closed form

$$g(n) = \frac{n}{\mathrm{rad}(n)}.$$

Proposition 11. *For every $n \in \mathbb{N}$,*

$$g(n) = \frac{n}{\operatorname{rad}(n)}.$$

Equivalently,

$$\sum_{d \leq_P n} \frac{d}{\operatorname{rad}(d)} = n \qquad (n \in \mathbb{N}).$$

Proof. Set

$$h(n) := \frac{n}{\operatorname{rad}(n)}.$$

It is enough to show that

$$\sum_{d \leq_P n} h(d) = n \qquad (n \in \mathbb{N}),$$

because the Mobius transform on a locally finite poset is uniquely determined by this zeta-summation identity.

We argue by induction on n. The case $n = 1$ is immediate. Assume $n > 1$, and write

$$n = uP^e,$$

where P is the largest prime divisor of n, $e \geq 1$, and $P \nmid u$.

Since P is the largest prime dividing n, no smaller prime contributes to the P-coordinate of the Pratt valuation. Hence

$$m_P(n) = v_P(n) = e.$$

Now let $d \leq_P n$, and write

$$d = P^k c, \qquad P \nmid c.$$

Then necessarily $0 \leq k \leq e$. Using complete additivity of the Pratt valuation together with

$$m(P) = e_P + m(P-1),$$

where e_P denotes the unit vector at the prime P, we obtain

$$m(d) = ke_P + km(P-1) + m(c),$$

while

$$m(n) = ee_P + em(P-1) + m(u).$$

Therefore

$$d \leq_P n \iff m(c) \leq m(u) + (e-k)m(P-1) \iff c \leq_P u(P-1)^{e-k}.$$

Thus every $d \leq_P n$ is uniquely of the form $d = P^k c$ with $0 \leq k \leq e$ and

$$c \leq_P u(P-1)^{e-k}.$$

Now sum $h(d)$ over all such pairs (k, c). Since $P \nmid c$, we have

$$h(P^k c) = \begin{cases} h(c), & k = 0, \\ P^{k-1} h(c), & k \geq 1. \end{cases}$$

Hence, if

$$H(n) := \sum_{d \leq_P n} h(d),$$

then

$$H(n) = H\Big(u(P-1)^e\Big) + \sum_{k=1}^{e} P^{k-1} H\Big(u(P-1)^{e-k}\Big).$$

Each argument on the right-hand side is strictly smaller than n, so the induction hypothesis gives

$$H\Big(u(P-1)^j\Big) = u(P-1)^j \qquad (0 \leq j \leq e).$$

Therefore

$$H(n) = u(P-1)^e + \sum_{k=1}^{e} P^{k-1} u(P-1)^{e-k} = u\left((P-1)^e + \sum_{k=1}^{e} P^{k-1}(P-1)^{e-k}\right).$$

The finite sum telescopes, because

$$P^{k-1}(P-1)^{e-k} = P^k(P-1)^{e-k} - P^{k-1}(P-1)^{e-k+1}.$$

Hence

$$\sum_{k=1}^{e} P^{k-1}(P-1)^{e-k} = P^e - (P-1)^e.$$

Substituting this into the previous display yields

$$H(n) = u\Big((P-1)^e + P^e - (P-1)^e\Big) = uP^e = n.$$

Thus $H(n) = n$ for all n, i.e.

$$\sum_{d \leq_P n} \frac{d}{\mathrm{rad}(d)} = n.$$

By uniqueness of Mobius inversion on $(\mathbb{N}, \leq_P)$, it follows that

$$g(n) = \frac{n}{\mathrm{rad}(n)}.$$

□

22 Feature vectors

Let $m_p(n)$ denote the Pratt valuation of $n \in \mathbb{N}$, and define the Pratt partial order by

$$a \le_P b \qquad \Longleftrightarrow \qquad m_p(a) \le m_p(b) \quad \text{for every prime } p.$$

For $a, b \in \mathbb{N}$, define their Pratt meet by

$$a \wedge_P b := \prod_p \left(1 - \frac{1}{p}\right)^{-\min\{m_p(a), m_p(b)\}}.$$

Equivalently, $a \wedge_P b$ is the unique natural number whose Pratt valuation is given by

$$m_p(a \wedge_P b) = \min\{m_p(a), m_p(b)\} \qquad \text{for all primes } p.$$

We consider the kernel

$$k(a, b) := a \wedge_P b.$$

Assume that the Möbius transform g of the identity function on the Pratt poset satisfies

$$g(n) = \frac{n}{\operatorname{rad}(n)}.$$

Then, since

$$a \wedge_P b = \sum_{\substack{d \le_P a \\ d \le_P b}} g(d),$$

we obtain an explicit feature map for k.

Proposition 12. *Define, for each $n \in \mathbb{N}$, the vector*

$$\phi(n) = \big(\phi_d(n)\big)_{d \ge 1}, \qquad \phi_d(n) := \sqrt{\frac{d}{\operatorname{rad}(d)}}\, \mathbf{1}_{\{d \le_P n\}}.$$

Then $\phi(n)$ has finite support and

$$\langle \phi(a), \phi(b) \rangle_{\ell^2(\mathbb{N})} = a \wedge_P b \qquad \text{for all } a, b \in \mathbb{N}.$$

In particular, $k(a, b) = a \wedge_P b$ is a positive semidefinite kernel on $\mathbb{N}$.

Proof. Since $d \le_P n$ implies $d \le n$, only finitely many coordinates of $\phi(n)$ are nonzero; hence $\phi(n) \in \ell^2(\mathbb{N})$. Now

$$\langle \phi(a), \phi(b) \rangle = \sum_{d \ge 1} \sqrt{\frac{d}{\operatorname{rad}(d)}}\, \mathbf{1}_{\{d \le_P a\}} \sqrt{\frac{d}{\operatorname{rad}(d)}}\, \mathbf{1}_{\{d \le_P b\}}.$$

Therefore

$$\langle \phi(a), \phi(b) \rangle = \sum_{d \geq 1} \frac{d}{\mathrm{rad}(d)} \mathbf{1}_{\{d \leq_P a\}} \mathbf{1}_{\{d \leq_P b\}} = \sum_{\substack{d \leq_P a \\ d \leq_P b}} \frac{d}{\mathrm{rad}(d)}.$$

Using $g(d) = d/\mathrm{rad}(d)$ and the Möbius inversion formula on the Pratt poset, we obtain

$$\sum_{\substack{d \leq_P a \\ d \leq_P b}} \frac{d}{\mathrm{rad}(d)} = a \wedge_P b.$$

Hence

$$\langle \phi(a), \phi(b) \rangle = a \wedge_P b = k(a, b),$$

as claimed. Positive semidefiniteness follows immediately from this Gram representation. □

Corollary 4. *The squared norm of the feature vector is*

$$\|\phi(n)\|^2 = k(n, n) = n.$$

Hence the normalized feature map

$$\widehat{\phi}(n) := \frac{1}{\sqrt{n}} \phi(n)$$

satisfies

$$\langle \widehat{\phi}(a), \widehat{\phi}(b) \rangle = \frac{a \wedge_P b}{\sqrt{ab}}.$$

Proof. Since $n \wedge_P n = n$, the first identity is immediate from the proposition. The second follows by rescaling. □

23 Properties of the Meet Kernel $a \wedge b$ and a New Kernel

23.1 Definitions and the Partial Order

Let $m_p(a)$ denote the multiplicity of the prime p appearing in all Pratt trees associated with the integer $a \in \mathbb{N}_{\geq 1}$. For example, evaluating the prime 2, we have $m_2(2) = 1$ and $m_2(6) = 2$.

We define a relation $\leq_p$ on $\mathbb{N}_{\geq 1}$ as follows:

$$a \leq_p b \iff m_p(a) \leq m_p(b) \quad \text{for all primes } p. \tag{44}$$

23.2 The Order Proposition

Proposition 1. If $a \leq_p b$, then $a \leq b$.

Proof. We take as given the representation of any integer $n \in \mathbb{N}_{\geq 1}$ in terms of its Pratt tree prime multiplicities $m_p(n)$:

$$n = \prod_p \left(1 - \frac{1}{p}\right)^{-m_p(n)} \tag{45}$$

Assume $a \leq_p b$. By our definition of the partial order $\leq_p$, this implies that $m_p(a) \leq m_p(b)$ for all primes p. For any prime $p \geq 2$, the fraction $\frac{1}{p}$ satisfies $0 < \frac{1}{p} \leq \frac{1}{2}$, which means $0 < 1 - \frac{1}{p} < 1$. We can rewrite the factors in the product by taking the reciprocal of the base:

$$\left(1 - \frac{1}{p}\right)^{-m_p(n)} = \left(\frac{p}{p-1}\right)^{m_p(n)} \tag{46}$$

Since $p \geq 2$, the new base $\frac{p}{p-1}$ is strictly greater than 1. For any base $B > 1$, the exponential function $f(x) = B^x$ is strictly monotonically increasing. Because $m_p(a) \leq m_p(b)$, it follows that:

$$\left(\frac{p}{p-1}\right)^{m_p(a)} \leq \left(\frac{p}{p-1}\right)^{m_p(b)} \implies \left(1 - \frac{1}{p}\right)^{-m_p(a)} \leq \left(1 - \frac{1}{p}\right)^{-m_p(b)} \tag{47}$$

This inequality holds for every individual prime factor p. Since all factors in the product are strictly positive, multiplying these inequalities together preserves the direction of the inequality over the entire product:

$$\prod_p \left(1 - \frac{1}{p}\right)^{-m_p(a)} \leq \prod_p \left(1 - \frac{1}{p}\right)^{-m_p(b)} \tag{48}$$

By substituting our given representation back into both sides, we immediately obtain $a \leq b$.
□

23.3 Properties of the Meet Operation

Let $a \wedge b$ denote the meet (greatest lower bound) of a and b in the poset $(\mathbb{N}_{\geq 1}, \leq_p)$. By the definition of the meet in this lattice, the prime valuations of $a \wedge b$ satisfy:

$$m_p(a \wedge b) \leq m_p(a) \quad \text{and} \quad m_p(a \wedge b) \leq m_p(b) \quad \text{for all } p. \tag{49}$$

This directly implies $a \wedge b \leq_p a$ and $a \wedge b \leq_p b$. Applying Proposition 1, we obtain:

$$a \wedge b \leq a \quad \text{and} \quad a \wedge b \leq b \tag{50}$$

Since $a \wedge b$ is less than or equal to both a and b in the standard integers, it must be bounded by their minimum:

$$a \wedge b \leq \min(a, b). \tag{51}$$

23.4 Kernel Definiteness and Feature Mapping

Definition. Let $K_c(a, b)$ be defined as the number of common predecessors in the $\leq_p$ poset:

$$K_c(a, b) := \sum_{\substack{d \leq_p a \\ d \leq_p b}} 1 = \#\{d \mid d \leq_p a \wedge b\} \tag{52}$$

Proposition 2. $K_c(a, b)$ is a positive definite (p.d.) kernel.

Proof. We construct a natural feature map $\phi(a)$ mapping into an infinite-dimensional sequence space. Let the d-th component of the vector $\phi(a)$ be the indicator function:

$$\phi(a)_d = \mathbf{1}_{\{d \leq_p a\}} \tag{53}$$

The inner product of two such feature vectors $\phi(a)$ and $\phi(b)$ is:

$$\langle \phi(a), \phi(b) \rangle = \sum_{d=1}^{\infty} \mathbf{1}_{\{d \leq_p a\}} \mathbf{1}_{\{d \leq_p b\}} = \sum_{\substack{d \leq_p a \\ d \leq_p b}} 1 = K_c(a, b) \tag{54}$$

Because the kernel function $K_c(a, b)$ can be expressed exactly as an inner product $\langle \phi(a), \phi(b) \rangle$ in a feature space, it is a positive semi-definite kernel. $\square$

23.5 The Gram Matrix and Lattice Structure

Let G_n be the $n \times n$ Gram matrix defined by $(G_n)_{i,j} = K_c(i, j)$ for $1 \leq i, j \leq n$.

Proposition 3. $\det(G_n) = 1$.

Proof. Define an $n \times n$ matrix L such that its entries are $L_{i,d} = \mathbf{1}_{\{d \leq_p i\}}$. From Proposition 1, we know that $d \leq_p i \implies d \leq i$. This structural property ensures that $L_{i,d} = 0$ whenever $d > i$. Thus, L is a lower-triangular matrix.

The diagonal elements are $L_{i,i} = \mathbf{1}_{\{i \leq_p i\}} = 1$. Since L is lower-triangular with 1s on the main diagonal, its determinant is the product of its diagonal elements, yielding $\det(L) = 1$. The Gram matrix G_n can be factorized as $G_n = LL^T$. Therefore:

$$\det(G_n) = \det(LL^T) = \det(L)\det(L^T) = 1 \cdot 1 = 1. \square \tag{55}$$

Theorem 1. The lattice $\mathcal{L}_n$ spanned by the basis vectors $\phi(1), \phi(2), \ldots, \phi(n)$ is unimodular and integral.

Proof.

1. **Integral:** The inner product of any two basis vectors is $\langle \phi(i), \phi(j) \rangle = K_c(i, j)$. Since K_c counts a finite set of integers, $K_c(i, j) \in \mathbb{Z}$. Therefore, $\mathcal{L}_n$ is an integral lattice.
2. **Unimodular:** The volume of the fundamental parallelotope of the lattice is given by $\sqrt{\det(G_n)}$. Since $\det(G_n) = 1$, the volume is 1, proving the lattice is unimodular. $\square$

23.6 Lattice Isometry and the Theta Series

Proposition 4. For any $n \geq 1$, the lattice $\mathcal{L}_n$ associated with the Gram matrix G_n is isometric to the standard hypercubic lattice $\mathbb{Z}^n$. Consequently, for $n = 6$, the coefficients of the theta series of $\mathcal{L}_6$ are exactly the sequence OEIS A000141.

Proof. Let G_n be the Gram matrix of the lattice $\mathcal{L}_n$. From Proposition 3, we have the exact factorization:

$$G_n = LL^T \tag{56}$$

where L is an $n \times n$ matrix with integer entries defined by $L_{i,d} = \mathbb{1}_{\{d \leq_p i\}}$. We established that L is a lower-triangular matrix with ones on its main diagonal.

Because L is an integer matrix with $\det(L) = 1$, it is unimodular. Therefore, its inverse L^{-1} is also an integer matrix, which implies that L is an element of the general linear group over the integers, $\mathrm{GL}_n(\mathbb{Z})$.

The matrix L can be viewed as a change of basis matrix. Since $G_n = LI_nL^T$ (where I_n is the $n \times n$ identity matrix), the Gram matrix G_n is integer-equivalent to the identity matrix. Geometrically, this means that the lattice $\mathcal{L}_n$ generated by G_n is exactly the standard lattice $\mathbb{Z}^n$ represented in a different basis. Thus, there exists a lattice isometry:

$$\mathcal{L}_n \cong \mathbb{Z}^n \tag{57}$$

Because the theta series $\Theta_{\mathcal{L}}(q)$ of a lattice is a geometric invariant that depends only on the lengths of its vectors, isometric lattices share the identical theta series. Therefore:

$$\Theta_{\mathcal{L}_n}(q) = \Theta_{\mathbb{Z}^n}(q) \tag{58}$$

The theta series of the standard lattice $\mathbb{Z}^n$ is given by the sum over the squared norms of all lattice vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$:

$$\Theta_{\mathbb{Z}^n}(q) = \sum_{\mathbf{x} \in \mathbb{Z}^n} q^{\|\mathbf{x}\|^2} = \sum_{(x_1,\ldots,x_n) \in \mathbb{Z}^n} q^{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{59}$$

By grouping the terms according to their exponent (the squared length m), we can rewrite this series as:

$$\Theta_{\mathbb{Z}^n}(q) = \sum_{m=0}^{\infty} r_n(m) q^m \tag{60}$$

where the coefficient $r_n(m)$ counts the number of distinct ways to represent the integer m as a sum of n squares.

Evaluating this for $n = 6$, the theta series becomes:

$$\Theta_{\mathcal{L}_6}(q) = \sum_{m=0}^{\infty} r_6(m) q^m \tag{61}$$

The sequence of coefficients $r_6(m)$ for $m \geq 0$ is the number of ways of writing m as a sum of 6 squares, which is precisely the defining property of the integer sequence OEIS A000141. □

24 Compatibility of $a \wedge_P b$ with multiplication

The Pratt meet interacts very naturally with ordinary multiplication because the Pratt valuation is additive:

$$m_p(xy) = m_p(x) + m_p(y) \qquad (x, y \in \mathbb{N}).$$

Since the meet is defined by taking the coordinatewise minimum,

$$m_p(a \wedge_P b) = \min\{m_p(a), m_p(b)\},$$

one expects identities of min-plus type. The next propositions record the basic ones.

Proposition 13 (Common factors factor out of the Pratt meet)**.** *For all $a, b, c \in \mathbb{N}$ one has*

$$(ac) \wedge_P (bc) = c\,(a \wedge_P b).$$

Proof. Fix a prime p. By additivity of the Pratt valuation,

$$m_p(ac) = m_p(a) + m_p(c), \qquad m_p(bc) = m_p(b) + m_p(c).$$

Hence

$$m_p\Big((ac) \wedge_P (bc)\Big) = \min\Big(m_p(a) + m_p(c),\, m_p(b) + m_p(c)\Big).$$

Using the elementary identity $\min(x + z, y + z) = z + \min(x, y)$, we obtain

$$m_p\Big((ac) \wedge_P (bc)\Big) = m_p(c) + \min\Big(m_p(a), m_p(b)\Big) = m_p\Big(c(a \wedge_P b)\Big).$$

Since this holds for every prime p, the two integers are equal. □

Corollary 5 (Powers)**.** *For every $a \in \mathbb{N}$ and every $m, n \in \mathbb{N}_0$,*

$$a^m \wedge_P a^n = a^{\min(m,n)}.$$

Proof. For each prime p,

$$m_p(a^m) = m\,m_p(a), \qquad m_p(a^n) = n\,m_p(a).$$

Therefore

$$m_p(a^m \wedge_P a^n) = \min\{m\,m_p(a), n\,m_p(a)\} = \min(m, n)\,m_p(a) = m_p\Big(a^{\min(m,n)}\Big),$$

and the claim follows. □

Proposition 14 (A submultiplicative bound)**.** *For all $a, b, c \in \mathbb{N}$,*

$$(ab) \wedge_P c \le (a \wedge_P c)(b \wedge_P c).$$

More generally, for all $a, b, c, d \in \mathbb{N}$,

$$(a \wedge_P c)(b \wedge_P d) \le (ab) \wedge_P (cd).$$

Proof. For the first inequality, fix a prime p. Then

$$m_p\Big((ab) \wedge_P c\Big) = \min\Big(m_p(a) + m_p(b), m_p(c)\Big),$$

whereas

$$m_p\Big((a \wedge_P c)(b \wedge_P c)\Big) = \min\Big(m_p(a), m_p(c)\Big) + \min\Big(m_p(b), m_p(c)\Big).$$

Since for nonnegative real numbers x, y, z one has

$$\min(x + y, z) \le \min(x, z) + \min(y, z),$$

it follows that

$$m_p\Big((ab) \wedge_P c\Big) \le m_p\Big((a \wedge_P c)(b \wedge_P c)\Big)$$

for every prime p. Hence

$$(ab) \wedge_P c \le_P (a \wedge_P c)(b \wedge_P c).$$

Because $x \le_P y$ implies $x \le y$, the desired inequality follows.

For the more general inequality, we similarly compute

$$m_p\Big((a \wedge_P c)(b \wedge_P d)\Big) = \min\Big(m_p(a), m_p(c)\Big) + \min\Big(m_p(b), m_p(d)\Big),$$

while

$$m_p\Big((ab) \wedge_P (cd)\Big) = \min\Big(m_p(a) + m_p(b), m_p(c) + m_p(d)\Big).$$

The elementary inequality

$$\min(x, u) + \min(y, v) \le \min(x + y, u + v)$$

for $x, y, u, v \ge 0$ yields

$$m_p\Big((a \wedge_P c)(b \wedge_P d)\Big) \le m_p\Big((ab) \wedge_P (cd)\Big)$$

for every prime p, and again the conclusion follows from the implication $x \le_P y \Rightarrow x \le y$. $\square$

Remark 16. These identities show that multiplication is the natural operation for the Pratt meet. In the valuation coordinates $\Phi(n) := (m_p(n))_p$, multiplication becomes vector addition and the Pratt meet becomes coordinatewise minimum. By contrast, there is no comparably simple rule involving the ordinary sum $a + b$.

25 Numerical factorization of the theta series for the Pratt meet kernel

Let

$$G_N = (i \wedge_P j)_{1 \le i,j \le N}$$

be the Gram matrix of the Pratt meet kernel restricted to $\{1, 2, \dots, N\}$. We saw earlier that, the diagonal weights are already identified via the Möbius-transform identity

$$g(n) = \frac{n}{\operatorname{rad}(n)}.$$

The purpose of this section is to record the resulting theta-factorization explicitly, to list the numerical values for $N = 1, 2, \dots, 20$, and to point out how this family fits the numerical strategy used by Simon Plouffe in *Numbers in the base e^π*.

Exact triangular factorization

For $1 \le a, d \le N$, define

$$L_N(a,d) = \mathbf{1}_{\{d \le_P a\}}, \qquad D_N = \operatorname{diag}\left(\frac{1}{\operatorname{rad}(1)}, \frac{2}{\operatorname{rad}(2)}, \dots, \frac{N}{\operatorname{rad}(N)}\right).$$

By the feature-map description given earlier,

$$\phi_d(a) = \sqrt{\frac{d}{\operatorname{rad}(d)}}\,\mathbf{1}_{\{d \le_P a\}},$$

and therefore the restricted Gram matrix satisfies

$$G_N = L_N D_N L_N^{\mathsf{T}}.$$

Since $d \le_P a$ implies $d \le a$, the matrix L_N is lower triangular. Since also $a \le_P a$, every diagonal entry of L_N equals 1, so L_N is unimodular. Hence

$$\det(G_N) = \prod_{n=1}^{N} \frac{n}{\operatorname{rad}(n)}.$$

Corollary 6 (Theta-series factorization)**.** *Let*

$$\Theta_{G_N}(q) = \sum_{x \in \mathbb{Z}^N} q^{x^{\mathsf{T}} G_N x}.$$

Then

$$\Theta_{G_N}(q) = \prod_{n=1}^{N} \theta_3\left(q^{n/\operatorname{rad}(n)}\right).$$

In particular, if $q = e^{-\pi}$ and $T_d := \theta_3(e^{-d\pi})$, then

$$\Theta_{G_N}(e^{-\pi}) = \prod_{n=1}^{N} T_{n/\operatorname{rad}(n)} = T_1^{e_1(N)} T_2^{e_2(N)} T_3^{e_3(N)} T_4^{e_4(N)} T_8^{e_8(N)},$$

where

$$e_d(N) = \#\Big\{1 \le n \le N : \ \frac{n}{\operatorname{rad}(n)} = d\Big\}.$$

For $N \le 20$, only the values $d \in \{1, 2, 3, 4, 8\}$ occur.

Proof. Because L_N is unimodular, the change of variables $y = L_N^\mathsf{T} x$ is a bijection of $\mathbb{Z}^N$. Hence

$$\Theta_{G_N}(q) = \sum_{x \in \mathbb{Z}^N} q^{x^\mathsf{T} L_N D_N L_N^\mathsf{T} x} = \sum_{y \in \mathbb{Z}^N} q^{y^\mathsf{T} D_N y} = \prod_{n=1}^{N} \left(\sum_{m \in \mathbb{Z}} q^{(n/\operatorname{rad}(n))m^2} \right) = \prod_{n=1}^{N} \theta_3\Big(q^{n/\operatorname{rad}(n)}\Big).$$

The specialization $q = e^{-\pi}$ is immediate. □

Thus the exact factorization with theta values is not a conjectural output of numerical fitting: it follows directly from the proved diagonal-weight formula.

Theta values used for $N \le 20$

For convenience, here are the theta constants needed in the range $N \le 20$:

d	symbol	decimal value
1	T_1	1.08643481121330801458
2	T_2	1.00373488548773909105
3	T_3	1.00016139903514069402
4	T_4	1.00000697468471241799
8	T_8	1.00000000002432311342

A few exact examples are:

$$\Theta_{G_5}(e^{-\pi}) = T_1^4 T_2,$$
$$\Theta_{G_9}(e^{-\pi}) = T_1^6 T_2 T_3 T_4,$$
$$\Theta_{G_{20}}(e^{-\pi}) = T_1^{13} T_2^3 T_3^2 T_4 T_8.$$

All Gram matrices, determinants, and LDL^T decompositions were computed in exact arithmetic in SymPy. The decimal values below were then evaluated from the rapidly convergent series

$$T_d = 1 + 2 \sum_{n \ge 1} e^{-d\pi n^2}.$$

Computed values for $N = 1, 2, \ldots, 20$

N	$\det(G_N)$	$(e_1, e_2, e_3, e_4, e_8)$	$\Theta_{G_N}(e^{-\pi})$
1	1	$(1, 0, 0, 0, 0)$	1.086434811213308
2	1	$(2, 0, 0, 0, 0)$	1.180340599016097
3	1	$(3, 0, 0, 0, 0)$	1.282363115859456
4	2	$(3, 1, 0, 0, 0)$	1.287152595250892
5	2	$(4, 1, 0, 0, 0)$	1.398407386824122
6	2	$(5, 1, 0, 0, 0)$	1.519278465303561
7	2	$(6, 1, 0, 0, 0)$	1.650597012632519
8	8	$(6, 1, 0, 1, 0)$	1.650608525026269
9	24	$(6, 1, 1, 1, 0)$	1.650874931649603
10	24	$(7, 1, 1, 1, 0)$	1.793567994703520
11	24	$(8, 1, 1, 1, 0)$	1.948594705723950
12	48	$(8, 2, 1, 1, 0)$	1.955872483811844
13	48	$(9, 2, 1, 1, 0)$	2.124927952707425
14	48	$(10, 2, 1, 1, 0)$	2.308595699141573
15	48	$(11, 2, 1, 1, 0)$	2.508138732564730
16	384	$(11, 2, 1, 1, 1)$	2.508138732625736
17	384	$(12, 2, 1, 1, 1)$	2.724929230477028
18	1152	$(12, 2, 2, 1, 1)$	2.725369031425653
19	1152	$(13, 2, 2, 1, 1)$	2.960935789143525
20	2304	$(13, 3, 2, 1, 1)$	2.971994545252526

26 Explicit theta constants, the constants C_N, and the coefficient sequences

Let

$$G_N := (\, i \wedge_P j \,)_{1 \le i,j \le N}$$

be the $N \times N$ Gram matrix attached to the Pratt-meet kernel $k(a, b) = a \wedge_P b$, and let

$$\Theta_N(q) := \sum_{x \in \mathbb{Z}^N} q^{x^\intercal G_N x}.$$

Write

$$d_n := \frac{n}{\operatorname{rad}(n)}, \qquad u_r(N) := \#\{1 \le n \le N : d_n = r\}.$$

For $N \leq 100$ the only values of d_n are

$$r \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 27, 32\}.$$

The diagonal decomposition proved in the Pratt-meet paper gives

$$G_N = L_N \operatorname{diag}(d_1, \dots, d_N)\, L_N^{\mathsf{T}}$$

with $L_N \in \mathrm{GL}_N(\mathbb{Z})$ lower unitriangular. Hence the theta series is invariant under the unimodular change of variables $x \mapsto L_N^{\mathsf{T}} x$, and therefore

$$\Theta_N(q) = \prod_{n=1}^{N} \theta_3(q^{d_n}) = \prod_r \theta_3(q^r)^{u_r(N)}.$$

At $q = e^{-\pi}$ we obtain the constants

$$C_N := \Theta_N(e^{-\pi}) = \prod_r T_r^{u_r(N)}, \qquad T_r := \theta_3(e^{-r\pi}).$$

Following the explicit evaluations collected on the Wikipedia page "Theta function - Explicit values" (which summarizes classical evaluations for Jacobi theta constants and Ramanujan's $\varphi(q) = \theta_3(q)$), set

$$A := \frac{\pi^{1/4}}{\Gamma(3/4)} = T_1$$

and write $T_r = A\beta_r$ for the explicitly known cases $r \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\}$. A convenient list is

$$\beta_2 = \frac{\sqrt{2+\sqrt{2}}}{2},$$
$$\beta_3 = \frac{\sqrt{1+\sqrt{3}}}{108^{1/8}},$$
$$\beta_4 = \frac{2+\sqrt[4]{8}}{4},$$
$$\beta_5 = \sqrt{\frac{2+\sqrt{5}}{5}},$$
$$\beta_6 = \frac{\sqrt{1+\sqrt[4]{3}+\sqrt{2}+\sqrt{3}}}{12^{3/8}},$$
$$\beta_7 = \frac{\sqrt{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}}{14^{3/8}7^{1/16}},$$
$$\beta_8 = \frac{\sqrt{2+\sqrt{2}}+2^{7/8}}{4},$$
$$\beta_9 = \frac{1+\sqrt[3]{2+2\sqrt{3}}}{3},$$
$$\beta_{10} = \frac{\sqrt{\sqrt[4]{64}+\sqrt[4]{80}+\sqrt[4]{81}+\sqrt[4]{100}}}{200^{1/4}},$$
$$\beta_{12} = \frac{\sqrt{1+\sqrt[4]{2}+\sqrt[4]{3}+\sqrt{2}+\sqrt{3}+\sqrt[4]{18}+\sqrt[4]{24}}}{2\cdot 108^{1/8}},$$
$$\beta_{16} = \frac{1}{4}+\frac{2^{3/4}}{8}+\frac{2^{9/16}\sqrt[4]{1+\sqrt{2}}}{4}.$$

We leave T_{27} and T_{32} as symbolic theta constants.

If

$$R_0 := \{1,2,3,4,5,6,7,8,9,10,12,16\}, \qquad \sigma_N := \sum_{r\in R_0} u_r(N),$$

then

$$C_N = A^{\sigma_N}\Big(\prod_{r\in\{2,3,4,5,6,7,8,9,10,12,16\}} \beta_r^{u_r(N)}\Big)T_{27}^{u_{27}(N)}T_{32}^{u_{32}(N)}.$$

In the catalogue below, each entry is written in a Plouffe-style format: first the exact constant C_N, then a decimal approximation, and then the first twenty coefficients

$$\Theta_N(q) = \sum_{m\ge 0} a_N(m)q^m, \qquad (a_N(0), a_N(1), \ldots, a_N(19)).$$

All exact factorizations below were simplified by collecting powers with SymPy.

Catalogue entries $N = 1, \ldots, 10$

$N = 1$**.**

$$C_1 = A.$$

$$C_1 \approx 1.0864348112133080145753 16.$$

Coefficients: 1, 2, 0, 0, 2, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0.

$N = 2$**.**

$$C_2 = A^2.$$

$$C_2 \approx 1.1803405990160962260453 38.$$

Coefficients: 1, 4, 4, 0, 4, 8, 0, 0, 4, 4, 8, 0, 0, 8, 0, 0, 4, 8, 4, 0.

$N = 3$**.**

$$C_3 = A^3.$$

$$C_3 \approx 1.2823631158594553990001 43.$$

Coefficients: 1, 6, 12, 8, 6, 24, 24, 0, 12, 30, 24, 24, 8, 24, 48, 0, 6, 48, 36, 24.

$N = 4$**.**

$$C_4 = A^4 \, \beta_2.$$

$$C_4 \approx 1.2871525952508907616003 98.$$

Coefficients: 1, 6, 14, 20, 30, 40, 36, 48, 62, 42, 72, 100, 68, 120, 112, 48, 126, 108, 98, 180.

$N = 5$**.**

$$C_5 = A^5 \, \beta_2.$$

$$C_5 \approx 1.3984073868241209667094 08.$$

Coefficients: 1, 8, 26, 48, 72, 112, 144, 160, 218, 248, 240, 368, 432, 400, 576, 544, 456, 736, 650, 656.

$N = 6$**.**

$$C_6 = A^6 \, \beta_2.$$

$$C_6 \approx 1.5192784653035592559589 82.$$

Coefficients: 1, 10, 42, 100, 170, 272, 420, 544, 682, 910, 1040, 1220, 1700, 1904, 2080, 2720, 2730, 2900, 3822, 3620.

$N = 7$**.**

$$C_7 = A^7 \, \beta_2.$$

$$C_7 \approx 1.6505970126325167308960 33.$$

Coefficients: 1, 12, 62, 184, 372, 632, 1048, 1584, 2110, 2820, 3720, 4472, 5704, 7464, 8512, 10160, 12660, 13552, 15686, 18984.

$N = 8$.

$$C_8 = A^8 \beta_2 \beta_4.$$

$$C_8 \approx 1.6506085250262671017 16507.$$

Coefficients: 1, 12, 62, 184, 374, 656, 1172, 1952, 2854, 4084, 5816, 7640, 9924, 13104, 15952, 19104, 24070, 28504, 32834, 39672.

$N = 9$.

$$C_9 = A^9 \beta_2 \beta_3 \beta_4.$$

$$C_9 \approx 1.6508749316496013453 27688.$$

Coefficients: 1, 12, 62, 186, 398, 780, 1540, 2700, 4166, 6428, 9720, 13348, 18094, 24760, 31356, 39320, 51026, 61720, 73386, 91716.

$N = 10$.

$$C_{10} = A^{10} \beta_2 \beta_3 \beta_4.$$

$$C_{10} \approx 1.7935679947035174098 1877.$$

Coefficients: 1, 14, 86, 310, 772, 1600, 3224, 6152, 10362, 16322, 25680, 38312, 53494, 74600, 101876, 131808, 171256, 221648, 272518, 336940.

Catalogue entries $N = 11, \ldots, 20$

$N = 11$.

$$C_{11} = A^{11} \beta_2 \beta_3 \beta_4.$$

$$C_{11} \approx 1.9485947057239473661 27703.$$

Coefficients: 1, 16, 114, 482, 1394, 3172, 6596, 13220, 24210, 40248, 64800, 102148, 151462, 215776, 305636, 418632, 554166, 734112, 952382, 1197572.

$N = 12$.

$$C_{12} = A^{12} \beta_2^2 \beta_3 \beta_4.$$

$$C_{12} \approx 1.9558724838118409618 76362.$$

Coefficients: 1, 16, 116, 514, 1622, 4136, 9384, 19564, 37404, 66720, 113448, 183608, 283850, 426416, 621752, 876624, 1213858, 1651872, 2190316, 2870124.

$N = 13$.

$$C_{13} = A^{13} \beta_2^2 \beta_3 \beta_4.$$

$$C_{13} \approx 2.1249279527074212712 62599.$$

Coefficients: 1, 18, 148, 746, 2652, 7412, 17888, 39360, 79776, 149802, 265688, 449864, 726902, 1130800, 1709752, 2506112, 3573936, 5007260, 6871236, 9231928.

$N = 14$.

$$C_{14} = A^{14}\,\beta_2^2\,\beta_3\,\beta_4.$$

$$C_{14} \approx 2.3085956991415683298491 73.$$

Coefficients: 1, 20, 184, 1042, 4146, 12752, 33008, 76628, 163800, 324180, 601104, 1060256, 1787674, 2889512, 4517552, 6861120, 10118686, 14576320, 20605160, 28519492.

$N = 15$.

$$C_{15} = A^{15}\,\beta_2^2\,\beta_3\,\beta_4.$$

$$C_{15} \approx 2.50813873256472461572 4273.$$

Coefficients: 1, 22, 224, 1410, 6232, 21084, 58880, 144728, 325348, 677286, 1315520, 2416088, 4237870, 7121512, 11524288, 18082752, 27569532, 40920356, 59441632, 84656344.

$N = 16$.

$$C_{16} = A^{16}\,\beta_2^2\,\beta_3\,\beta_4\,\beta_8.$$

$$C_{16} \approx 2.5081387326257303585865 77.$$

Coefficients: 1, 22, 224, 1410, 6232, 21084, 58880, 144728, 325350, 677330, 1315968, 2418908, 4250334, 7163680, 11642048, 18372208, 28220228, 42274928, 62072672, 89488520.

$N = 17$.

$$C_{17} = A^{17}\,\beta_2^2\,\beta_3\,\beta_4\,\beta_8.$$

$$C_{17} \approx 2.7249292304770209892038 14.$$

Coefficients: 1, 24, 268, 1858, 9054, 33592, 101496, 265308, 627270, 1370200, 2788432, 5340748, 9741670, 17031472, 28643512, 46611880, 73754770, 113693488, 171261732, 253013036.

$N = 18$.

$$C_{18} = A^{18}\,\beta_2^2\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{18} \approx 2.7253690314256466542328 91.$$

Coefficients: 1, 24, 268, 1860, 9102, 34128, 105212, 283416, 694454, 1573192, 3319048, 6595288, 12482072, 22608384, 39325544, 66098936, 107835822, 171047696, 264688484, 401053192.

$N = 19$.

$$C_{19} = A^{19}\,\beta_2^2\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{19} \approx 2.9609357891435185404227 5.$$

Coefficients: 1, 26, 316, 2396, 12824, 52380, 174004, 497560, 1279490, 3030358, 6675904, 13800752, 27065276, 50737116, 91248664, 158151024, 265564672, 433325064, 688581884, 1069269848.

$N = 20$.

$$C_{20} = A^{20}\,\beta_2^3\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{20} \approx 2.9719945452525179611140 01.$$

Coefficients: 1, 26, 318, 2448, 13456, 57172, 199652, 602320, 1627500, 4025530, 9235516, 19866260, 40442732, 78443380, 145727224, 260620376, 450620980, 755687828, 1233063038, 1963521532.

Catalogue entries $N = 21, \ldots, 30$

$N = 21$.

$$C_{21} = A^{21}\,\beta_2^3\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{21} \approx 3.2288783326984005541334 79.$$

Coefficients: 1, 28, 370, 3084, 18354, 84136, 314632, 1006520, 2859052, 7394876, 17685932, 39542568, 83435148, 167406816, 321199360, 592206648, 1053951838, 1817071600, 3043944838, 4969364288.

$N = 22$.

$$C_{22} = A^{22}\,\beta_2^3\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{22} \approx 3.5079658218159275524854 54.$$

Coefficients: 1, 30, 426, 3824, 24524, 120900, 483644, 1641952, 4908800, 13281254, 33105004, 76928212, 168244556, 349103572, 691553128, 1314319768, 2407248472, 4265507068, 7335277250, 12277045292.

$N = 23$.

$$C_{23} = A^{23}\,\beta_2^3\,\beta_3^2\,\beta_4\,\beta_8.$$

$$C_{23} \approx 3.8111761853673241519238 5.$$

Coefficients: 1, 32, 486, 4676, 32174, 170008, 726296, 2616888, 8241752, 23340656, 60634860, 146422976, 331926228, 712204240, 1456212080, 2852249736, 5375661026, 9788028816, 17275961002, 29642456984.

$N = 24$.

$$C_{24} = A^{24}\,\beta_2^3\,\beta_3^2\,\beta_4^2\,\beta_8.$$

$$C_{24} \approx 3.8112027671196005649161 26.$$

Coefficients: 1, 32, 486, 4676, 32176, 170072, 727268, 2626240, 8306100, 23680672, 62087452, 151656752, 348409732, 758885552, 1577481800, 3145095688, 6039513484, 11212437360, 20188386134, 35346965808.

$N = 25$.

$$C_{25} = A^{25}\,\beta_2^3\,\beta_3^2\,\beta_4^2\,\beta_5\,\beta_8.$$

$$C_{25} \approx 3.8112039158292825776471 99.$$

Coefficients: 1, 32, 486, 4676, 32176, 170074, 727332, 2627212, 8315452, 23745024, 62427596, 153111288, 353662212, 775497752, 1624843144, 3269270592, 6342826988, 11909256824, 21706157238, 38501929408.

$N = 26$.

$$C_{26} = A^{26}\,\beta_2^3\,\beta_3^2\,\beta_4^2\,\beta_5\,\beta_8.$$

$$C_{26} \approx 4.1406246067894068659389 42.$$

Coefficients: 1, 34, 550, 5648, 41530, 234490, 1068452, 4091228, 13634228, 40716078, 111372372, 283221876, 676525044, 1530376576, 3301033988, 6826634120, 13593947022, 26162537272, 48821848194, 88577649612.

$N = 27$.

$$C_{27} = A^{27} \beta_2^3 \beta_3^2 \beta_4^2 \beta_5 \beta_8 \beta_9.$$

$$C_{27} \approx 4.14062460679375906418702.$$

Coefficients: 1, 34, 550, 5648, 41530, 234490, 1068452, 4091228, 13634228, 40716080, 111372440, 283222976, 676536340, 1530459636, 3301502968, 6828771024, 13602129478, 26189805728, 48903280350, 88800394356.

$N = 28$.

$$C_{28} = A^{28} \beta_2^4 \beta_3^2 \beta_4^2 \beta_5 \beta_8 \beta_9.$$

$$C_{28} \approx 4.15608936554784845509671.$$

Coefficients: 1, 34, 552, 5716, 42630, 245786, 1151512, 4560208, 15771134, 48898604, 138641996, 364666432, 899364280, 2097374568, 4656712552, 9897872752, 20232403870, 39928779936, 76330284188, 141746451832.

$N = 29$.

$$C_{29} = A^{29} \beta_2^4 \beta_3^2 \beta_4^2 \beta_5 \beta_8 \beta_9.$$

$$C_{29} \approx 4.51532016524461381873223 3.$$

Coefficients: 1, 36, 620, 6820, 54064, 331114, 1644188, 6874664, 24976810, 80932446, 238742296, 651071944, 1660250844, 3993985596, 9129237252, 19942933744, 41835998352, 84619879148, 165599067476, 314480061136.

$N = 30$.

$$C_{30} = A^{30} \beta_2^4 \beta_3^2 \beta_4^2 \beta_5 \beta_8 \beta_9.$$

$$C_{30} \approx 4.90560101129517476254255 9.$$

Coefficients: 1, 38, 692, 8060, 67706, 439314, 2307656, 10176680, 38834266, 131548296, 403895636, 1142307104, 3012361992, 7476460304, 17595355264, 39506840512, 85056116858, 176329800736, 353259166408, 686041561808.

Catalogue entries $N = 31, \ldots, 40$

$N = 31$.

$$C_{31} = A^{31} \beta_2^4 \beta_3^2 \beta_4^2 \beta_5 \beta_8 \beta_9.$$

$$C_{31} \approx 5.32961570859428607037293 1.$$

Coefficients: 1, 40, 768, 9444, 83828, 574802, 3187668, 14808112, 59323038, 210095458, 671607616, 1970453120, 5374660852, 13764416292, 33356945772, 76986780560, 170114875228, 361472623668, 741372576384, 1472381383040.

$N = 32$.

$$C_{32} = A^{32} \beta_2^4 \beta_3^2 \beta_4^2 \beta_5 \beta_8 \beta_9 \beta_{16}.$$

$$C_{32} \approx 5.32961570859428607037450 8.$$

Coefficients: 1, 40, 768, 9444, 83828, 574802, 3187668, 14808112, 59323038, 210095458, 671607616, 1970453120, 5374660852, 13764416292, 33356945772, 76986780560, 170114875230, 361472623748, 741372577920, 1472381401928.

$N = 33.$

$$C_{33} = A^{33}\, \beta_2^4\, \beta_3^2\, \beta_4^2 \\ \beta_5\, \beta_8\, \beta_9\, \beta_{16}.$$

$$C_{33} \approx 5.790280036206114007796925.$$

Coefficients: 1, 42, 848, 10980, 102718, 742538, 4338808, 21202336, 89106918, 329891140, 1098173948, 3343286112, 9434232056, 24934096568, 62230143192, 147647953680, 334867374280, 729349852948, 1531451909412, 3110443353008.

$N = 34.$

$$C_{34} = A^{34}\, \beta_2^4\, \beta_3^2\, \beta_4^2 \\ \beta_5\, \beta_8\, \beta_9\, \beta_{16}.$$

$$C_{34} \approx 6.290761798007775767463 52.$$

Coefficients: 1, 44, 932, 12676, 124680, 948058, 5825580, 29901912, 131717026, 509590054, 1766633928, 5582040376, 16299040076, 44462548396, 114296169300, 278803489904, 649074150426, 1449131008564, 3115271685668, 6470839449048.

$N = 35.$

$$C_{35} = A^{35}\, \beta_2^4\, \beta_3^2\, \beta_4^2 \\ \beta_5\, \beta_8\, \beta_9\, \beta_{16}.$$

$$C_{35} \approx 6.834502606406467951753894.$$

Coefficients: 1, 46, 1020, 14540, 150034, 1197506, 7723560, 41578424, 191770210, 774920224, 2797465284, 9175113920, 27726580232, 78080058016, 206756430064, 518571560416, 1239339014212, 2836467840348, 6243145223368, 13262523093400.

$N = 36.$

$$C_{36} = A^{36}\, \beta_2^4\, \beta_3^2\, \beta_4^2 \\ \beta_5\, \beta_6\, \beta_8\, \beta_9\, \beta_{16}.$$

$$C_{36} \approx 6.834502695424663387816284.$$

Coefficients: 1, 46, 1020, 14540, 150034, 1197506, 7723562, 41578516, 191772250, 774949304, 2797765352, 9177508932, 27742027352, 78163214864, 207139970484, 520121400864, 1244933944780, 2854818068188, 6298598383832, 13418683209432.

$N = 37.$

$$C_{37} = A^{37}\, \beta_2^4\, \beta_3^2\, \beta_4^2 \\ \beta_5\, \beta_6\, \beta_8\, \beta_9\, \beta_{16}.$$

$$C_{37} \approx 7.425241645640538933051485.$$

Coefficients: 1, 48, 1112, 16580, 179116, 1497666, 10120614, 57054720, 275229350, 1160888818, 4363111176, 14856198708, 46480618796, 135197468244, 369064325928, 952771806820, 2340743958246, 5501395932068, 12424064361824, 27061718338608.

$N = 38$.

$$C_{38} = A^{38}\,\beta_2^4\,\beta_3^2\,\beta_4^2$$
$$\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$
$$C_{38} \approx 8.0670410054946714428 41469.$$

Coefficients: 1, 50, 1208, 18804, 212278, 1855994, 13118170, 77329108, 389697022, 1714342852, 6705130136, 23696532724, 76743508072, 230480841704, 648188480100, 1720633097320, 4339362918920, 10453829243844, 24167306657676, 53824116931408.

$N = 39$.

$$C_{39} = A^{39}\,\beta_2^4\,\beta_3^2\,\beta_4^2$$
$$\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$
$$C_{39} \approx 8.7643141718546178308 90298.$$

Coefficients: 1, 52, 1308, 21220, 249888, 2280650, 16832574, 103603056, 544779794, 2497448886, 10160052280, 37261453628, 124916005172, 387396968108, 1122564135768, 3064429359308, 7934270787922, 19594296159236, 46374770793684, 105613406739280.

$N = 40$.

$$C_{40} = A^{40}\,\beta_2^4\,\beta_3^2\,\beta_4^3$$
$$\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$
$$C_{40} \approx 8.7643753001826870936 40863.$$

Coefficients: 1, 52, 1308, 21220, 249890, 2280754, 16835190, 103645496, 545279570, 2502010186, 10193717428, 37468659740, 126005564760, 392391865880, 1142884240328, 3138952266564, 8184102798268, 20369090095556, 48619899067836, 111742265500336.

Catalogue entries $N = 41, \ldots, 50$

$N = 41$.

$$C_{41} = A^{41}\,\beta_2^4\,\beta_3^2\,\beta_4^3$$
$$\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$
$$C_{41} \approx 9.5219224246565574123 2676.$$

Coefficients: 1, 54, 1412, 23836, 292332, 2780638, 21399314, 137358316, 753070342, 3597130836, 15231408284, 58063388204, 202033485820, 649407515552, 1948059968452, 5499691737080, 14714225751910, 37523169983096, 91648851762592, 215280355646432.

$N = 42$.

$$C_{42} = A^{42}\,\beta_2^4\,\beta_3^2\,\beta_4^3$$
$$\beta_5\,\beta_6\,\beta_8\,\beta_9\,\beta_{16}.$$
$$C_{42} \approx 10.344947991819511059 51554.$$

Coefficients: 1, 56, 1520, 26660, 340006, 3365410, 26963414, 180204616, 1028371638, 5108832798, 22468468692, 88800924228, 319666450584, 1060669333528, 3277343377400, 9511981249020, 26117950914344, 68251942658812, 170598505930184, 409607905510016.

$N = 43$.

$$C_{43} = A^{43}\, \beta_2^4\, \beta_3^2\, \beta_4^3$$
$$\beta_5\, \beta_6\, \beta_8\, \beta_9\, \beta_{16}.$$
$$C_{43} \approx 11.239111618503920361077 1.$$

Coefficients: 1, 58, 1632, 29700, 393328, 4045534, 33697274, 234184764, 1389460882, 7172306896, 32740061228, 134098273884, 499325095636, 1710220580304, 5443625712660, 16244323779104, 45781606722786, 122611239897944, 313667295671244, 769873816859128.

$N = 44$.

$$C_{44} = A^{44}\, \beta_2^5\, \beta_3^2\, \beta_4^3$$
$$\beta_5\, \beta_6\, \beta_8\, \beta_9\, \beta_{16}.$$
$$C_{44} \approx 11.281088413382950460667 4.$$

Coefficients: 1, 58, 1634, 29816, 396592, 4104934, 34483930, 242275832, 1456855432, 7640676540, 35518986256, 148442947076, 564806004748, 1978425219140, 6442343298480, 19665233309240, 56671637069870, 155114232069944, 405295989239274, 1015364493202900.

$N = 45$.

$$C_{45} = A^{45}\, \beta_2^5\, \beta_3^3\, \beta_4^3$$
$$\beta_5\, \beta_6\, \beta_8\, \beta_9\, \beta_{16}.$$
$$C_{45} \approx 11.282909170168207331653.$$

Coefficients: 1, 58, 1634, 29818, 396708, 4108202, 34543562, 243069016, 1465065300, 7709644400, 36003537920, 151356657940, 580087357830, 2049463191768, 6739229195900, 20794845378368, 60628488301334, 167998926876772, 444626524825614, 1128708251894304.

$N = 46$.

$$C_{46} = A^{46}\, \beta_2^5\, \beta_3^3\, \beta_4^3$$
$$\beta_5\, \beta_6\, \beta_8\, \beta_9\, \beta_{16}.$$
$$C_{46} \approx 12.258145294228598124295 45.$$

Coefficients: 1, 60, 1750, 33086, 456346, 4901734, 42763234, 312215776, 1951996748, 10647991406, 51491913960, 223849875080, 885730863946, 3225057989644, 10910170871680, 34576086173172, 103378839911764, 293357759993692, 794118256263026, 2059622999437744.

$N = 47$.

$$C_{47} = A^{47}\, \beta_2^5\, \beta_3^3\, \beta_4^3$$
$$\beta_5\, \beta_6\, \beta_8\, \beta_9\, \beta_{16}.$$
$$C_{47} \approx 13.317675768560547029051 54.$$

Coefficients: 1, 62, 1870, 36586, 522520, 5814546, 52570202, 397808416, 2577340992, 14561788372, 72873423360, 327458138052, 1337334673774, 5017816613040, 17463280482356, 56844213193160, 174303098417554, 506569459790124, 1402675413980082, 3717114668204232.

$N = 48$.

$$C_{48} = A^{48}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5\,\beta_6\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{48} \approx 13.3176757688844743 6724529.$$

Coefficients: 1, 62, 1870, 36586, 522520, 5814546, 52570202, 397808416, 2577340994, 14561788496, 72873427100, 327458211224, 1337335718814, 5017828242132, 17463385622760, 56845008809992, 174308253099538, 506598583366868, 1402821160826802, 3717769584480336.

$N = 49$.

$$C_{49} = A^{49}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{49} \approx 13.31767577638037733592354.$$

Coefficients: 1, 62, 1870, 36586, 522520, 5814546, 52570202, 397808418, 2577341118, 14561792236, 72873500272, 327459256264, 1337347347906, 5017933382536, 17464181239592, 56850163491980, 174337376676530, 506744330221068, 1403476077249250, 3720444255917964.

$N = 50$.

$$C_{50} = A^{50}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{50} \approx 13.31767979037386994572747.$$

Coefficients: 1, 62, 1870, 36586, 522520, 5814548, 52570326, 397812158, 2577414290, 14562837276, 72885129364, 327564396668, 1338142964742, 5023088064772, 17493304824064, 56995910492524, 174992295189058, 509419024916880, 1413511944014322, 3755372618397148.

Catalogue entries $N = 51, \ldots, 60$

$N = 51$.

$$C_{51} = A^{51}\,\beta_2^5\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{51} \approx 14.46879092885412284865946.$$

Coefficients: 1, 64, 1994, 40326, 595694, 6859712, 64203162, 503025982, 3374083646, 19729294954, 102115944692, 474130283452, 1998426659830, 7728500713848, 27685262841432, 92637754074640, 291661197727908, 869454946253244, 2467365729174502, 6696534097742740.

$N = 52$.

$$C_{52} = A^{52}\,\beta_2^6\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{52} \approx 14.52283020611943111154733.$$

Coefficients: 1, 64, 1996, 40454, 599682, 6940364, 65394550, 516745406, 3502489972, 20735347046, 108864115972, 513588954012, 2202659740602, 8676775000176, 31682244567416, 108095761554300, 347038471578064, 1054769912992432, 3050892356519704, 8436392250816260.

$N = 53$.

$$C_{53} = A^{53}\,\beta_2^6\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9\,\beta_{16}.$$

$$C_{53} \approx 15.778108293268291264648 52.$$

Coefficients: 1, 66, 2124, 44446, 680592, 8139856, 79279270, 647615414, 4537180148, 27754207720, 150465599292, 732350680760, 3236842709478, 13123566374836, 49253536680440, 172487559386256, 567636347658682, 1766207411128984, 5223838142337484, 14754586215277120.

$N = 54$.

$$C_{54} = A^{54}\,\beta_2^6\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{54} \approx 15.7781082932848755874966 6.$$

Coefficients: 1, 66, 2124, 44446, 680592, 8139856, 79279270, 647615414, 4537180148, 27754207722, 150465599424, 732350685008, 3236842798370, 13123567736020, 49253552960152, 172487717944796, 567637642889510, 1766216485489280, 5223893650752924, 14754887146475704.

$N = 55$.

$$C_{55} = A^{55}\,\beta_2^6\,\beta_3^3\,\beta_4^3\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{55} \approx 17.1418861049180833318553 7.$$

Coefficients: 1, 68, 2256, 48694, 769486, 9501172, 95563230, 806262846, 5833772160, 36844847732, 206132573540, 1034577118932, 4710618617574, 19652763109388, 75801635910752, 272459683793656, 919088059606672, 2927747981100768, 8854889236071480, 25547950815158884.

$N = 56$.

$$C_{56} = A^{56}\,\beta_2^6\,\beta_3^3\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{56} \approx 17.1420056641690413143981.$$

Coefficients: 1, 68, 2256, 48694, 769488, 9501308, 95567742, 806360234, 5835311132, 36863850076, 206323700000, 1036189644624, 4722286161894, 19726452804852, 76213901057832, 274528838031520, 928509296841822, 2967053507319680, 9006492507897496, 26092870182843584.

$N = 57$.

$$C_{57} = A^{57}\,\beta_2^6\,\beta_3^3\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{57} \approx 18.6236716875689490665220 2.$$

Coefficients: 1, 70, 2392, 53206, 866878, 11040420, 114574870, 997593106, 7449570576, 48553474958, 280242535772, 1450449769604, 6806336170794, 29244754367768, 116079473070152, 429029210571916, 1487013157949120, 4863536677235428, 15093101052357184, 44655325522199004.

$N = 58$.

$$C_{58} = A^{58}\,\beta_2^6\,\beta_3^3\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{58} \approx 20.233405233982600660333333.$$

Coefficients: 1, 72, 2532, 57990, 973292, 12774316, 136660494, 1226849258, 9446490544, 63474696952, 377578635568, 2012930032144, 9722134957566, 42954535393028, 175129488958072, 664089285401168, 2358686246620754, 7896067401010496, 25052430459923044, 75700146533235160.

$N = 59$.

$$C_{59} = A^{59}\,\beta_2^6\,\beta_3^3\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{59} \approx 21.982275795584245024256336.$$

Coefficients: 1, 74, 2676, 63054, 1089274, 14721044, 162214190, 1500286226, 11902135644, 82393226674, 504801350604, 2770541006860, 13766888118922, 62525756648648, 261793742563896, 1018374396702588, 3706311541036740, 12699367858019292, 41194951189259148, 127133941181270700.

$N = 60$.

$$C_{60} = A^{60}\,\beta_2^7\,\beta_3^3\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{60} \approx 22.064377078440650902931444.$$

Coefficients: 1, 74, 2678, 63202, 1094626, 14847152, 164392738, 1529728314, 12226564026, 85393799274, 528605627244, 2935327586316, 14776492998678, 68066868104456, 289327843230120, 1143428910572336, 4229922830435820, 14736281437877816, 48608583874033838, 152538217979323152.

Catalogue entries $N = 61, \ldots, 70$

$N = 61$.

$$C_{61} = A^{61}\,\beta_2^7\,\beta_3^3\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{61} \approx 23.971507345754909205886659.$$

Coefficients: 1, 76, 2826, 68558, 1221032, 17036552, 194092398, 1858640194, 15288209906, 109876621632, 699722011416, 3995598302788, 20671601425766, 97790643889612, 426518820387824, 1727955580990684, 6546336697034478, 23332285288086568, 78659973223853614, 252043300759916392.

$N = 62$.

$$C_{62} = A^{62}\,\beta_2^7\,\beta_3^3\,\beta_4^4\,\beta_5^2 \\ \beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{62} \approx 26.043480057683661073720015.$$

Coefficients: 1, 78, 2978, 74210, 1358150, 19478768, 228171154, 2246962106, 19007932358, 140487114550, 919863439628, 5398759611660, 28693374588270, 139353602426472, 623499586262984, 2588984806556704, 10043594779147768, 36620570546354712, 126177801194051314, 412820557813764936.

$N = 63$.

$$C_{63} = A^{63}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}.$$

$$C_{63} \approx 26.04768345023667712289594.$$

Coefficients: 1, 78, 2978, 74212, 1358306, 19484724, 228319574, 2249678406, 19046889894, 140943456858, 924357363840, 5436775476376, 28974348817372, 141193329305884, 634297105492260, 2646371555881664, 10322301986717012, 37867569757838216, 131355771263507030, 432907751865984684.

$N = 64$.

$$C_{64} = A^{63}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{64} \approx 26.04768345023667712289594.$$

Coefficients: 1, 78, 2978, 74212, 1358306, 19484724, 228319574, 2249678406, 19046889894, 140943456858, 924357363840, 5436775476376, 28974348817372, 141193329305884, 634297105492260, 2646371555881664, 10322301986717012, 37867569757838216, 131355771263507030, 432907751865984684.

$N = 65$.

$$C_{65} = A^{64}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{65} \approx 28.29911005180189185633777.$$

Coefficients: 1, 80, 3134, 80168, 1506732, 22201492, 267294978, 2706465978, 23548963318, 179076206096, 1206700916860, 7289989566824, 39885993698336, 199423916570956, 918532517801156, 3925839774458084, 15672998295471898, 58794598483663952, 208359786877087654, 700913886219638176.

$N = 66$.

$$C_{66} = A^{65}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{66} \approx 30.7451382866340155677781.$$

Coefficients: 1, 82, 3294, 86436, 1667070, 25215116, 311704230, 3241216270, 28964908738, 226218535718, 1565387919168, 9708804338768, 54513070918956, 279554059393284, 1319793797179772, 5777485323784000, 23604455244716696, 90539490005676456, 327786407032436330, 1125487552924723708.

$N = 67$.

$$C_{67} = A^{66}\,\beta_2^7\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{67} \approx 33.40258851016627493531865.$$

Coefficients: 1, 84, 3458, 93024, 1839944, 28549420, 362141050, 3864797602, 35450675418, 284198783428, 2018448399228, 12846062616232, 73988609586840, 389032641636772, 1882032742234908, 8436491150229540, 35268458516555150, 138307566543714056, 511505427075226810, 1792618468413175576.

$N = 68$.

$$C_{68} = A^{67}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{68} \approx 33.527343353245815461668 61.$$

Coefficients: 1, 84, 3460, 93192, 1846860, 28735468, 365820938, 3921896442, 36174957520, 291928378800, 2089349756980, 13414460369136, 78025510065184, 414724823968076, 2030010685690688, 9214564163098288, 39032594902375802, 155181117241739992, 582046381005135568, 2069259293625836320.

$N = 69$.

$$C_{69} = A^{68}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{69} \approx 36.4252729464673748023046.$$

Coefficients: 1, 86, 3628, 100112, 2033246, 32429356, 423298794, 4653724702, 44022444124, 364335764778, 2673938156624, 17601003682900, 104926780904880, 571359704549764, 2863639090611736, 13301415186859812, 57617782092495632, 234075829044342956, 896469220716761448, 3251785362662004376.

$N = 70$.

$$C_{70} = A^{69}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{70} \approx 39.5736845369884981135 8289.$$

Coefficients: 1, 88, 3800, 107368, 2233472, 36496020, 488164762, 5500522514, 53333960020, 452445511740, 3403456283940, 22958187452808, 140216833359152, 781941941955572, 4011706440883224, 19063896222046672, 84430475335474422, 350454200683322168, 1370348885658207644, 5071331982345760368.

Catalogue entries $N = 71, \ldots, 80$

$N = 71$.

$$C_{71} = A^{70}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{16}\,T_{32}.$$

$$C_{71} \approx 42.9942284889581055 3521919.$$

Coefficients: 1, 90, 3976, 114968, 2448210, 40963140, 561164402, 6477066774, 64339471992, 559186423822, 4309323637120, 29776101073316, 186239876399544, 1063280504164300, 5582397310354288, 27133226455048260, 122838712447331100, 520879141906102372, 2079281604797649508, 7850164353019051616.

$N = 72$.

$$C_{72} = A^{71}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{72} \approx 42.9942284889581091 8211965.$$

Coefficients: 1, 90, 3976, 114968, 2448210, 40963140, 561164402, 6477066774, 64339471992, 559186423822, 4309323637120, 29776101073316, 186239876399546, 1063280504164480, 5582397310362240, 27133226455278196, 122838712452227520, 520879141988028652, 2079281605919978312, 7850164365973185164.

$N = 73$.

$$C_{73} = A^{72}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6$$
$$\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{73} \approx 46.710426511663032453370 68.$$

Coefficients: 1, 92, 4156, 122920, 2678148, 45859740, 643098634, 7599625514, 77298501960, 687947294088, 5428818813748, 38407702489056, 245920757720098, 1436878634707636, 7717577047891720, 38357574400478112, 177477658069716554, 768683256579756816, 3132205802889615692, 12063002649371202356.

$N = 74$.

$$C_{74} = A^{73}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^2\,\beta_6$$
$$\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{74} \approx 50.7478334088917242983 3369.$$

Coefficients: 1, 94, 4340, 131232, 2923990, 51216220, 734826426, 8886068622, 92503109284, 842636017490, 6805999599376, 49280539375892, 322890759947970, 1930096050092304, 10602192046653968, 53869545187436932, 254684663585364004, 1126512484585609300, 4685008846039509252, 18404140261589263300.

$N = 75$.

$$C_{75} = A^{74}\,\beta_2^8\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6$$
$$\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{75} \approx 50.747848704464051461713 89.$$

Coefficients: 1, 94, 4340, 131232, 2923990, 51216222, 734826614, 8886077302, 92503371748, 842641865470, 6806102031816, 49282009028744, 322908532085214, 1930281056310872, 10603877318688948, 53883157186635684, 254783224664115788, 1127158266105505240, 4688869038139693860, 18425344645682571236.

$N = 76$.

$$C_{76} = A^{75}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6$$
$$\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{76} \approx 50.937386108124333280297 94.$$

Coefficients: 1, 94, 4342, 131420, 2932670, 51478686, 740674594, 8988509746, 93973024978, 860414020262, 6991108783992, 50967293022148, 336520741996826, 2028845176800804, 11249695852512604, 57743737071412032, 275991164308237180, 1234926265762507548, 5198449099671989070, 20679759741911639392.

$N = 77$.

$$C_{77} = A^{76}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6$$
$$\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{77} \approx 55.340149460079438290261 79.$$

Coefficients: 1, 96, 4530, 140104, 3195512, 57344214, 843640650, 10470121774, 111955909810, 1048463027592, 8713418173892, 64967487618308, 438643274353918, 2703607494700320, 15321368526639568, 80345064843830724, 392151697912074390, 1790966472678633660, 7690802743730078582, 31192159397616272420.

$N = 78$.

$$C_{78} = A^{77}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{78} \approx 60.123464831177653991497 53.$$

Coefficients: 1, 98, 4722, 149164, 3475722, 63735430, 958338138, 12157683282, 132902544382, 1272489535642, 10812031510568, 82415264218700, 568802161690362, 3582990975854364, 20746010467076420, 111117738559627776, 553719135088687224, 2580677307404002892, 11303380523066689282, 46734472441600719024.

$N = 79$.

$$C_{79} = A^{78}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^2\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{79} \approx 65.3202251633504583342 0336.$$

Coefficients: 1, 100, 4918, 158608, 3774052, 70687070, 1085818442, 14074657886, 157224862390, 1538422095268, 13358927258324, 104063642615844, 733898495514854, 4723140285257816, 27933616609277144, 152774591938894292, 777092240846690066, 3695281825338175028, 16506229703787928634, 69563490588916672604.

$N = 80$.

$$C_{80} = A^{79}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{32}.$$

$$C_{80} \approx 65.3202251649392495793943.$$

Coefficients: 1, 100, 4918, 158608, 3774052, 70687070, 1085818442, 14074657886, 157224862392, 1538422095468, 13358927268160, 104063642933060, 733898503062958, 4723140426631956, 27933618780914028, 152774620088210064, 777092555296414846, 3695284902182365564, 16506256421642445282, 69563698716201904292.

Catalogue entries $N = 81, \ldots, 90$

$N = 81$.

$$C_{81} = A^{79}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{81} \approx 65.3202251649392495793943.$$

Coefficients: 1, 100, 4918, 158608, 3774052, 70687070, 1085818442, 14074657886, 157224862392, 1538422095468, 13358927268160, 104063642933060, 733898503062958, 4723140426631956, 27933618780914028, 152774620088210064, 777092555296414846, 3695284902182365564, 16506256421642445282, 69563698716201904292.

$N = 82$.

$$C_{82} = A^{80}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{82} \approx 70.9661664954815449848 1687.$$

Coefficients: 1, 102, 5118, 168444, 4091270, 78235374, 1227202418, 16246611986, 185381726268, 1853013194394, 16437943096180, 130809646794988, 942340238971078, 6194014284496912, 37406617630088400, 208849987107541124, 1084109620628276664, 5258916608078184152, 23952696540413205238, 102881787517518068520.

$N = 83$.

$$C_{83} = A^{81}\,\beta_2^9\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{83} \approx 77.10011369905067675674.$$

Coefficients: 1, 104, 5322, 178680, 4428160, 86418118, 1383683402, 18701353710, 217883132780, 2223933117680, 20146423890008, 163718026221556, 1204330296350478, 8082400797010396, 49827522241745332, 283924844115712736, 1503694307814525042, 7439524248667184044, 34545346697856149366, 151204913448446090492.

$N = 84$.

$$C_{84} = A^{82}\,\beta_2^{10}\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{84} \approx 77.38807379480829501884633.$$

Coefficients: 1, 104, 5324, 178888, 4438804, 86775478, 1392539722, 18874189946, 220650499586, 2261335825308, 20582190166212, 168165892814276, 1244623152986814, 8409837022289744, 52236185601813092, 300089683112440948, 1603349788064281266, 8007378384764844876, 37552775606332979468, 166084289381832901900.

$N = 85$.

$$C_{85} = A^{83}\,\beta_2^{10}\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{85} \approx 84.07709734342409915301116.$$

Coefficients: 1, 106, 5532, 189536, 4796582, 95653294, 1566101326, 21659627166, 258407757086, 2702810375438, 25107646896480, 209368021537240, 1581396239972314, 10903606008791596, 69097024200275960, 404898388886775128, 2206018438343516684, 11230898076238986276, 53672009269737956668, 241790061125104432932.

$N = 86$.

$$C_{86} = A^{84}\,\beta_2^{10}\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{86} \approx 91.34428537966588196172832.$$

Coefficients: 1, 108, 5744, 200600, 5175656, 105246670, 1757418978, 24792208890, 301736604582, 3219817196200, 30516399850220, 259626634595596, 2000649098940038, 14071804119080264, 90954451702958700, 543511176462604180, 3018978051916265902, 15664742681759117220, 76272004876237243080, 349943926657648068556.

$N = 87$.

$$C_{87} = A^{85}\,\beta_2^{10}\,\beta_3^4\,\beta_4^4\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{87} \approx 99.23961144187183386937622.$$

Coefficients: 1, 110, 5960, 212088, 5576858, 115598198, 1967923806, 28307448046, 351331373674, 3823500898706, 36959549080792, 320709018725304, 2520505841741594, 18079541961704052, 119159092951313008, 725939336652550728, 4110001752623772120, 21730842997303018932, 107783405582795798808, 503575019795847864716.

$N = 88$.

$$C_{88} = A^{86}\,\beta_2^{10}\,\beta_3^{4}\,\beta_4^{5}\,\beta_5^{3}\,\beta_6\\ \beta_7\,\beta_8^{3}\,\beta_9^{2}\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{88} \approx 99.240303606872623794424 72.$$

Coefficients: 1, 110, 5960, 212088, 5576860, 115598418, 1967935726, 28307872222, 351342527390, 3823732095102, 36963484928404, 320765633621396, 2521208504488942, 18087188963501464, 119233012049474592, 726580754690001336, 4115042764307255310, 21767002081226427256, 108021723768698436744, 505026898469153390348.

$N = 89$.

$$C_{89} = A^{87}\,\beta_2^{10}\,\beta_3^{4}\,\beta_4^{5}\,\beta_5^{3}\,\beta_6\\ \beta_7\,\beta_8^{3}\,\beta_9^{2}\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{89} \approx 107.81812051388402946140 35.$$

Coefficients: 1, 112, 6180, 224008, 6001038, 126752358, 2199144482, 32244167850, 407969425554, 4526648346720, 44614884990280, 394749219234568, 3163442457210690, 23137253447823272, 155481317177531164, 965688313992064764, 5573246747311980312, 30033262690452995804, 151794201602714442564, 722523581442900547492.

$N = 90$.

$$C_{90} = A^{88}\,\beta_2^{10}\,\beta_3^{5}\,\beta_4^{5}\,\beta_5^{3}\,\beta_6\\ \beta_7\,\beta_8^{3}\,\beta_9^{2}\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{90} \approx 107.83552225450565341282 64.$$

Coefficients: 1, 112, 6180, 224010, 6001262, 126764718, 2199592498, 32256169926, 408222930270, 4531046635684, 44679373325980, 395565158085676, 3172495753904132, 23226483217804056, 156270815616012660, 972015198906934160, 5619521254219628932, 30344225325061562848, 153725578235096861056, 733670075002012843816.

Catalogue entries $N = 91, \ldots, 100$

$N = 91$.

$$C_{91} = A^{89}\,\beta_2^{10}\,\beta_3^{5}\,\beta_4^{5}\,\beta_5^{3}\,\beta_6\\ \beta_7\,\beta_8^{3}\,\beta_9^{2}\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{91} \approx 117.15626526266232461679 65.$$

Coefficients: 1, 114, 6404, 236370, 6449284, 138767466, 2453134294, 36655802942, 472747272646, 5347746025662, 53745865782568, 484988417089848, 3964442516384044, 29580536830886212, 202813141051802168, 1285347964854315828, 7569896708053645370, 41629721616382289588, 214726579578545295800, 1043065351228767534228.

$N = 92$.

$$C_{92} = A^{90}\,\beta_2^{11}\,\beta_3^{5}\,\beta_4^{5}\,\beta_5^{3}\,\beta_6\\ \beta_7\,\beta_8^{3}\,\beta_9^{2}\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{92} \approx 117.59383049758955352282 46.$$

Coefficients: 1, 114, 6406, 236598, 6462092, 139240206, 2466032862, 36933337874, 477653541236, 5421057631774, 54691360340668, 495683909613912, 4071934260847748, 30550513942600840, 210742030990838844, 1344509111827694136, 7975523935651794998, 44200428241582972568, 229866480486384151678, 1126325764438366293328.

$N = 93$.

$$C_{93} = A^{91}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{93} \approx 127.7580310364984490468914.$$

Coefficients: 1, 116, 6634, 249410, 6935290, 152164618, 2744526086, 41865876794, 551533141168, 6376643194660, 65538407670168, 605140496983808, 5064257387631240, 38705224592484068, 271952441875202272, 1766984546560665372, 10672686101695554516, 60212578096078846944, 318688831873647050862, 1588747853017511139488.

$N = 94$.

$$C_{94} = A^{92}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{94} \approx 138.8007723301221385384997.$$

Coefficients: 1, 118, 6866, 262678, 7434112, 166035430, 3048868590, 47355427786, 635278765336, 7480013806234, 78297183111892, 736301044091000, 6275641448380012, 48846492668006448, 349493968179839980, 2312099716794089704, 14216783793323901330, 81635361851721206680, 439657905702841551882, 2229659616934742411112.

$N = 95$.

$$C_{95} = A^{93}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}\,T_{27}\,T_{32}.$$

$$C_{95} \approx 150.7979088273759235724 96.$$

Coefficients: 1, 120, 7102, 276410, 7959470, 180903890, 3380953182, 53453690322, 730004489132, 8750903407768, 93263308461776, 892990121184088, 7749514094618040, 61412735607247164, 447343548214147520, 3012560261339688844, 18853534604519696336, 110166623694262553144, 603627632302671271402, 3113599784368380443424.

$N = 96$.

$$C_{96} = A^{94}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}^2\,T_{27}\,T_{32}.$$

$$C_{96} \approx 150.7979088273759235729 42.$$

Coefficients: 1, 120, 7102, 276410, 7959470, 180903890, 3380953182, 53453690322, 730004489132, 8750903407768, 93263308461776, 892990121184088, 7749514094618040, 61412735607247164, 447343548214147520, 3012560261339688844, 18853534604519696338, 110166623694262553384, 603627632302671285606, 3113599784368380996244.

$N = 97$.

$$C_{97} = A^{95}\,\beta_2^{11}\,\beta_3^5\,\beta_4^5\,\beta_5^3\,\beta_6\,\beta_7\,\beta_8^3\,\beta_9^2\,\beta_{12}\,\beta_{16}^2\,T_{27}\,T_{32}.$$

$$C_{97} \approx 163.8321867560331593524354.$$

Coefficients: 1, 122, 7342, 290614, 8512292, 196823070, 3742775166, 60216149506, 836927788716, 10211274193814, 110771877183916, 1079623645502488, 9536954346517300, 76929265619217720, 570355546407373180, 3909033344772258424, 24894154262295690752, 147996519834525418892, 824855584289431517154, 4326880356023020421516.

$N = 98$.

$$\begin{aligned} C_{98} = A^{96}\, &\beta_2^{11}\, \beta_3^5\, \beta_4^5\, \beta_5^3\, \beta_6 \\ &\beta_7^2\, \beta_8^3\, \beta_9^2\, \beta_{12}\, \beta_{16}^2\, T_{27}\, T_{32}. \end{aligned}$$

$$C_{98} \approx 163.83218684824671045411 26.$$

Coefficients: 1, 122, 7342, 290614, 8512292, 196823070, 3742775166, 60216149508, 836927788960, 10211274208498, 110771877765144, 1079623662527072, 9536954740163440, 76929273104768052, 570355666839672192, 3909035018627835856, 24894174684844078380, 147996741378279786724, 824857743536722522130, 4326899429931713456116.

$N = 99$.

$$\begin{aligned} C_{99} = A^{97}\, &\beta_2^{11}\, \beta_3^6\, \beta_4^5\, \beta_5^3\, \beta_6 \\ &\beta_7^2\, \beta_8^3\, \beta_9^2\, \beta_{12}\, \beta_{16}^2\, T_{27}\, T_{32}. \end{aligned}$$

$$C_{99} \approx 163.858629205129007373841.$$

Coefficients: 1, 122, 7342, 290616, 8512536, 196837754, 3743356394, 60233174092, 837321435100, 10218759758830, 110892310064160, 1081297518104992, 9557377288580438, 77150816860298584, 572514914164741020, 3928108928108743964, 25048033231070639068, 149137452712352777248, 832675813581463744174, 4376687779421833911892.

$N = 100$.

$$\begin{aligned} C_{100} = A^{98}\, &\beta_2^{11}\, \beta_3^6\, \beta_4^5\, \beta_5^3\, \beta_6\, \beta_7^2 \\ &\beta_8^3\, \beta_9^2\, \beta_{10}\, \beta_{12}\, \beta_{16}^2\, T_{27}\, T_{32}. \end{aligned}$$

$$C_{100} \approx 163.858629205136450163 9979.$$

Coefficients: 1, 122, 7342, 290616, 8512536, 196837754, 3743356394, 60233174092, 837321435100, 10218759758830, 110892310064162, 1081297518105236, 9557377288595122, 77150816860879816, 572514914181766092, 3928108928502419472, 25048033238557351856, 149137452832819125432, 832675815256106614374, 4376687799859353429552.

Relation to Simon Plouffe's paper

Simon Plouffe's paper *Numbers in the base* e^{π} studies high-precision evaluations of series at points of the form $e^{-m\pi}$ and then applies integer-relation detection (for instance, `lindep`/`algdep`) to guess exact formulas. The present family fits that framework naturally, because each value $\Theta_{G_N}(e^{-\pi})$ is already an explicit finite product of classical theta constants T_d. In that sense, the Plouffe-style numerical step is useful as a confirmation and discovery heuristic, but in the present setting the exact factorization follows directly from the proved triangular Gram decomposition.

27 A quadratic Pratt-poset decomposition for $1/\operatorname{rad}(n)^2$

Let

$$g(n) := \frac{n}{\operatorname{rad}(n)}.$$

Recall that on the Pratt poset $(\mathbb{N}, \leq_P)$ one has the identity

$$n = \sum_{d \leq_P n} \frac{d}{\operatorname{rad}(d)}.$$

Define

$$h(d) := \frac{d}{\operatorname{rad}(d)}.$$

Then

$$n = \sum_{d \leq_P n} h(d).$$

Squaring this identity gives

$$n^2 = \left(\sum_{d \leq_P n} h(d) \right)^2 = \sum_{d,e \leq_P n} h(d)h(e).$$

Grouping the terms according to the join $d \vee_P e$ in the Pratt poset, define

$$C_2(t) := \sum_{d \vee_P e = t} h(d)h(e).$$

Then

$$n^2 = \sum_{t \leq_P n} C_2(t).$$

Equivalently, by Möbius inversion on the Pratt poset,

$$C_2(n) = \sum_{d \leq_P n} \mu_P(d, n)\, d^2,$$

where μ_P denotes the incidence Möbius function of $(\mathbb{N}, \leq_P)$.

The main point is that $C_2(n)$ admits a completely explicit closed formula.

Proposition 15. *For every $n \geq 1$,*

$$C_2(n) = \prod_{p^e \| n} (2p - 1)p^{2e-2}.$$

In particular, $C_2(n) > 0$ for all n.

Proof. Define

$$\widetilde{C}_2(n) := \prod_{p^e \| n} (2p - 1)p^{2e-2}.$$

We prove that

$$\sum_{d \le_P n} \widetilde{C}_2(d) = n^2$$

for all n. By uniqueness of Möbius inversion, this implies $\widetilde{C}_2(n) = C_2(n)$.

Write

$$n = uP^e,$$

where P is the largest prime divisor of n, $e \ge 1$, and $P \nmid u$. The structure of the Pratt poset implies that every $d \le_P n$ can be written uniquely as

$$d = P^k c \qquad (0 \le k \le e),$$

with

$$c \le_P u(P-1)^{e-k}.$$

Therefore

$$\sum_{d \le_P n} \widetilde{C}_2(d) = \sum_{k=0}^{e} \sum_{c \le_P u(P-1)^{e-k}} \widetilde{C}_2(P^k c).$$

Since $P \nmid c$, the function $\widetilde{C}_2$ is multiplicative here, so

$$\widetilde{C}_2(P^k c) = \widetilde{C}_2(P^k)\widetilde{C}_2(c).$$

Hence

$$\sum_{d \le_P n} \widetilde{C}_2(d) = \sum_{k=0}^{e} \widetilde{C}_2(P^k) \sum_{c \le_P u(P-1)^{e-k}} \widetilde{C}_2(c).$$

By induction on n,

$$\sum_{c \le_P u(P-1)^{e-k}} \widetilde{C}_2(c) = \left(u(P-1)^{e-k}\right)^2.$$

Thus

$$\sum_{d \le_P n} \widetilde{C}_2(d) = u^2 \sum_{k=0}^{e} \widetilde{C}_2(P^k)(P-1)^{2(e-k)}.$$

It remains to verify that

$$\sum_{k=0}^{e} \widetilde{C}_2(P^k)(P-1)^{2(e-k)} = P^{2e}.$$

Now

$$\widetilde{C}_2(1) = 1, \qquad \widetilde{C}_2(P^k) = (2P-1)P^{2k-2} \quad (k \ge 1).$$

So

$$\sum_{k=0}^{e} \widetilde{C}_2(P^k)(P-1)^{2(e-k)} = (P-1)^{2e} + (2P-1)\sum_{k=1}^{e} P^{2k-2}(P-1)^{2(e-k)}.$$

Set
$$r := \frac{(P-1)^2}{P^2}.$$
Then
$$\sum_{k=1}^{e} P^{2k-2}(P-1)^{2(e-k)} = P^{2e-2}\sum_{j=0}^{e-1} r^j = P^{2e-2}\frac{1-r^e}{1-r}.$$
Since
$$1-r = \frac{P^2-(P-1)^2}{P^2} = \frac{2P-1}{P^2},$$
we get
$$(2P-1)\sum_{k=1}^{e} P^{2k-2}(P-1)^{2(e-k)} = P^{2e}(1-r^e).$$
Also
$$(P-1)^{2e} = P^{2e}r^e.$$
Therefore
$$(P-1)^{2e} + (2P-1)\sum_{k=1}^{e} P^{2k-2}(P-1)^{2(e-k)} = P^{2e}r^e + P^{2e}(1-r^e) = P^{2e}.$$
Hence
$$\sum_{d\le_P n} \widetilde{C}_2(d) = u^2P^{2e} = n^2,$$
as required. □

As a consequence, if
$$g(m) = \frac{m}{\operatorname{rad}(m)},$$
then
$$g(m)^2 = \sum_{d\le_P g(m)} C_2(d),$$
and therefore
$$\frac{1}{\operatorname{rad}(m)^2} = \frac{g(m)^2}{m^2} = \frac{1}{m^2}\sum_{d\le_P g(m)}\prod_{p^e\| d}(2p-1)p^{2e-2}.$$

In particular, for
$$m = xy(x+y), \qquad x = \frac{a}{\gcd(a,b)}, \qquad y = \frac{b}{\gcd(a,b)},$$
we obtain the explicit positive Pratt-poset decomposition
$$\frac{1}{\operatorname{rad}(xy(x+y))^2} = \frac{1}{(xy(x+y))^2}\sum_{d\le_P g(xy(x+y))}\prod_{p^e\| d}(2p-1)p^{2e-2}.$$
This gives a completely explicit expansion with strictly positive coefficients.

28 Application to sum representation of $1/\operatorname{rad}(n)^k$

Let

$$g(n) := \frac{n}{\operatorname{rad}(n)}.$$

In the Pratt-poset Möbius inversion discussion, we proved the basic identity

$$\sum_{d \leq_P n} \frac{d}{\operatorname{rad}(d)} = n.$$

Equivalently, if

$$C_1(d) := \frac{d}{\operatorname{rad}(d)},$$

then

$$\sum_{d \leq_P n} C_1(d) = n.$$

The quadratic case suggests a natural higher-power generalization.

Proposition 16. *For each integer $k \geq 1$, define*

$$C_k(n) := \prod_{p^e \| n} \Big(p^k - (p-1)^k\Big) p^{k(e-1)} \qquad (n \geq 1),$$

with the empty product for $n = 1$ equal to 1. Then for every $n \geq 1$ one has

$$\sum_{d \leq_P n} C_k(d) = n^k.$$

Consequently,

$$g(m)^k = \sum_{d \leq_P g(m)} C_k(d) \qquad (m \geq 1),$$

and therefore

$$\frac{1}{\operatorname{rad}(m)^k} = \frac{1}{m^k} \sum_{d \leq_P g(m)} \prod_{p^e \| d} \Big(p^k - (p-1)^k\Big) p^{k(e-1)}.$$

Proof. Fix $k \geq 1$, and write

$$\widetilde{C}_k(n) := \prod_{p^e \| n} \Big(p^k - (p-1)^k\Big) p^{k(e-1)}.$$

We prove that

$$\sum_{d \leq_P n} \widetilde{C}_k(d) = n^k$$

for all $n \geq 1$. By Möbius inversion on the Pratt poset, this will identify $\widetilde{C}_k$ with the unique coefficient function attached to the expansion of n^k.

We argue by induction on n. The case $n = 1$ is immediate. Now write

$$n = uP^e,$$

where P is the largest prime divisor of n, $e \geq 1$, and $P \nmid u$. As in the quadratic case, every $d \leq_P n$ can be written uniquely as

$$d = P^j c \qquad (0 \leq j \leq e),$$

with

$$c \leq_P u(P-1)^{e-j}.$$

Hence

$$\sum_{d \leq_P n} \widetilde{C}_k(d) = \sum_{j=0}^{e} \sum_{c \leq_P u(P-1)^{e-j}} \widetilde{C}_k(P^j c).$$

Since $P \nmid c$, the function $\widetilde{C}_k$ is multiplicative on these terms, so

$$\widetilde{C}_k(P^j c) = \widetilde{C}_k(P^j)\widetilde{C}_k(c).$$

Therefore

$$\sum_{d \leq_P n} \widetilde{C}_k(d) = \sum_{j=0}^{e} \widetilde{C}_k(P^j) \sum_{c \leq_P u(P-1)^{e-j}} \widetilde{C}_k(c).$$

By the induction hypothesis,

$$\sum_{c \leq_P u(P-1)^{e-j}} \widetilde{C}_k(c) = \Big(u(P-1)^{e-j}\Big)^k.$$

Thus

$$\sum_{d \leq_P n} \widetilde{C}_k(d) = u^k \sum_{j=0}^{e} \widetilde{C}_k(P^j)(P-1)^{k(e-j)}.$$

It remains to prove that

$$\sum_{j=0}^{e} \widetilde{C}_k(P^j)(P-1)^{k(e-j)} = P^{ke}.$$

Now

$$\widetilde{C}_k(1) = 1, \qquad \widetilde{C}_k(P^j) = \Big(P^k - (P-1)^k\Big)P^{k(j-1)} \quad (j \geq 1).$$

Hence

$$\sum_{j=0}^{e} \widetilde{C}_k(P^j)(P-1)^{k(e-j)} = (P-1)^{ke} + \Big(P^k - (P-1)^k\Big) \sum_{j=1}^{e} P^{k(j-1)}(P-1)^{k(e-j)}.$$

Set

$$r := \frac{(P-1)^k}{P^k}.$$

Then

$$\sum_{j=1}^{e} P^{k(j-1)}(P-1)^{k(e-j)} = P^{k(e-1)} \sum_{t=0}^{e-1} r^t = P^{k(e-1)} \frac{1-r^e}{1-r}.$$

But

$$1 - r = \frac{P^k - (P-1)^k}{P^k},$$

so

$$\Big(P^k - (P-1)^k\Big) \sum_{j=1}^{e} P^{k(j-1)}(P-1)^{k(e-j)} = P^{ke}(1-r^e).$$

Also,

$$(P-1)^{ke} = P^{ke} r^e.$$

Therefore

$$(P-1)^{ke} + \Big(P^k - (P-1)^k\Big) \sum_{j=1}^{e} P^{k(j-1)}(P-1)^{k(e-j)} = P^{ke} r^e + P^{ke}(1-r^e) = P^{ke}.$$

This proves

$$\sum_{d \le_P n} \widetilde{C}_k(d) = u^k P^{ke} = n^k,$$

as required.

Finally, taking $n = g(m)$ gives

$$g(m)^k = \sum_{d \le_P g(m)} C_k(d).$$

Dividing by m^k yields

$$\frac{1}{\operatorname{rad}(m)^k} = \frac{g(m)^k}{m^k} = \frac{1}{m^k} \sum_{d \le_P g(m)} C_k(d),$$

which is the claimed sum representation. □

Remark 17. The cases $k = 1$ and $k = 2$ recover the earlier formulas exactly. Indeed,

$$C_1(n) = \prod_{p^e \| n} p^{e-1} = \frac{n}{\operatorname{rad}(n)},$$

while

$$C_2(n) = \prod_{p^e \| n} \Big(p^2 - (p-1)^2\Big) p^{2(e-1)} = \prod_{p^e \| n} (2p-1) p^{2e-2}.$$

Thus the quadratic decomposition is the special case $k = 2$ of the general formula.

In particular, for

$$m = xy(x+y), \qquad x = \frac{a}{\gcd(a,b)}, \qquad y = \frac{b}{\gcd(a,b)},$$

one obtains the explicit positive Pratt-poset expansion

$$\frac{1}{\mathrm{rad}(xy(x+y))^k} = \frac{1}{(xy(x+y))^k} \sum_{d \le_P g(xy(x+y))} \prod_{p^e \| d} \Big(p^k - (p-1)^k\Big) p^{k(e-1)}.$$

All coefficients in this expansion are strictly positive.

29 Formulas for $C_n(d)$

In this section we collect several useful formulas for the coefficients

$$C_n(d) := \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)}.$$

These formulas are all elementary, but they reveal different aspects of the same object: local structure at prime powers, multiplicativity, binomial expansions, squarefree corrections, Euler products, and asymptotic behavior. The guiding principle is that $C_n(d)$ behaves like d^n, but with a correction factor depending only on the distinct prime divisors of d.

Throughout this section, $n \ge 1$ and $d \ge 1$ are integers. We use the standard notation

$$d = \prod_{p^e \| d} p^e, \qquad \mathrm{rad}(d) := \prod_{p \mid d} p, \qquad \omega(d) := \#\{p : p \mid d\}.$$

29.1 The defining formula and the local prime-power formula

We begin with the definition

$$C_n(d) = \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)}.$$

This is already a factorization over the prime powers dividing d. In particular, for a prime power $d = p^e$ one immediately gets the local formula

$$\boxed{C_n(p^e) = \Big(p^n - (p-1)^n\Big) p^{n(e-1)}.}$$

Equivalently,

$$C_n(p^e) = p^{ne} \left(1 - \left(1 - \frac{1}{p}\right)^n\right).$$

This second form is often more transparent: it shows that the entire dependence on the exponent e is the obvious factor p^{ne}, while the correction term

$$1 - \left(1 - \frac{1}{p}\right)^n$$

depends only on the prime p and the parameter n.

Thus the prime-power case already suggests the general philosophy:

$$C_n(d) = d^n \times (\text{correction factor depending only on the distinct prime divisors of } d).$$

29.2 Multiplicativity

The definition makes it immediate that C_n is multiplicative.

Proposition 17. *For each fixed $n \geq 1$, the function $d \mapsto C_n(d)$ is multiplicative: if $(a, b) = 1$, then*

$$C_n(ab) = C_n(a)C_n(b).$$

Proof. If $(a, b) = 1$, then the sets of primes dividing a and b are disjoint. Writing

$$a = \prod_{p^{e_p} \| a} p^{e_p}, \qquad b = \prod_{q^{f_q} \| b} q^{f_q},$$

we have

$$ab = \prod_{p^{e_p} \| a} p^{e_p} \prod_{q^{f_q} \| b} q^{f_q},$$

and therefore

$$C_n(ab) = \prod_{p^{e_p} \| a} \Big(p^n - (p-1)^n\Big) p^{n(e_p - 1)} \prod_{q^{f_q} \| b} \Big(q^n - (q-1)^n\Big) q^{n(f_q - 1)}.$$

This is exactly $C_n(a)C_n(b)$. □

As usual, multiplicativity means that one can understand C_n completely from the prime-power values.

29.3 The global product formula

We now rewrite the definition in a form which separates the dominant term d^n from a squarefree correction factor.

Proposition 18. *For every $d \geq 1$,*

$$\boxed{C_n(d) = \frac{d^n}{\operatorname{rad}(d)^n} \prod_{p \mid d} \Big(p^n - (p-1)^n\Big).}$$

Equivalently,

$$\boxed{C_n(d) = d^n \prod_{p \mid d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right).}$$

Proof. From the definition,

$$C_n(d) = \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)}.$$

Now

$$\prod_{p^e \| d} p^{n(e-1)} = \prod_{p^e \| d} \frac{p^{ne}}{p^n} = \frac{d^n}{\mathrm{rad}(d)^n}.$$

Hence

$$C_n(d) = \frac{d^n}{\mathrm{rad}(d)^n} \prod_{p | d} \Big(p^n - (p-1)^n\Big).$$

Factoring out p^n from each term in the product gives

$$p^n - (p-1)^n = p^n \left(1 - \left(1 - \frac{1}{p}\right)^n\right),$$

so that

$$\prod_{p|d} \Big(p^n - (p-1)^n\Big) = \mathrm{rad}(d)^n \prod_{p|d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right).$$

Substituting into the previous formula yields

$$C_n(d) = d^n \prod_{p|d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right).$$

□

This formula is one of the most useful ones. It shows immediately that:

- $C_n(d) > 0$ for all d;
- $C_n(d) \le d^n$ for all d, since each factor satisfies

$$0 < 1 - \left(1 - \frac{1}{p}\right)^n \le 1;$$

- the ratio $C_n(d)/d^n$ depends only on the squarefree kernel $\mathrm{rad}(d)$.

Thus the non-squarefree part of d contributes only through the factor d^n itself.

29.4 Binomial expansion at prime powers

The term $p^n - (p-1)^n$ admits an explicit binomial expansion, which gives a polynomial formula in the prime p.

Proposition 19. *For every prime power p^e,*

$$\boxed{C_n(p^e) = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} p^{ne-j}.}$$

Equivalently,

$$\boxed{C_n(p^e) = p^{n(e-1)} \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} p^{n-j}.}$$

Proof. By the binomial theorem,

$$(p-1)^n = \sum_{j=0}^{n} \binom{n}{j} p^{n-j} (-1)^j.$$

Hence

$$p^n - (p-1)^n = -\sum_{j=1}^{n} \binom{n}{j} p^{n-j} (-1)^j = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} p^{n-j}.$$

Multiplying by $p^{n(e-1)}$ gives

$$C_n(p^e) = p^{n(e-1)} \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} p^{n-j} = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} p^{ne-j}.$$

□

This gives explicit local formulas for small n:

$$C_1(p^e) = p^{e-1},$$

$$C_2(p^e) = (2p-1)p^{2e-2},$$
$$C_3(p^e) = (3p^2 - 3p + 1)p^{3e-3},$$
$$C_4(p^e) = (4p^3 - 6p^2 + 4p - 1)p^{4e-4}.$$

These are often useful for experimentation and for proving basic inequalities.

29.5 Special cases for small n

It is helpful to record the first few instances explicitly.

Case $n = 1$. Since

$$p - (p-1) = 1,$$

we get

$$C_1(p^e) = p^{e-1}.$$

Therefore

$$\boxed{C_1(d) = \prod_{p^e \| d} p^{e-1} = \frac{d}{\operatorname{rad}(d)}.}$$

Case $n = 2$. Since

$$p^2 - (p-1)^2 = 2p - 1,$$

we obtain

$$C_2(p^e) = (2p-1)p^{2e-2},$$

and hence

$$\boxed{C_2(d) = d^2 \prod_{p|d} \frac{2p-1}{p^2}.}$$

Case $n = 3$. Since

$$p^3 - (p-1)^3 = 3p^2 - 3p + 1,$$

we get

$$C_3(p^e) = (3p^2 - 3p + 1)p^{3e-3},$$

and therefore

$$\boxed{C_3(d) = d^3 \prod_{p|d} \frac{3p^2 - 3p + 1}{p^3}.}$$

In general, for fixed n, each local factor is a degree-$(n-1)$ polynomial in p times $p^{n(e-1)}$.

29.6 Squarefree correction factors

A conceptually useful way to package the previous formula is to isolate the squarefree correction factor.

Definition 6. For fixed $n \geq 1$, define

$$\kappa_n(m) := \prod_{p|m} \left(1 - \left(1 - \frac{1}{p}\right)^n\right)$$

for squarefree m (or more generally for any $m \geq 1$, noting that the right-hand side depends only on $\operatorname{rad}(m)$).

Then we may write

$$\boxed{C_n(d) = d^n \, \kappa_n(\operatorname{rad}(d)).}$$

This shows explicitly that all deviation from the pure monomial d^n is controlled by the squarefree part of d. In particular, if two numbers d_1 and d_2 have the same squarefree kernel, then

$$\frac{C_n(d_1)}{d_1^n} = \frac{C_n(d_2)}{d_2^n}.$$

29.7 A divisor-expansion point of view

Since

$$1-\left(1-\frac{1}{p}\right)^n=\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}p^{-j},$$

the product

$$\prod_{p\mid d}\left(1-\left(1-\frac{1}{p}\right)^n\right)$$

is a finite expansion in inverse powers of the primes dividing d. Thus $C_n(d)$ may be expressed as a finite linear combination of terms of the shape d^n/m, where m runs through divisors of powers of $\mathrm{rad}(d)$.

More explicitly, choosing for each prime $p\mid d$ an index $j_p\in\{1,\dots,n\}$, one gets

$$C_n(d)=d^n\sum_{(j_p)_{p\mid d}}\prod_{p\mid d}\left((-1)^{j_p+1}\binom{n}{j_p}p^{-j_p}\right).$$

Equivalently,

$$C_n(d)=\sum_{(j_p)_{p\mid d}}\left(\prod_{p\mid d}(-1)^{j_p+1}\binom{n}{j_p}\right)\frac{d^n}{\prod_{p\mid d}p^{j_p}}.$$

This is not always the most elegant formula for computation, but it makes clear that $C_n(d)$ is built from finitely many divisor-like corrections to the main term d^n.

29.8 Dirichlet series and Euler products

Since C_n is multiplicative, one naturally studies its Dirichlet series

$$F_n(s):=\sum_{d\geq 1}\frac{C_n(d)}{d^s}.$$

This admits a clean Euler product.

Proposition 20. *For $\Re(s)$ sufficiently large,*

$$\boxed{\sum_{d\geq 1}\frac{C_n(d)}{d^s}=\prod_p\frac{1-(p-1)^n p^{-s}}{1-p^{n-s}}.}$$

Equivalently,

$$\boxed{\sum_{d\geq 1}\frac{C_n(d)}{d^s}=\zeta(s-n)\prod_p\left(1-(p-1)^n p^{-s}\right).}$$

Proof. By multiplicativity,

$$\sum_{d\geq 1} \frac{C_n(d)}{d^s} = \prod_p \left(1 + \sum_{e\geq 1} \frac{C_n(p^e)}{p^{es}}\right).$$

Using the local formula,

$$C_n(p^e) = \Big(p^n - (p-1)^n\Big)p^{n(e-1)},$$

we compute

$$\sum_{e\geq 1} \frac{C_n(p^e)}{p^{es}} = \sum_{e\geq 1} \Big(p^n - (p-1)^n\Big)p^{n(e-1)-es}.$$

Factor out the term independent of e:

$$= \Big(p^n - (p-1)^n\Big)p^{-n} \sum_{e\geq 1} p^{e(n-s)}.$$

Since

$$\sum_{e\geq 1} p^{e(n-s)} = \frac{p^{n-s}}{1-p^{n-s}},$$

we get

$$\sum_{e\geq 1} \frac{C_n(p^e)}{p^{es}} = \left(1 - \left(1 - \frac{1}{p}\right)^n\right) \frac{p^{n-s}}{1-p^{n-s}}.$$

Therefore the Euler factor is

$$1 + \sum_{e\geq 1} \frac{C_n(p^e)}{p^{es}} = 1 + \left(1 - \left(1 - \frac{1}{p}\right)^n\right) \frac{p^{n-s}}{1-p^{n-s}}.$$

Putting over the common denominator $1 - p^{n-s}$,

$$= \frac{1 - p^{n-s} + \left(1 - \left(1-\frac{1}{p}\right)^n\right) p^{n-s}}{1-p^{n-s}} = \frac{1 - \left(1-\frac{1}{p}\right)^n p^{n-s}}{1-p^{n-s}}.$$

Now

$$\left(1 - \frac{1}{p}\right)^n p^{n-s} = (p-1)^n p^{-s},$$

so

$$1 + \sum_{e\geq 1} \frac{C_n(p^e)}{p^{es}} = \frac{1-(p-1)^n p^{-s}}{1-p^{n-s}}.$$

Taking the product over all primes gives

$$\sum_{d\geq 1} \frac{C_n(d)}{d^s} = \prod_p \frac{1-(p-1)^n p^{-s}}{1-p^{n-s}}.$$

Finally, since

$$\prod_p \frac{1}{1-p^{n-s}} = \zeta(s-n),$$

we obtain

$$\sum_{d\geq 1} \frac{C_n(d)}{d^s} = \zeta(s-n)\prod_p \left(1-(p-1)^n p^{-s}\right).$$

□

This is perhaps the most "Eulerian" formula in the whole theory. It expresses C_n simultaneously as:

- a local prime-power object,
- a multiplicative function,
- an Euler product,
- and a deformation of the shifted zeta function $\zeta(s-n)$.

29.9 Basic inequalities

The product formula immediately yields elementary bounds.

Proposition 21. *For every $d \geq 1$,*

$$0 < C_n(d) \leq d^n.$$

More precisely,

$$\boxed{C_n(d) = d^n \prod_{p\mid d}\left(1-\left(1-\frac{1}{p}\right)^n\right) \leq d^n.}$$

Proof. Each factor satisfies

$$0 < 1-\left(1-\frac{1}{p}\right)^n \leq 1.$$

Multiplying over $p \mid d$ gives

$$0 < \prod_{p\mid d}\left(1-\left(1-\frac{1}{p}\right)^n\right) \leq 1.$$

Multiplying by d^n yields the claim. □

A second useful bound comes from Bernoulli's inequality: for $0 \leq x \leq 1$,

$$(1-x)^n \geq 1-nx,$$

hence

$$1-(1-x)^n \le nx.$$

Substituting $x = 1/p$ gives

$$1-\left(1-\frac{1}{p}\right)^n \le \frac{n}{p}.$$

Therefore:

Proposition 22. *For every $d \ge 1$,*

$$\boxed{C_n(d) \le d^n \frac{n^{\omega(d)}}{\operatorname{rad}(d)}.}$$

More sharply,

$$\boxed{C_n(d) \le d^n \prod_{p|d} \min\left(1, \frac{n}{p}\right).}$$

Proof. Applying

$$1-\left(1-\frac{1}{p}\right)^n \le \frac{n}{p}$$

termwise in the product formula yields

$$C_n(d) = d^n \prod_{p|d}\left(1-\left(1-\frac{1}{p}\right)^n\right) \le d^n \prod_{p|d} \frac{n}{p} = d^n \frac{n^{\omega(d)}}{\operatorname{rad}(d)}.$$

Since each factor also satisfies

$$1-\left(1-\frac{1}{p}\right)^n \le 1,$$

we may combine the two upper bounds to get

$$1-\left(1-\frac{1}{p}\right)^n \le \min\left(1, \frac{n}{p}\right),$$

and thus

$$C_n(d) \le d^n \prod_{p|d} \min\left(1, \frac{n}{p}\right).$$

□

These inequalities are often much more convenient in applications than the exact formula.

29.10 A symmetric upper bound via the geometric mean of the prime divisors

Another useful upper bound is obtained by observing that, for fixed product of the distinct prime divisors, the correction factor is maximized when the prime divisors are all equal.

Let

$$f_n(x) := 1 - \left(1 - \frac{1}{x}\right)^n, \qquad x \ge 1.$$

Then

$$C_n(d) = d^n \prod_{p \mid d} f_n(p).$$

If $\omega(d) = r$ and $\operatorname{rad}(d) = s$, then the product of the distinct prime divisors of d equals s, and one may show that

$$\prod_{i=1}^{r} f_n(x_i)$$

under the constraint $x_1 \cdots x_r = s$ is maximized when all x_i are equal. Applying this with $x_i = p_i$ the distinct primes dividing d yields:

Proposition 23. *For every $d \ge 1$,*

$$\boxed{C_n(d) \le d^n \left(1 - \left(1 - \operatorname{rad}(d)^{-1/\omega(d)}\right)^n\right)^{\omega(d)}.}$$

This bound is exact for prime powers (where $\omega(d) = 1$), and is often much sharper than the rough estimate

$$C_n(d) \le d^n \, \frac{n^{\omega(d)}}{\operatorname{rad}(d)}.$$

29.11 Asymptotics for fixed d and large n

The product formula makes the large-n behavior immediate.

Proposition 24. *For each fixed $d \ge 1$,*

$$\boxed{C_n(d) \sim d^n \qquad (n \to \infty).}$$

Proof. Fix d. Then the set of primes dividing d is finite. For each such prime p,

$$\left(1 - \frac{1}{p}\right)^n \to 0 \qquad (n \to \infty),$$

hence

$$1 - \left(1 - \frac{1}{p}\right)^n \to 1.$$

Therefore

$$\prod_{p \mid d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right) \to 1.$$

Using

$$C_n(d) = d^n \prod_{p \mid d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right),$$

we obtain

$$C_n(d) \sim d^n.$$

□

Thus for fixed d, the correction factor tends to 1 as n grows.

29.12 Asymptotics for fixed n and large primes

If p is large compared to n, then

$$\left(1 - \frac{1}{p}\right)^n = 1 - \frac{n}{p} + O_n\left(\frac{1}{p^2}\right),$$

so

$$1 - \left(1 - \frac{1}{p}\right)^n = \frac{n}{p} + O_n\left(\frac{1}{p^2}\right).$$

Hence for fixed n and large prime p,

$$C_n(p^e) = p^{ne}\left(\frac{n}{p} + O_n\left(\frac{1}{p^2}\right)\right) = n\, p^{ne-1} + O_n(p^{ne-2}).$$

More generally, if all distinct prime divisors of d are large compared to n, then

$$C_n(d) \approx d^n \frac{n^{\omega(d)}}{\operatorname{rad}(d)}.$$

So the rough bound

$$C_n(d) \leq d^n \frac{n^{\omega(d)}}{\operatorname{rad}(d)}$$

is not merely formal; it is asymptotically of the correct shape when the prime divisors are large.

29.13 Summary of the main formulas

For convenience, we collect the most important identities in one place.

$$
\begin{aligned}
C_n(d) &= \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)}, \\
C_n(p^e) &= \Big(p^n - (p-1)^n\Big) p^{n(e-1)}, \\
C_n(d) &= \frac{d^n}{\operatorname{rad}(d)^n} \prod_{p \mid d} \Big(p^n - (p-1)^n\Big), \\
C_n(d) &= d^n \prod_{p \mid d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right), \\
C_n(p^e) &= \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} p^{ne-j}, \\
\sum_{d \ge 1} \frac{C_n(d)}{d^s} &= \prod_p \frac{1 - (p-1)^n p^{-s}}{1 - p^{n-s}}, \\
\sum_{d \ge 1} \frac{C_n(d)}{d^s} &= \zeta(s-n) \prod_p \Big(1 - (p-1)^n p^{-s}\Big), \\
0 < C_n(d) &\le d^n, \\
C_n(d) &\le d^n \frac{n^{\omega(d)}}{\operatorname{rad}(d)}, \\
C_n(d) &\le d^n \left(1 - \left(1 - \operatorname{rad}(d)^{-1/\omega(d)}\right)^n\right)^{\omega(d)}, \\
C_n(d) &\sim d^n \qquad (n \to \infty,\ d \text{ fixed}).
\end{aligned}
$$

Taken together, these formulas show that $C_n(d)$ is a multiplicative deformation of the monomial d^n, with local factors controlled by the simple difference $p^n - (p-1)^n$. This makes $C_n(d)$ particularly suitable for Euler-product methods, asymptotic arguments, and structural comparisons with classical multiplicative functions.

30 Integral and differential formulas for $C_n(d)$

In this section we derive several integral and differential formulas for the coefficients

$$
C_n(d) := \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)}.
$$

These formulas complement the multiplicative and Euler-product identities from the previous section. They show that the coefficients $C_n(d)$ admit a very natural analytic interpretation in terms of moments over short intervals, logarithmic derivatives, and curvature properties in the parameter n.

The philosophy is simple: the local factor

$$p^n - (p-1)^n$$

should be viewed not only as a finite difference of monomials, but also as an integral of the derivative of the monomial t^n. This leads to integral representations which are particularly well suited for asymptotic estimates and for studying how $C_n(d)$ varies with n.

Throughout this section, $n \geq 1$ and $d \geq 1$ are integers, although several formulas extend naturally to real $n > 0$.

30.1 The basic integral identity

We begin with the elementary identity

$$a^n - b^n = n \int_b^a t^{n-1}\, dt \qquad (a > b > 0).$$

Applying this with $a = p$ and $b = p-1$, we obtain

$$p^n - (p-1)^n = n \int_{p-1}^{p} t^{n-1}\, dt.$$

This immediately yields an integral formula for $C_n(p^e)$.

Proposition 25 (Local integral formula)**.** *For every prime power p^e,*

$$\boxed{C_n(p^e) = n\, p^{n(e-1)} \int_{p-1}^{p} t^{n-1}\, dt.}$$

Equivalently, after scaling $t = pu$,

$$\boxed{C_n(p^e) = n\, p^{ne} \int_{1-1/p}^{1} u^{n-1}\, du.}$$

Proof. By definition,

$$C_n(p^e) = \Big(p^n - (p-1)^n\Big) p^{n(e-1)}.$$

Using

$$p^n - (p-1)^n = n \int_{p-1}^{p} t^{n-1}\, dt,$$

we obtain

$$C_n(p^e) = n\, p^{n(e-1)} \int_{p-1}^{p} t^{n-1}\, dt.$$

Now make the change of variables $t = pu$, so $dt = p\, du$, and as t runs from $p-1$ to p, u runs from $1 - 1/p$ to 1. Then

$$\int_{p-1}^{p} t^{n-1}\, dt = \int_{1-1/p}^{1} (pu)^{n-1} p\, du = p^n \int_{1-1/p}^{1} u^{n-1}\, du.$$

Substituting into the previous identity yields

$$C_n(p^e) = n\, p^{ne} \int_{1-1/p}^{1} u^{n-1}\, du.$$

$\square$

This formula already gives a useful interpretation: $C_n(p^e)$ is essentially the n-th power p^{ne} multiplied by an averaged $(n-1)$-st moment over the short interval $[1 - 1/p, 1]$.

30.2 A global product-integral formula

The local formula extends immediately to arbitrary d by multiplicativity.

Proposition 26 (Product-integral formula)**.** *For every $d \geq 1$,*

$$\boxed{C_n(d) = n^{\omega(d)} \prod_{p^e \| d} \left(p^{n(e-1)} \int_{p-1}^{p} t^{n-1}\, dt \right).}$$

Equivalently,

$$\boxed{C_n(d) = d^n\, n^{\omega(d)} \prod_{p \mid d} \int_{1-1/p}^{1} u^{n-1}\, du.}$$

Proof. Using the local integral formula for each prime power $p^e \| d$, we have

$$C_n(d) = \prod_{p^e \| d} C_n(p^e) = \prod_{p^e \| d} \left(n\, p^{n(e-1)} \int_{p-1}^{p} t^{n-1}\, dt \right).$$

Since there are $\omega(d)$ distinct primes dividing d, this becomes

$$C_n(d) = n^{\omega(d)} \prod_{p^e \| d} \left(p^{n(e-1)} \int_{p-1}^{p} t^{n-1}\, dt \right).$$

Now apply the scaling $t = pu$ separately at each prime:

$$\int_{p-1}^{p} t^{n-1}\, dt = p^n \int_{1-1/p}^{1} u^{n-1}\, du.$$

Thus

$$p^{n(e-1)} \int_{p-1}^{p} t^{n-1}\, dt = p^{n(e-1)} p^n \int_{1-1/p}^{1} u^{n-1}\, du = p^{ne} \int_{1-1/p}^{1} u^{n-1}\, du.$$

Taking the product over $p^e \parallel d$, we get

$$C_n(d) = n^{\omega(d)} \prod_{p^e \| d} \left(p^{ne} \int_{1-1/p}^{1} u^{n-1}\, du \right) = d^n\, n^{\omega(d)} \prod_{p|d} \int_{1-1/p}^{1} u^{n-1}\, du.$$

$\square$

Using the identity

$$1 - \left(1 - \frac{1}{p}\right)^n = n \int_{1-1/p}^{1} u^{n-1}\, du,$$

this product-integral formula is exactly equivalent to the previously established product formula

$$C_n(d) = d^n \prod_{p|d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right),$$

but it has a different flavor: it shows that $C_n(d)$ is built from moments over short intervals, one interval for each prime divisor of d.

30.3 A multiple-integral representation

It is sometimes convenient to combine the product of one-dimensional integrals into a single multiple integral.

Proposition 27 (Multiple-integral formula)**.** *For every $d \geq 1$,*

$$\boxed{C_n(d) = d^n\, n^{\omega(d)} \int_{\prod_{p|d}[1-1/p,\,1]} \prod_{p|d} u_p^{n-1}\, du_p.}$$

Equivalently,

$$\boxed{C_n(d) = n^{\omega(d)} \frac{d^n}{\mathrm{rad}(d)^n} \int_{\prod_{p|d}[p-1,\,p]} \prod_{p|d} t_p^{n-1}\, dt_p.}$$

Proof. This is simply the product-integral formula written as an integral over the Cartesian product of the intervals corresponding to the distinct prime divisors of d. Indeed,

$$\prod_{p|d} \int_{1-1/p}^{1} u^{n-1}\, du = \int_{\prod_{p|d}[1-1/p,\,1]} \prod_{p|d} u_p^{n-1}\, du_p.$$

Multiplying by $d^n n^{\omega(d)}$ gives the first formula. The second formula follows analogously from the unscaled version. $\square$

This formula makes the geometry of the coefficient particularly transparent: the domain of integration is a rectangular box, one side for each distinct prime dividing d, and the integrand is simply the monomial

$$\prod_{p|d} u_p^{n-1}.$$

30.4 An alternative short-interval form

The interval $[1 - 1/p, 1]$ can also be shifted to the interval $[0, 1/p]$ by writing $u = 1 - v$. This yields another useful integral form.

Proposition 28 (Shifted short-interval formula)**.** *For every $d \geq 1$,*

$$\boxed{C_n(d) = d^n\, n^{\omega(d)} \prod_{p|d} \int_0^{1/p} (1 - v)^{n-1}\, dv.}$$

Proof. Starting from

$$\int_{1-1/p}^{1} u^{n-1}\, du,$$

let $u = 1 - v$. Then $du = -dv$, and as u runs from $1 - 1/p$ to 1, v runs from $1/p$ to 0. Hence

$$\int_{1-1/p}^{1} u^{n-1}\, du = \int_0^{1/p} (1 - v)^{n-1}\, dv.$$

Substituting into the product-integral formula gives the claim. □

This form is especially useful for analytic estimates, because the interval of integration is now small and starts at the origin. For example, since

$$(1 - v)^{n-1} \leq e^{-(n-1)v} \qquad (0 \leq v \leq 1),$$

we immediately obtain exponential upper bounds.

30.5 A differential view: the logarithmic derivative

The product formula also allows us to regard n as a continuous real parameter and differentiate with respect to it. For fixed d, define

$$L_d(n) := \log C_n(d).$$

Using

$$C_n(d) = d^n \prod_{p|d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right),$$

we obtain

$$L_d(n) = n \log d + \sum_{p|d} \log \left(1 - \left(1 - \frac{1}{p}\right)^n\right).$$

Proposition 29 (Logarithmic derivative)**.** *For fixed $d \geq 1$ and real $n > 0$,*

$$\boxed{\frac{\partial}{\partial n} \log C_n(d) = \log d - \sum_{p|d} \frac{\left(1 - \frac{1}{p}\right)^n \log\left(\frac{p}{p-1}\right)}{1 - \left(1 - \frac{1}{p}\right)^n}.}$$

Consequently,

$$\boxed{\frac{\partial}{\partial n} C_n(d) = C_n(d) \left[\log d - \sum_{p|d} \frac{\left(1-\frac{1}{p}\right)^n \log\left(\frac{p}{p-1}\right)}{1-\left(1-\frac{1}{p}\right)^n} \right].}$$

Proof. Differentiating

$$L_d(n) = n \log d + \sum_{p|d} \log \left(1 - \left(1 - \frac{1}{p}\right)^n\right)$$

term by term gives

$$\frac{\partial}{\partial n} L_d(n) = \log d + \sum_{p|d} \frac{-\left(1-\frac{1}{p}\right)^n \log\left(1-\frac{1}{p}\right)}{1-\left(1-\frac{1}{p}\right)^n}.$$

Since

$$-\log\left(1 - \frac{1}{p}\right) = \log\left(\frac{p}{p-1}\right),$$

this becomes

$$\frac{\partial}{\partial n} L_d(n) = \log d - \sum_{p|d} \frac{\left(1-\frac{1}{p}\right)^n \log\left(\frac{p}{p-1}\right)}{1-\left(1-\frac{1}{p}\right)^n}.$$

Finally,

$$\frac{\partial}{\partial n} C_n(d) = C_n(d) \frac{\partial}{\partial n} \log C_n(d),$$

which gives the second formula. □

This identity makes precise the heuristic statement that $C_n(d)$ behaves more and more like d^n as n grows: the correction term is a finite sum of positive quantities which tends to 0 as $n \to \infty$.

30.6 Concavity in the parameter n

A particularly pleasant consequence of the logarithmic formula is that $C_n(d)$ is log-concave as a function of the real variable n.

Proposition 30 (Strict log-concavity)**.** *For each fixed $d \geq 2$, the function $n \mapsto \log C_n(d)$ is strictly concave on $(0, \infty)$. More precisely,*

$$\boxed{\frac{\partial^2}{\partial n^2} \log C_n(d) = -\sum_{p|d} \frac{\left(1-\frac{1}{p}\right)^n \left(\log\frac{p}{p-1}\right)^2}{\left(1-\left(1-\frac{1}{p}\right)^n\right)^2} < 0.}$$

Proof. Differentiate the logarithmic derivative. For a fixed prime $p \mid d$, write

$$a_p := 1 - \frac{1}{p} \in (0, 1).$$

Then the contribution of p to $\frac{\partial}{\partial n} \log C_n(d)$ is

$$-\frac{a_p^n(-\log a_p)}{1 - a_p^n}.$$

Since $-\log a_p = \log \frac{p}{p-1} > 0$, differentiating gives

$$-\left(\log \frac{p}{p-1}\right)^2 \frac{a_p^n}{(1 - a_p^n)^2}.$$

Summing over $p \mid d$ yields

$$\frac{\partial^2}{\partial n^2} \log C_n(d) = -\sum_{p \mid d} \frac{\left(1 - \frac{1}{p}\right)^n \left(\log \frac{p}{p-1}\right)^2}{\left(1 - \left(1 - \frac{1}{p}\right)^n\right)^2}.$$

Each summand is strictly negative, so the whole sum is strictly negative. □

Thus, although $C_n(d)$ grows quickly with n, it does so in a highly regular way: its logarithm bends downward.

30.7 Asymptotics from the differential point of view

The integral and differential formulas make the asymptotic relation

$$C_n(d) \sim d^n \qquad (n \to \infty)$$

particularly transparent.

Indeed, from the product formula,

$$\frac{C_n(d)}{d^n} = \prod_{p \mid d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right),$$

and every factor tends to 1 as $n \to \infty$. Equivalently, from the logarithmic derivative,

$$\frac{\partial}{\partial n} \log C_n(d) = \log d + o(1) \qquad (n \to \infty),$$

so the slope of $\log C_n(d)$ approaches the slope of $\log d^n = n \log d$.

One may also recover the same asymptotic from the integral formula: for fixed p,

$$n \int_{1-1/p}^{1} u^{n-1}\, du = 1 - \left(1 - \frac{1}{p}\right)^{n} \longrightarrow 1.$$

Since only finitely many primes divide d, the product tends to 1.

30.8 Large-prime asymptotics

The integral formulas also give another route to the approximation

$$C_n(d) \approx d^n \frac{n^{\omega(d)}}{\mathrm{rad}(d)}$$

when the prime divisors of d are large compared to n.

Indeed, for fixed n and large p,

$$\int_0^{1/p} (1-v)^{n-1}\, dv = \frac{1}{p} + O_n\!\left(\frac{1}{p^2}\right),$$

because $(1-v)^{n-1} = 1 + O_n(v)$ uniformly on $[0, 1/p]$. Therefore

$$n \int_0^{1/p} (1-v)^{n-1}\, dv = \frac{n}{p} + O_n\!\left(\frac{1}{p^2}\right).$$

Substituting into the shifted short-interval formula gives

$$\frac{C_n(d)}{d^n} = \prod_{p \mid d} \left(\frac{n}{p} + O_n\!\left(\frac{1}{p^2}\right)\right),$$

which heuristically leads to

$$C_n(d) \approx d^n \frac{n^{\omega(d)}}{\mathrm{rad}(d)}.$$

Thus the integral representation naturally explains the rough upper bound

$$C_n(d) \le d^n \frac{n^{\omega(d)}}{\mathrm{rad}(d)},$$

and shows why this estimate has the correct shape when all relevant primes are large.

30.9 Finite-difference interpretation

The local factor

$$p^n - (p-1)^n$$

may also be read as a forward difference of the monomial x^n:

$$p^n - (p-1)^n = \Delta(x^n)\Big|_{x=p-1},$$

where

$$\Delta f(x) := f(x+1) - f(x).$$

Hence

$$\boxed{C_n(p^e) = p^{n(e-1)}\,\Delta(x^n)\Big|_{x=p-1}.}$$

This makes it clear that the local building blocks of $C_n(d)$ are simultaneously:

- differences of monomials,
- integrals of derivatives of monomials,
- and moments over short intervals.

In this sense, the coefficients $C_n(d)$ lie naturally at the meeting point of finite differences, calculus, and multiplicative number theory.

30.10 A generating-function remark

For a fixed prime p, the sequence $n \mapsto p^n - (p-1)^n$ also admits a simple generating function:

$$\sum_{n\geq 1}\Big(p^n - (p-1)^n\Big)z^n = \sum_{n\geq 1}(pz)^n - \sum_{n\geq 1}((p-1)z)^n = \frac{pz}{1-pz} - \frac{(p-1)z}{1-(p-1)z}.$$

Thus the local factors appearing in $C_n(p^e)$ can be studied not only via products and integrals, but also via ordinary generating functions in the parameter n. We do not pursue this here, but it may be useful in future work.

30.11 Summary of the analytic formulas

We conclude by collecting the most useful integral and differential formulas in one place:

$$C_n(p^e) = n\,p^{n(e-1)} \int_{p-1}^{p} t^{n-1}\,dt,$$

$$C_n(p^e) = n\,p^{ne} \int_{1-1/p}^{1} u^{n-1}\,du,$$

$$C_n(d) = n^{\omega(d)} \prod_{p^e \| d} \left(p^{n(e-1)} \int_{p-1}^{p} t^{n-1}\,dt \right),$$

$$C_n(d) = d^n\, n^{\omega(d)} \prod_{p \mid d} \int_{1-1/p}^{1} u^{n-1}\,du,$$

$$C_n(d) = d^n\, n^{\omega(d)} \prod_{p \mid d} \int_{0}^{1/p} (1-v)^{n-1}\,dv,$$

$$\frac{\partial}{\partial n} \log C_n(d) = \log d - \sum_{p \mid d} \frac{\left(1-\frac{1}{p}\right)^n \log\!\left(\frac{p}{p-1}\right)}{1-\left(1-\frac{1}{p}\right)^n},$$

$$\frac{\partial^2}{\partial n^2} \log C_n(d) = -\sum_{p \mid d} \frac{\left(1-\frac{1}{p}\right)^n \left(\log \frac{p}{p-1}\right)^2}{\left(1-\left(1-\frac{1}{p}\right)^n\right)^2} < 0.$$

These formulas show that the coefficients $C_n(d)$ are not merely multiplicative combinatorial objects. They also possess a rich analytic structure: each local factor is a short-interval moment, and the global coefficient is a product of such moments. In particular, they admit the kind of integral, differential, and asymptotic analysis which are entirely natural from the viewpoint analysis.

31 A Pratt–Möbius Operator for Polynomial Values

In this section we extend the decomposition

$$x^k = \sum_{d \le_P x} C_k(d)$$

from monomials to arbitrary polynomials. This yields a natural linear operator from the polynomial ring to the space of arithmetic functions, obtained by Möbius inversion on the Pratt poset.

31.1 The operator

Recall that the Pratt order is defined by

$$a \leq_P b \iff m_q(a) \leq m_q(b) \qquad \text{for all primes } q.$$

For each integer $k \geq 0$, define

$$C_k(n) := \prod_{p^e \| n} \Big(p^k - (p-1)^k\Big) p^{k(e-1)}.$$

For $k = 0$, this gives

$$C_0(1) = 1, \qquad C_0(n) = 0 \quad (n > 1),$$

since for every prime p one has

$$p^0 - (p-1)^0 = 1 - 1 = 0.$$

By the main decomposition theorem, for every $k \geq 0$ and every $x \in \mathbb{N}$,

$$x^k = \sum_{d \leq_P x} C_k(d).$$

Now let

$$P(T) = \sum_{k=0}^{m} a_k T^k \in R[T],$$

where R is any commutative ring. We define an arithmetic function

$$\mathcal{C}(P) : \mathbb{N} \to R$$

by

$$\mathcal{C}(P)(d) := \sum_{k=0}^{m} a_k \, C_k(d).$$

Thus $\mathcal{C}$ sends a polynomial to the corresponding linear combination of the basic arithmetic functions C_k.

31.2 The reconstruction formula

Theorem 6. *For every polynomial $P(T) \in R[T]$ and every $x \in \mathbb{N}$,*

$$P(x) = \sum_{d \leq_P x} \mathcal{C}(P)(d).$$

Proof. Write

$$P(T) = \sum_{k=0}^{m} a_k T^k.$$

Then

$$P(x) = \sum_{k=0}^{m} a_k x^k.$$

Using the decomposition of the monomials,

$$x^k = \sum_{d \leq_P x} C_k(d),$$

we obtain

$$P(x) = \sum_{k=0}^{m} a_k \sum_{d \leq_P x} C_k(d).$$

Since all sums are finite, we may interchange them:

$$P(x) = \sum_{d \leq_P x} \sum_{k=0}^{m} a_k C_k(d) = \sum_{d \leq_P x} \mathcal{C}(P)(d).$$

This proves the claim. □

31.3 Interpretation via Möbius inversion

Let

$$E(P)(x) := P(x)$$

be the usual evaluation operator from $R[T]$ to the space of arithmetic functions on $\mathbb{N}$. Let μ_P denote the Möbius function of the Pratt poset $(\mathbb{N}, \leq_P)$, and let $*_P$ denote convolution in the corresponding incidence algebra.

Then the operator $\mathcal{C}$ is precisely the Möbius transform of the evaluation function:

$$\mathcal{C}(P) = \mu_P *_P E(P).$$

Equivalently,

$$E(P) = \zeta_P *_P \mathcal{C}(P),$$

where ζ_P is the zeta function of the Pratt poset.

In particular, for the monomials T^k one has

$$\mathcal{C}(T^k) = C_k.$$

31.4 Basic properties

The operator $\mathcal{C}$ has several immediate properties.

Proposition 31. *The map*

$$\mathcal{C} : R[T] \to R^{\mathbb{N}}$$

is linear.

Proof. If

$$P(T) = \sum_k a_k T^k, \qquad Q(T) = \sum_k b_k T^k,$$

then for $\alpha, \beta \in R$,

$$\mathcal{C}(\alpha P + \beta Q)(d) = \sum_k (\alpha a_k + \beta b_k) C_k(d) = \alpha \sum_k a_k C_k(d) + \beta \sum_k b_k C_k(d).$$

Hence

$$\mathcal{C}(\alpha P + \beta Q) = \alpha \mathcal{C}(P) + \beta \mathcal{C}(Q).$$

□

Proposition 32. *Assume that R is an integral domain. Then $\mathcal{C}$ is injective.*

Proof. Suppose that $\mathcal{C}(P) = 0$. Then for every $x \in \mathbb{N}$,

$$P(x) = \sum_{d \leq_P x} \mathcal{C}(P)(d) = 0.$$

Thus P vanishes at every positive integer. Since a nonzero polynomial over an integral domain can have only finitely many roots, it follows that $P = 0$. Therefore $\mathcal{C}$ is injective. □

Remark 18. In general, $\mathcal{C}$ is not a ring homomorphism. One should not expect a simple identity of the form

$$\mathcal{C}(PQ) = \mathcal{C}(P)\mathcal{C}(Q),$$

whether pointwise or with respect to a standard convolution law. The operator $\mathcal{C}$ is linear, but not naturally multiplicative in the polynomial variable.

31.5 A natural basis viewpoint

Let

$$\mathbf{C}_k := \mathcal{C}(T^k).$$

Then

$$\mathbf{C}_k = C_k$$

for every $k \geq 0$, and for every polynomial

$$P(T) = \sum_{k=0}^{m} a_k T^k$$

we have

$$\mathcal{C}(P) = \sum_{k=0}^{m} a_k \, \mathbf{C}_k.$$

Thus the operator $\mathcal{C}$ sends the monomial basis

$$1, T, T^2, T^3, \ldots$$

of $R[T]$ to the family of arithmetic functions

$$C_0, C_1, C_2, C_3, \ldots.$$

Since $\mathcal{C}$ is injective, these functions are linearly independent over R whenever R is an integral domain.

Corollary 7. *If R is an integral domain and*

$$\sum_{k=0}^{m} a_k C_k = 0$$

as an arithmetic function, then

$$a_0 = a_1 = \cdots = a_m = 0.$$

Proof. The identity

$$\sum_{k=0}^{m} a_k C_k = 0$$

means precisely that

$$\mathcal{C}\left(\sum_{k=0}^{m} a_k T^k\right) = 0.$$

By injectivity of $\mathcal{C}$, we obtain

$$\sum_{k=0}^{m} a_k T^k = 0,$$

hence all coefficients vanish. □

32 Comparison with Other Operators from Polynomials to Arithmetic Functions

The operator $\mathcal{C}$ fits naturally into a general pattern: one starts with a polynomial P, evaluates it on the positive integers, and then applies an inversion procedure adapted to a chosen combinatorial structure. In the present case, that structure is the Pratt poset.

32.1 The ordinary evaluation operator

The most basic operator is the evaluation map

$$E : R[T] \to R^{\mathbb{N}}, \qquad E(P)(n) = P(n).$$

This operator remembers the values of the polynomial on the positive integers. The Pratt operator $\mathcal{C}$ is obtained from E by Möbius inversion on the Pratt poset:

$$\mathcal{C}(P) = \mu_P *_P E(P).$$

Thus $E(P)$ gives the accumulated values, while $\mathcal{C}(P)$ records the corresponding atomic contributions with respect to the order $\leq_P$.

32.2 Comparison with divisor Möbius inversion

On the divisibility poset $(\mathbb{N}, |)$, one often studies an arithmetic function f via its Möbius transform $g = \mu * f$, so that

$$f(n) = \sum_{d|n} g(d).$$

The operator $\mathcal{C}$ is the exact analogue of this construction, but with the Pratt order in place of divisibility:

$$P(x) = \sum_{d \leq_P x} \mathcal{C}(P)(d).$$

Hence $\mathcal{C}$ should be viewed as a Möbius transform adapted not to divisors, but to the Pratt structure of the integer x.

32.3 Comparison with Newton and Mahler expansions

Another classical way to pass from polynomials to arithmetic data is via finite differences. Every polynomial P admits a Newton expansion

$$P(x) = \sum_{k=0}^{m} (\Delta^k P)(0) \binom{x}{k},$$

where Δ is the forward difference operator.

This expansion is adapted to the total order

$$0 < 1 < 2 < \cdots,$$

and to the combinatorics of successive differences.

By contrast, the operator $\mathcal{C}$ is adapted to the partial order $(\mathbb{N}, \leq_P)$. In this sense, Newton expansion diagonalizes the difference structure of the integers, while the Pratt operator extracts the contributions attached to the Pratt lower set

$$\{d : d \leq_P x\}.$$

32.4 Incidence algebra viewpoint

More generally, if X is any locally finite poset and $f : X \to R$ is a function, then one may pass between f and its cumulative sums by using the zeta and Möbius functions of the incidence algebra of X.

The operator $\mathcal{C}$ is a concrete example of this general mechanism:

$$\mathcal{C}(P) = \mu_P *_P (x \mapsto P(x)).$$

Thus the construction is not ad hoc. It belongs naturally to the general theory of Möbius inversion on locally finite posets.

33 Why the Pratt Operator is Especially Natural

The operator $\mathcal{C}$ has three features that make it especially useful.

33.1 It is canonical

The definition depends only on the polynomial P and the intrinsic order structure of the Pratt poset. No auxiliary choices are involved.

33.2 It is injective

When R is an integral domain, the operator $\mathcal{C}$ loses no information. A polynomial is uniquely determined by its associated Pratt coefficient function.

33.3 It is explicit on monomials

The images of the monomials are given by a closed formula:

$$\mathcal{C}(T^k)(n) = C_k(n) = \prod_{p^e \| n} \Big(p^k - (p-1)^k\Big) p^{k(e-1)}.$$

This is particularly valuable, since in many Möbius-inversion settings one knows the abstract existence of the inverse transform, but not such a concrete formula on a natural basis.

33.4 A concise formulation

The discussion above may be summarized as follows.

Theorem 7. *Let R be an integral domain. Let*

$$E : R[T] \to R^{\mathbb{N}}, \qquad E(P)(x) = P(x),$$

be the evaluation map, and let μ_P be the Möbius function of the Pratt poset $(\mathbb{N}, \leq_P)$. Then

$$\mathcal{C}(P) := \mu_P *_P E(P)$$

defines a linear injective operator

$$\mathcal{C} : R[T] \to R^{\mathbb{N}}$$

such that, for every $P \in R[T]$ and every $x \in \mathbb{N}$,

$$P(x) = \sum_{d \leq_P x} \mathcal{C}(P)(d).$$

Moreover,

$$\mathcal{C}(T^k) = C_k \qquad (k \geq 0).$$

This theorem shows that the family $C_0, C_1, C_2, \ldots$ is the natural Pratt–Möbius image of the monomial basis of the polynomial ring.

34 The Multivariate Pratt–Möbius Operator

In this section we extend the Pratt–Möbius operator from one variable to several variables. The resulting construction is completely natural: it is simply the product version of the one-variable operator, and it may be viewed as Möbius inversion on the product Pratt poset.

34.1 Definition

Let R be a commutative ring, and let

$$P(T_1, \ldots, T_r) = \sum_{\alpha \in \mathbb{N}_0^r} a_\alpha T^\alpha$$

be a polynomial in r variables, where

$$T^\alpha := T_1^{\alpha_1} \cdots T_r^{\alpha_r}, \qquad \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}_0^r,$$

and only finitely many coefficients $a_\alpha \in R$ are nonzero.

Recall that for each $k \geq 0$, the one-variable Pratt coefficient function $C_k : \mathbb{N} \to \mathbb{Z}$ satisfies

$$x^k = \sum_{d \leq_P x} C_k(d) \qquad \text{for all } x \in \mathbb{N}.$$

We define the *multivariate Pratt–Möbius operator*

$$\mathcal{C}_r : R[T_1, \ldots, T_r] \longrightarrow R^{\mathbb{N}^r}$$

by

$$\mathcal{C}_r(P)(d_1, \ldots, d_r) := \sum_{\alpha \in \mathbb{N}_0^r} a_\alpha \prod_{j=1}^{r} C_{\alpha_j}(d_j).$$

Since P has only finitely many nonzero coefficients, this sum is finite.

34.2 Monomials

The basic case is the monomial

$$T^{\alpha} = T_1^{\alpha_1} \cdots T_r^{\alpha_r}.$$

Its image under $\mathcal{C}_r$ is

$$\mathcal{C}_r(T^{\alpha})(d_1, \ldots, d_r) = \prod_{j=1}^{r} C_{\alpha_j}(d_j).$$

Thus the multivariate operator sends monomials to pure tensor products of the one-variable coefficient functions.

Lemma 8. *For every $\alpha \in \mathbb{N}_0^r$ and every $x_1, \ldots, x_r \in \mathbb{N}$,*

$$x_1^{\alpha_1} \cdots x_r^{\alpha_r} = \sum_{\substack{d_1 \leq_P x_1 \\ \cdots \\ d_r \leq_P x_r}} \prod_{j=1}^{r} C_{\alpha_j}(d_j).$$

Proof. For each j one has

$$x_j^{\alpha_j} = \sum_{d_j \leq_P x_j} C_{\alpha_j}(d_j).$$

Multiplying these identities gives

$$x_1^{\alpha_1} \cdots x_r^{\alpha_r} = \prod_{j=1}^{r} \left(\sum_{d_j \leq_P x_j} C_{\alpha_j}(d_j) \right).$$

Expanding the product yields

$$x_1^{\alpha_1} \cdots x_r^{\alpha_r} = \sum_{\substack{d_1 \leq_P x_1 \\ \cdots \\ d_r \leq_P x_r}} \prod_{j=1}^{r} C_{\alpha_j}(d_j),$$

as claimed. □

34.3 Reconstruction formula

The monomial identity immediately implies the general polynomial case.

Theorem 8. *For every polynomial*

$$P(T_1, \ldots, T_r) \in R[T_1, \ldots, T_r]$$

and every $x_1, \ldots, x_r \in \mathbb{N}$,

$$P(x_1, \ldots, x_r) = \sum_{\substack{d_1 \leq_P x_1 \\ \cdots \\ d_r \leq_P x_r}} \mathcal{C}_r(P)(d_1, \ldots, d_r).$$

Proof. Write

$$P(T_1, \dots, T_r) = \sum_{\alpha \in \mathbb{N}_0^r} a_\alpha T^\alpha.$$

Then

$$P(x_1, \dots, x_r) = \sum_{\alpha \in \mathbb{N}_0^r} a_\alpha x_1^{\alpha_1} \cdots x_r^{\alpha_r}.$$

By the previous lemma,

$$x_1^{\alpha_1} \cdots x_r^{\alpha_r} = \sum_{\substack{d_1 \le_P x_1 \\ \cdots \\ d_r \le_P x_r}} \prod_{j=1}^{r} C_{\alpha_j}(d_j).$$

Substituting this into the expression for $P(x_1, \dots, x_r)$, we obtain

$$P(x_1, \dots, x_r) = \sum_{\alpha} a_\alpha \sum_{\substack{d_1 \le_P x_1 \\ \cdots \\ d_r \le_P x_r}} \prod_{j=1}^{r} C_{\alpha_j}(d_j).$$

Since all sums are finite, we may interchange the order of summation:

$$P(x_1, \dots, x_r) = \sum_{\substack{d_1 \le_P x_1 \\ \cdots \\ d_r \le_P x_r}} \sum_{\alpha} a_\alpha \prod_{j=1}^{r} C_{\alpha_j}(d_j).$$

The inner sum is precisely $\mathcal{C}_r(P)(d_1, \dots, d_r)$. Hence

$$P(x_1, \dots, x_r) = \sum_{\substack{d_1 \le_P x_1 \\ \cdots \\ d_r \le_P x_r}} \mathcal{C}_r(P)(d_1, \dots, d_r),$$

as required. □

34.4 Linearity

The multivariate operator is linear.

Proposition 33. *For all polynomials $P, Q \in R[T_1, \dots, T_r]$ and all $\lambda, \mu \in R$,*

$$\mathcal{C}_r(\lambda P + \mu Q) = \lambda \mathcal{C}_r(P) + \mu \mathcal{C}_r(Q).$$

Proof. This follows immediately from the definition, since the coefficients of $\lambda P + \mu Q$ are the corresponding linear combinations of the coefficients of P and Q. □

34.5 Injectivity

The multivariate Pratt–Möbius operator is injective over an integral domain.

Proposition 34. *If R is an integral domain, then*

$$\mathcal{C}_r : R[T_1, \dots, T_r] \to R^{\mathbb{N}^r}$$

is injective.

Proof. Suppose that $\mathcal{C}_r(P) = 0$. Then for every $x_1, \dots, x_r \in \mathbb{N}$,

$$P(x_1, \dots, x_r) = \sum_{\substack{d_1 \leq_P x_1 \\ \cdots \\ d_r \leq_P x_r}} \mathcal{C}_r(P)(d_1, \dots, d_r) = 0.$$

So P vanishes on all of $\mathbb{N}^r$.

We prove that this implies $P = 0$. The case $r = 1$ is the usual fact that a nonzero polynomial over an integral domain has only finitely many roots.

Assume the statement known for $r - 1$ variables, and write

$$P(T_1, \dots, T_r) = \sum_{m=0}^{M} P_m(T_1, \dots, T_{r-1})\, T_r^m,$$

with $P_M \neq 0$. Since P_M is a nonzero polynomial in $r-1$ variables, there exist $x_1, \dots, x_{r-1} \in \mathbb{N}$ such that

$$P_M(x_1, \dots, x_{r-1}) \neq 0.$$

Then the one-variable polynomial

$$Q(T_r) := P(x_1, \dots, x_{r-1}, T_r)$$

is nonzero, yet it vanishes for every $T_r \in \mathbb{N}$, contradiction. Hence $P = 0$.

Therefore $\mathcal{C}_r$ is injective. □

34.6 Tensor product structure

The multivariate operator is simply the product version of the one-variable operator. On monomials one has

$$\mathcal{C}_r(T_1^{\alpha_1} \cdots T_r^{\alpha_r}) = C_{\alpha_1} \otimes \cdots \otimes C_{\alpha_r},$$

in the sense that

$$\mathcal{C}_r(T_1^{\alpha_1} \cdots T_r^{\alpha_r})(d_1, \dots, d_r) = C_{\alpha_1}(d_1) \cdots C_{\alpha_r}(d_r).$$

Thus $\mathcal{C}_r$ is the natural tensor extension of the one-variable operator $\mathcal{C}$.

34.7 Interpretation via the product Pratt poset

The construction can also be understood as Möbius inversion on the product poset

$$(\mathbb{N}, \leq_P)^r.$$

The order is given by

$$(d_1, \dots, d_r) \le (x_1, \dots, x_r) \iff d_j \le_P x_j \quad \text{for all } j.$$

Then the reconstruction formula becomes

$$P(x_1, \dots, x_r) = \sum_{(d_1,\dots,d_r) \le (x_1,\dots,x_r)} \mathcal{C}_r(P)(d_1, \dots, d_r).$$

This is exactly the product version of the one-variable identity. Since the Möbius function of a product poset factorizes, the multivariate operator is the Möbius transform attached to the product Pratt order.

34.8 Example

Consider the polynomial

$$P(X, Y) = X^2 + XY + 1.$$

Then

$$\mathcal{C}_2(P)(d, e) = C_2(d)C_0(e) + C_1(d)C_1(e) + C_0(d)C_0(e).$$

Since

$$C_0(1) = 1, \qquad C_0(n) = 0 \quad (n > 1),$$

this can also be written as

$$\mathcal{C}_2(P)(d, e) = C_2(d)\, \delta_{e,1} + C_1(d)C_1(e) + \delta_{d,1}\delta_{e,1}.$$

Accordingly,

$$x^2 + xy + 1 = \sum_{d \le_P x} \sum_{e \le_P y} \mathcal{C}_2(P)(d, e).$$

34.9 Summary

The multivariate Pratt–Möbius operator

$$\mathcal{C}_r : R[T_1, \dots, T_r] \to R^{\mathbb{N}^r}$$

is a natural extension of the one-variable construction. It is linear, injective over an integral domain, explicit on monomials, and it reconstructs polynomial values by summing over the corresponding product Pratt lower set:

$$P(x_1, \dots, x_r) = \sum_{\substack{d_1 \le_P x_1 \\ \cdots \\ d_r \le_P x_r}} \mathcal{C}_r(P)(d_1, \dots, d_r).$$

In this way, the family of functions

$$(d_1, \dots, d_r) \longmapsto \prod_{j=1}^{r} C_{\alpha_j}(d_j)$$

plays, for multivariate polynomials, the same role that the functions C_k play in one variable.

35 Reminder: Setup and notation

For $a, b \in \mathbb{N}$, write

$$a \le_P b \iff m_q(a) \le m_q(b) \qquad \text{for every prime } q,$$

where $m_q(\cdot)$ denotes the q-th Pratt exponent. The Pratt coefficients are characterized by the product formula

$$C_n(d) = \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)} = d^n \prod_{p|d} \Big(1 - \Big(1 - \tfrac{1}{p}\Big)^n\Big) \qquad (n \ge 1),$$

with the convention $C_0(1) = 1$ and $C_0(d) = 0$ for $d > 1$.

We also use the standard incidence-algebra notation

$$C_n(d) = \sum_{x \le_P d} \mu_P(x, d) x^n,$$

where $\mu_P(x, d)$ denotes the incidence Möbius function of the locally finite Pratt poset.

36 Tuple counting modulo d

36.1 A counting interpretation

Theorem 9 (Tuple-counting formula)**.** *Let $d \ge 1$ and $n \ge 1$. Then $C_n(d)$ is the number of tuples*

$$(a_1, \dots, a_n) \in (\mathbb{Z}/d\mathbb{Z})^n$$

such that for every prime $p \mid d$ there exists at least one index $i \in \{1, \dots, n\}$ with

$$a_i \equiv 0 \pmod{p}.$$

Equivalently, $C_n(d)$ counts the n-tuples modulo d that hit the residue class $0 \pmod{p}$ for every prime divisor p of d.

Proof. We first treat the prime-power case $d = p^e$. There are p^e residue classes modulo p^e, of which exactly p^{e-1} are divisible by p and exactly

$$p^e - p^{e-1} = p^{e-1}(p-1)$$

are not divisible by p. Hence the total number of n-tuples modulo p^e is p^{en}, while the number of tuples in which no coordinate is divisible by p is

$$(p^e - p^{e-1})^n = p^{n(e-1)}(p-1)^n.$$

Therefore the number of tuples in which at least one coordinate is divisible by p equals

$$p^{en} - p^{n(e-1)}(p-1)^n = p^{n(e-1)}\Big(p^n - (p-1)^n\Big) = C_n(p^e).$$

Now let $d = \prod_{j=1}^r p_j^{e_j}$. By the Chinese remainder theorem,

$$\mathbb{Z}/d\mathbb{Z} \cong \prod_{j=1}^r \mathbb{Z}/p_j^{e_j}\mathbb{Z},$$

and hence

$$(\mathbb{Z}/d\mathbb{Z})^n \cong \prod_{j=1}^r (\mathbb{Z}/p_j^{e_j}\mathbb{Z})^n.$$

The condition

$$\forall p_j \mid d \ \exists i \colon a_i \equiv 0 \pmod{p_j}$$

is independent in the prime-power components. Therefore the number of admissible tuples modulo d is the product of the corresponding prime-power counts:

$$\prod_{j=1}^r C_n(p_j^{e_j}) = C_n(d)$$

by multiplicativity of the closed formula. ☐

36.2 Probabilistic reformulation

Corollary 8 (Covering probability)**.** *If a tuple $(a_1, \dots, a_n)$ is chosen uniformly at random from $(\mathbb{Z}/d\mathbb{Z})^n$, then*

$$\frac{C_n(d)}{d^n} = \mathbf{P}\Big(\forall p \mid d \ \exists i \colon p \mid a_i\Big) = \prod_{p \mid d}\Big(1 - \Big(1 - \tfrac{1}{p}\Big)^n\Big).$$

Thus $C_n(d)/d^n$ is the exact probability that the n sampled residues collectively cover all prime divisors of d.

Proof. The first identity is immediate from Theorem 9 by dividing by the total number d^n of tuples. The second is exactly the normalized product formula for $C_n(d)$. ☐

37 Inclusion–exclusion and the exponential spectrum

37.1 A finite exponential expansion

Proposition 35 (Radical expansion)**.** *For every $d \geq 1$ and $n \geq 1$,*

$$C_n(d) = \sum_{m \mid \mathrm{rad}(d)} \mu(m)\Big(d\frac{\varphi(m)}{m}\Big)^n.$$

Equivalently,

$$C_n(d) = \sum_{S \subseteq \{p : p \mid d\}} (-1)^{|S|}\Big(d\prod_{p\in S}\big(1-\tfrac{1}{p}\big)\Big)^n.$$

Proof. For each prime $p \mid d$, let E_p be the event that none of the coordinates is congruent to 0 (mod p). By Theorem 9, $C_n(d)$ counts the tuples outside the union $\bigcup_{p\mid d} E_p$. Hence inclusion–exclusion gives

$$C_n(d) = \sum_{S \subseteq \{p : p \mid d\}} (-1)^{|S|}\, N(S),$$

where $N(S)$ denotes the number of tuples for which every event E_p with $p \in S$ holds simultaneously.

If S is fixed, then for each coordinate there are exactly

$$d\prod_{p\in S}\big(1-\tfrac{1}{p}\big)$$

allowed residue classes modulo d, namely those not divisible by any prime in S. Therefore

$$N(S) = \Big(d\prod_{p\in S}\big(1-\tfrac{1}{p}\big)\Big)^n.$$

This proves the subset expansion. Now write $m = \prod_{p\in S} p$. Then m runs through the squarefree divisors of $\mathrm{rad}(d)$, one has $\mu(m) = (-1)^{|S|}$, and

$$\frac{\varphi(m)}{m} = \prod_{p\mid m}\big(1-\tfrac{1}{p}\big).$$

Substituting yields the stated formula. □

37.2 Linear recurrences in the parameter n

Theorem 10 (Finite exponential spectrum)**.** *Fix $d \geq 1$ and define*

$$\Lambda(d) := \Big\{d\frac{\varphi(m)}{m} : m \mid \mathrm{rad}(d)\Big\}.$$

Then the sequence $u_n := C_n(d)$ is a finite linear combination of exponentials,

$$u_n = \sum_{m|\mathrm{rad}(d)} \mu(m)\lambda_m^n, \qquad \lambda_m := d\frac{\varphi(m)}{m}.$$

In particular, u_n satisfies a linear recurrence with constant coefficients whose characteristic polynomial divides

$$P_d(T) := \prod_{\lambda\in\Lambda(d)} (T-\lambda).$$

If the elements of $\Lambda(d)$ are pairwise distinct, then P_d is the minimal characteristic polynomial.

Proof. The exponential decomposition is exactly Proposition 35. Any finite linear combination of sequences of the form $n \mapsto \lambda^n$ is annihilated by the shift operator polynomial obtained by replacing T with the forward-shift operator. Concretely, if

$$P_d(T) = T^r - e_1T^{r-1} + e_2T^{r-2} - \cdots + (-1)^r e_r,$$

then

$$u_{n+r} - e_1u_{n+r-1} + e_2u_{n+r-2} - \cdots + (-1)^r e_r u_n = 0$$

for all $n \geq 1$.

If the $\lambda \in \Lambda(d)$ are distinct, then the exponential sequences λ^n are linearly independent over any field of characteristic 0, so no proper factor of P_d can annihilate u_n. □

Corollary 9 (Ordinary generating function)**.** *For fixed d, the ordinary generating function*

$$F_d(z) := \sum_{n\geq 1} C_n(d)z^n$$

is rational and is given by

$$F_d(z) = \sum_{m|\mathrm{rad}(d)} \mu(m)\frac{\lambda_m z}{1-\lambda_m z}, \qquad \lambda_m = d\frac{\varphi(m)}{m}.$$

If one starts the sum at $n = 0$, then the corresponding generating function is obtained by adding the constant term $C_0(d)$.

Proof. From Proposition 35,

$$F_d(z) = \sum_{m|\mathrm{rad}(d)} \mu(m) \sum_{n\geq 1} (\lambda_m z)^n,$$

and each inner series is geometric. □

37.3 Asymptotics from the second-largest eigenvalue

Proposition 36 (Dominant error term). *Let $d > 1$, and let $p_{\max}$ be the largest prime divisor of d. Then*

$$C_n(d) = d^n - \Big(d\Big(1 - \tfrac{1}{p_{\max}}\Big)\Big)^n + O(\rho_d^n),$$

where $\rho_d < d(1-1/p_{\max})$ is the third-largest element of $\Lambda(d)$ in absolute value (or any number strictly between that third-largest value and $d(1 - 1/p_{\max})$). In particular,

$$d^n - C_n(d) \asymp \Big(d\Big(1 - \tfrac{1}{p_{\max}}\Big)\Big)^n.$$

Proof. The term corresponding to $m = 1$ in Proposition 35 is d^n. Among the remaining terms,

$$d\frac{\varphi(m)}{m} = d\prod_{p|m}\Big(1 - \frac{1}{p}\Big)$$

is maximized when $m = p_{\max}$. Indeed, each additional prime factor decreases the product, and among single-prime factors the quantity $1 - 1/p$ is largest when p is largest. Thus the second-largest spectral value is

$$d\Big(1 - \frac{1}{p_{\max}}\Big),$$

with coefficient $\mu(p_{\max}) = -1$. The remainder estimate follows immediately from the finite spectral expansion in Proposition 35. □

37.4 Examples

Example 11 (Prime powers). *If $d = p^e$, then $\mathrm{rad}(d) = p$, hence*

$$C_n(p^e) = p^{en} - \Big(p^{e-1}(p-1)\Big)^n.$$

Therefore $u_n = C_n(p^e)$ satisfies the second-order recurrence

$$u_{n+2} - \Big(2p^e - p^{e-1}\Big)u_{n+1} + p^{2e-1}(p-1)u_n = 0.$$

Example 12 ($d = 6$). *Here*

$$\Lambda(6) = \{6, 4, 3, 2\},$$

so

$$C_n(6) = 6^n - 4^n - 3^n + 2^n.$$

Hence $u_n = C_n(6)$ satisfies

$$u_{n+4} - 15u_{n+3} + 80u_{n+2} - 180u_{n+1} + 144u_n = 0.$$

Example 13 ($d = 30$)**.** *Since the divisors of* $\mathrm{rad}(30) = 30$ *are* $1, 2, 3, 5, 6, 10, 15, 30$*, one finds*

$$\Lambda(30) = \{30, 15, 20, 24, 10, 12, 16, 8\},$$

and therefore

$$C_n(30) = 30^n - 15^n - 20^n - 24^n + 10^n + 12^n + 16^n - 8^n.$$

This illustrates the rigid finite-spectrum phenomenon particularly clearly.

38 Explicit formula for the incidence Möbius function on the Pratt poset

38.1 The Boolean layer inside an interval

Definition 7. Fix $d \in \mathbb{N}$. For each squarefree divisor $m \mid \mathrm{rad}(d)$, define

$$x_m := d\frac{\varphi(m)}{m} = d\prod_{p\mid m}\Big(1 - \frac{1}{p}\Big).$$

Lemma 9. *For every squarefree divisor* $m \mid \mathrm{rad}(d)$ *and every prime* q*,*

$$m_q(x_m) = m_q(d) - \mathbf{1}_{\{q\mid m\}}.$$

In particular, $x_m \leq_P d$*.*

Proof. By the Pratt product representation, multiplying an integer by $(1 - 1/p)$ lowers the p-th Pratt coordinate by exactly one and leaves all other coordinates unchanged. Since

$$x_m = d\prod_{p\mid m}\Big(1 - \frac{1}{p}\Big)$$

and m is squarefree, each prime divisor of m contributes exactly one such lowering step. Therefore

$$m_q(x_m) = m_q(d) - \mathbf{1}_{\{q\mid m\}}$$

for every prime q. □

Remark 19. Lemma 9 shows that the points x_m are obtained from d by moving down by at most one step in each nonzero Pratt coordinate. Thus the collection $\{x_m : m \mid \mathrm{rad}(d)\}$ is a Boolean layer inside the interval $[1, d]_P$.

38.2 Explicit support and sign pattern of the Möbius function

Theorem 14 (Explicit formula for $\mu_P(x,d)$)**.** *Let $x \le_P d$. Then*

$$\mu_P(x,d) = \begin{cases} \mu(m), & \text{if } x = d\,\dfrac{\varphi(m)}{m} \text{ for some squarefree } m \mid \operatorname{rad}(d), \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$\mu_P(x,d) \neq 0 \iff m_q(d) - m_q(x) \in \{0,1\} \quad \text{for every prime } q,$$

and in that case

$$\mu_P(x,d) = (-1)^{\sum_q (m_q(d) - m_q(x))}.$$

In particular,

$$\mu_P(x,d) \in \{0, \pm 1\} \qquad \text{for all } x \le_P d.$$

Proof. We compare two expressions for the same sequence in the parameter n.

On the one hand, by Pratt–Möbius inversion,

$$C_n(d) = \sum_{x \le_P d} \mu_P(x,d) x^n.$$

On the other hand, by Proposition 35,

$$C_n(d) = \sum_{m \mid \operatorname{rad}(d)} \mu(m) \Big(d \frac{\varphi(m)}{m} \Big)^n = \sum_{m \mid \operatorname{rad}(d)} \mu(m) x_m^n.$$

By Lemma 9, each x_m satisfies $x_m \le_P d$, so both are expansions of $C_n(d)$ in terms of powers x^n with $x \le_P d$.

Now the numbers $x \le_P d$ form a finite set, and the sequences $n \mapsto x^n$ attached to distinct positive real numbers are linearly independent. Indeed, if

$$\sum_{j=1}^{r} a_j \lambda_j^n = 0 \qquad (n \ge 1)$$

with distinct positive λ_j, then dividing by the largest λ_r^n and letting $n \to \infty$ shows $a_r = 0$; repeating gives all $a_j = 0$.

Therefore the two exponential expansions must coincide termwise. It follows that the support of $x \mapsto \mu_P(x,d)$ is exactly the set $\{x_m : m \mid \operatorname{rad}(d)\}$, and on this set the coefficients are precisely $\mu(m)$. This proves the first formula.

For the equivalent coordinate criterion, note from Lemma 9 that

$$x = x_m \iff m_q(x) = m_q(d) - \mathbf{1}_{\{q \mid m\}} \quad \text{for every prime } q.$$

Thus $x = x_m$ for some squarefree $m \mid \operatorname{rad}(d)$ exactly when each coordinate drop $m_q(d) - m_q(x)$

is either 0 or 1. In that case,

$$\mu_P(x,d) = \mu(m) = (-1)^{\omega(m)} = (-1)^{\sum_q (m_q(d)-m_q(x))}.$$

The final assertion $\mu_P(x,d) \in \{0, \pm 1\}$ is immediate. □

38.3 Interpretation: squarefree steps in Pratt coordinates

Corollary 10 (Boolean/squarefree character)**.** *The incidence Möbius function on the Pratt poset is supported exactly on squarefree coordinate descents: it vanishes unless one moves from d to x by lowering each nonzero Pratt coordinate by at most one, and on that Boolean layer the sign is $(-1)^{\text{number of lowered coordinates}}$.*

Proof. This is a restatement of Theorem 14. □

Remark 20. This parallels the classical divisor-poset Möbius function. There, nonvanishing occurs exactly for squarefree drops in the p-adic valuation coordinates v_p. Here, the same Boolean pattern appears, but with the Pratt coordinates m_p in place of the ordinary valuation coordinates.

38.4 Examples

Example 15 ($d = 6$)**.** *The squarefree divisors of* $\mathrm{rad}(6) = 6$ *are* $1, 2, 3, 6$*, yielding*

$$x_1 = 6, \qquad x_2 = 3, \qquad x_3 = 4, \qquad x_6 = 2.$$

Hence

$$\mu_P(6,6) = 1, \qquad \mu_P(3,6) = -1, \qquad \mu_P(4,6) = -1, \qquad \mu_P(2,6) = 1,$$

and $\mu_P(x,6) = 0$ for every other $x \leq_P 6$.

Example 16 ($d = p^e$)**.** *Here* $\mathrm{rad}(d) = p$*, so the only nonzero Möbius values on the interval below d are*

$$\mu_P(d,d) = 1, \qquad \mu_P\left(d\frac{p-1}{p}, d\right) = -1.$$

Every other $x \leq_P d$ has $\mu_P(x,d) = 0$.

39 Pairing the radical expansion into differences of powers

The radical expansion

$$C_n(d) = \sum_{S \subseteq \{p : p \mid d\}} (-1)^{|S|} \left(d \prod_{p \in S} \left(1 - \frac{1}{p}\right)\right)^n$$

already exhibits $C_n(d)$ as an alternating sum of n-th powers. The next observation shows that, as soon as $d > 1$, the positive and negative terms can be paired *canonically*. As a consequence, every $C_n(d)$ is a sum of differences of n-th powers.

39.1 Canonical pairing

Let

$$P(d) := \{p : \ p \mid d\}, \qquad \omega(d) := |P(d)|.$$

For each subset $S \subseteq P(d)$, define

$$X_S(d) := d \prod_{p \in S} \left(1 - \frac{1}{p}\right).$$

Then Proposition 34 gives

$$C_n(d) = \sum_{S \subseteq P(d)} (-1)^{|S|} X_S(d)^n.$$

Assume now that $d > 1$, and let

$$p_0 := \min P(d)$$

be the smallest prime divisor of d. This choice is canonical. The involution

$$S \longmapsto S \cup \{p_0\}$$

defines a bijection between subsets of $P(d) \setminus \{p_0\}$ and subsets of $P(d)$ containing p_0. Since adjoining p_0 flips parity, we may regroup the expansion into pairs:

$$C_n(d) = \sum_{S \subseteq P(d) \setminus \{p_0\}} (-1)^{|S|} \Big(X_S(d)^n - X_{S \cup \{p_0\}}(d)^n\Big).$$

Equivalently, if we define

$$(A_S, B_S) := \begin{cases} \Big(X_S(d), X_{S \cup \{p_0\}}(d)\Big), & |S| \text{ even}, \\ \Big(X_{S \cup \{p_0\}}(d), X_S(d)\Big), & |S| \text{ odd}, \end{cases}$$

then

$$\boxed{C_n(d) = \sum_{S \subseteq P(d) \setminus \{p_0\}} \Big(A_S^n - B_S^n\Big).}$$

In particular, $C_n(d)$ is always a sum of exactly $2^{\omega(d)-1}$ differences of n-th powers.

Proof. Starting from

$$C_n(d) = \sum_{S \subseteq P(d)} (-1)^{|S|} X_S(d)^n,$$

split the sum according to whether $p_0 \in S$ or $p_0 \notin S$. Every subset $T \subseteq P(d)$ with $p_0 \in T$ is uniquely of the form

$$T = S \cup \{p_0\}, \qquad S \subseteq P(d) \setminus \{p_0\}.$$

Hence

$$C_n(d) = \sum_{S \subseteq P(d) \setminus \{p_0\}} \Big((-1)^{|S|} X_S(d)^n + (-1)^{|S|+1} X_{S \cup \{p_0\}}(d)^n \Big),$$

which is

$$C_n(d) = \sum_{S \subseteq P(d) \setminus \{p_0\}} (-1)^{|S|} \Big(X_S(d)^n - X_{S \cup \{p_0\}}(d)^n \Big).$$

Now absorb the sign into the ordering of the pair: if $|S|$ is even, keep the order as written; if $|S|$ is odd, reverse the two terms. This gives the claimed formula. □

39.2 Interpretation

Thus $C_n(d)$ has two complementary faces:

- a multiplicative face,
$$C_n(d) = \prod_{p^e \| d} \Big(p^n - (p-1)^n \Big) p^{n(e-1)},$$
built from local differences $a^n - b^n$;
- an additive face,
$$C_n(d) = \sum_{S \subseteq P(d) \setminus \{p_0\}} \Big(A_S^n - B_S^n \Big),$$
built from canonically paired differences of global n-th powers.

The pairing is especially natural because each pair differs only by adjoining the single distinguished prime p_0. Indeed,

$$X_{S \cup \{p_0\}}(d) = X_S(d) \left(1 - \frac{1}{p_0} \right),$$

so each summand is of the form

$$X_S(d)^n - \left(X_S(d) \left(1 - \frac{1}{p_0} \right) \right)^n$$

or its negative, depending on the parity of S.

39.3 Examples (computed and checked in SymPy)

Example 1: $d = 6$. Here

$$P(6) = \{2, 3\}, \qquad p_0 = 2.$$

The subsets of $P(6) \setminus \{2\} = \{3\}$ are $\varnothing$ and $\{3\}$. The canonical pairs are

$$(6, 3), \qquad (2, 4).$$

Hence

$$\boxed{C_n(6) = (6^n - 3^n) + (2^n - 4^n).}$$

Equivalently,

$$C_n(6) = 6^n - 3^n - 4^n + 2^n.$$

Example 2: $d = 12$. Here

$$P(12) = \{2, 3\}, \qquad p_0 = 2.$$

The canonical pairs are

$$(12, 6), \qquad (4, 8).$$

Therefore

$$\boxed{C_n(12) = (12^n - 6^n) + (4^n - 8^n).}$$

Expanding signs gives

$$C_n(12) = 12^n - 6^n - 8^n + 4^n.$$

Example 3: $d = 18$. Again

$$P(18) = \{2, 3\}, \qquad p_0 = 2.$$

The canonical pairs are

$$(18, 9), \qquad (6, 12).$$

Thus

$$\boxed{C_n(18) = (18^n - 9^n) + (6^n - 12^n).}$$

Example 4: $d = 30$. Now

$$P(30) = \{2, 3, 5\}, \qquad p_0 = 2.$$

The subsets of $\{3, 5\}$ are

$$\varnothing, \ \{3\}, \ \{5\}, \ \{3, 5\}.$$

The corresponding canonical pairs are

$$(30, 15), \qquad (10, 20), \qquad (12, 24), \qquad (16, 8).$$

Hence

$$\boxed{C_n(30) = (30^n - 15^n) + (10^n - 20^n) + (12^n - 24^n) + (16^n - 8^n).}$$

After expanding signs, this becomes

$$C_n(30) = 30^n - 15^n - 20^n - 24^n + 10^n + 12^n + 16^n - 8^n,$$

which is exactly the radical expansion regrouped into canonical differences.

39.4 A concise corollary

For every $d > 1$ and $n \geq 1$, there exist explicitly defined integers

$$A_1, \dots, A_{2^{\omega(d)-1}}, \qquad B_1, \dots, B_{2^{\omega(d)-1}}$$

such that

$$\boxed{C_n(d) = \sum_{j=1}^{2^{\omega(d)-1}} (A_j^n - B_j^n).}$$

Moreover, the construction is canonical: it depends only on the distinguished choice $p_0 = \min\{p : p \mid d\}$.

39.5 Cyclotomic factorization and the difference-of-powers expansion

The canonical pairing above shows that every $C_n(d)$ is a sum of differences of n-th powers. At the same time, the local product formula

$$C_n(d) = \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)}$$

shows that $C_n(d)$ is built multiplicatively from factors of the form $a^n - b^n$, namely with $a = p$ and $b = p - 1$.

These local differences admit a standard cyclotomic factorization. Recall that

$$a^n - b^n = \prod_{m \mid n} \Phi_m(a, b),$$

where $\Phi_m(a, b)$ denotes the homogeneous cyclotomic polynomial. Applying this with $a = p$ and $b = p - 1$ gives

$$p^n - (p-1)^n = \prod_{m \mid n} \Phi_m(p, p-1).$$

Therefore

$$\boxed{C_n(d) = \prod_{p^e \| d} \left(\prod_{m \mid n} \Phi_m(p, p-1) \right) p^{n(e-1)}.}$$

So $C_n(d)$ admits two complementary structural descriptions:

1. a *multiplicative cyclotomic decomposition*,

$$C_n(d) = \prod_{p^e \| d} \left(\prod_{m|n} \Phi_m(p, p-1) \right) p^{n(e-1)};$$

2. an *additive paired decomposition*,

$$C_n(d) = \sum_{j=1}^{2^{\omega(d)-1}} (A_j^n - B_j^n).$$

In this sense, $C_n(d)$ may be viewed as an object that sits at the intersection of two classical themes:

- *cyclotomic factorization*, through the decomposition of each local factor $p^n - (p-1)^n$;
- *Waring-type additive structure*, through the representation of $C_n(d)$ as a finite sum of differences of n-th powers.

What is striking here is that both descriptions are completely explicit and canonical. The product side is dictated by the prime factorization of d, while the additive side is dictated by the radical expansion together with the distinguished choice of the smallest prime divisor p_0.

Example: $d = 6$. We have both

$$C_n(6) = (2^n - 1)(3^n - 2^n)$$

and

$$C_n(6) = (6^n - 3^n) + (2^n - 4^n).$$

Thus a product of two local differences of n-th powers becomes a sum of two global differences of n-th powers.

Example: $d = 30$. On the multiplicative side,

$$C_n(30) = (2^n - 1)(3^n - 2^n)(5^n - 4^n).$$

On the additive side,

$$C_n(30) = (30^n - 15^n) + (10^n - 20^n) + (12^n - 24^n) + (16^n - 8^n).$$

Thus the same quantity is simultaneously a product of three local differences and a sum of four canonical differences.

This suggests the following viewpoint: $C_n(d)$ provides a natural bridge between multiplicative factorizations of the shape $a^n - b^n$ and additive expansions of the shape

$$A_1^n + \cdots + A_r^n - B_1^n - \cdots - B_r^n.$$

Unlike an arbitrary identity of this type, the one attached to $C_n(d)$ is canonically generated by the arithmetic of the radical of d.

Remark 21. The additive decomposition is especially rigid: the number of paired differences is always exactly $2^{\omega(d)-1}$, and the terms are explicitly given by

$$X_S(d) = d \prod_{p \in S} \left(1 - \frac{1}{p}\right), \qquad S \subseteq P(d) \setminus \{p_0\}.$$

Thus the passage from the multiplicative local factors to the additive global differences is controlled entirely by inclusion–exclusion on the prime divisors of d.

39.6 Cyclotomic structure on both sides

The previous discussion shows that $C_n(d)$ admits a canonical decomposition

$$C_n(d) = \sum_{j=1}^{2^{\omega(d)-1}} (A_j^n - B_j^n),$$

where the pairs (A_j, B_j) are explicitly determined by the radical of d. What makes this especially striking is that each individual difference $A_j^n - B_j^n$ is itself cyclotomically factorable.

Indeed, for any integers A, B one has

$$A^n - B^n = \prod_{m \mid n} \Phi_m(A, B),$$

where $\Phi_m(A, B)$ denotes the homogeneous cyclotomic polynomial. Applying this to each canonical pair (A_j, B_j) gives

$$\boxed{A_j^n - B_j^n = \prod_{m \mid n} \Phi_m(A_j, B_j).}$$

Hence the additive decomposition may be rewritten as

$$\boxed{C_n(d) = \sum_{j=1}^{2^{\omega(d)-1}} \prod_{m \mid n} \Phi_m(A_j, B_j).}$$

This should be compared with the local multiplicative factorization

$$C_n(d) = \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)} = \prod_{p^e \| d} \left(\prod_{m|n} \Phi_m(p, p-1) \right) p^{n(e-1)}.$$

Thus $C_n(d)$ carries two parallel cyclotomic descriptions:

1. a *local multiplicative cyclotomic factorization*,

$$C_n(d) = \prod_{p^e \| d} \left(\prod_{m|n} \Phi_m(p, p-1) \right) p^{n(e-1)};$$

2. a *global additive cyclotomic expansion*,

$$C_n(d) = \sum_{j=1}^{2^{\omega(d)-1}} \prod_{m|n} \Phi_m(A_j, B_j).$$

In other words, $C_n(d)$ is not merely a sum of differences of n-th powers: it is a sum of quantities each possessing its own full cyclotomic factorization. This places the radical expansion and the local product expansion on remarkably symmetric footing.

Example 17. *Take* $n = 1234$ *and* $d = 4321 = 29 \cdot 149$. *Then*

$$C_{1234}(4321) = (29^{1234} - 28^{1234})(149^{1234} - 148^{1234}).$$

Since $1234 = 2 \cdot 617$, *the divisors of* 1234 *are* $1, 2, 617, 1234$, *and hence*

$$A^{1234} - B^{1234} = \Phi_1(A, B)\Phi_2(A, B)\Phi_{617}(A, B)\Phi_{1234}(A, B).$$

Therefore the local factorization becomes

$$C_{1234}(4321) = \Big(\prod_{m|1234} \Phi_m(29, 28) \Big) \Big(\prod_{m|1234} \Phi_m(149, 148) \Big).$$

Equivalently,

$$C_{1234}(4321) = 16929\, \Phi_{617}(29, 28)\Phi_{1234}(29, 28)\Phi_{617}(149, 148)\Phi_{1234}(149, 148),$$

since $\Phi_1(29, 28) = 1$, $\Phi_2(29, 28) = 57$, $\Phi_1(149, 148) = 1$, *and* $\Phi_2(149, 148) = 297$.

On the other hand, the radical expansion gives

$$C_{1234}(4321) = 4321^{1234} - 4172^{1234} - 4292^{1234} + 4144^{1234},$$

which may be canonically paired as

$$C_{1234}(4321) = (4321^{1234} - 4172^{1234}) + (4144^{1234} - 4292^{1234}).$$

Each difference again factors cyclotomically:

$$4321^{1234} - 4172^{1234} = \prod_{m|1234} \Phi_m(4321, 4172),$$

$$4144^{1234} - 4292^{1234} = \prod_{m|1234} \Phi_m(4144, 4292).$$

Hence one also has the global cyclotomic expansion

$$C_{1234}(4321) = \prod_{m|1234} \Phi_m(4321, 4172) - \prod_{m|1234} \Phi_m(4292, 4144).$$

Thus the same integer is simultaneously a product of local cyclotomic blocks and a difference of global cyclotomic blocks.

Lemma 10 (T-chain lower bound for Pratt–Möbius coefficients)**.** *Let $n \geq 1$ and define*

$$C_n(d) = \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)}.$$

For $x \geq 1$, set

$$x_0 := x, \qquad x_{j+1} := T(x_j) := \frac{x_j}{\mathrm{spf}(x_j)}$$

until $x_\ell = 1$. Then $x_{j+1} \leq_P x_j$ for all j, hence

$$\{x_0, x_1, \ldots, x_\ell\} \subseteq \downarrow_P x.$$

Consequently,

$$x^n = \sum_{d \leq_P x} C_n(d) \geq \sum_{j=0}^{\ell} C_n(x_j).$$

Moreover, for every $d \geq 1$,

$$C_n(d) \geq \frac{d^n}{\mathrm{rad}(d)},$$

and therefore

$$x^n \geq \sum_{j=0}^{\ell} \frac{x_j^n}{\mathrm{rad}(x_j)}.$$

Proof. Write $p_j := \mathrm{spf}(x_j)$, so that

$$x_j = p_j \, x_{j+1}.$$

Since the Pratt exponent vectors are additive under multiplication, for every prime q we have

$$m_q(x_j) = m_q(p_j) + m_q(x_{j+1}) \geq m_q(x_{j+1}),$$

because $m_q(p_j) \geq 0$. Hence

$$x_{j+1} \leq_P x_j$$

for every j. By transitivity it follows that

$$x_j \le_P x \qquad (0 \le j \le \ell),$$

so indeed

$$\{x_0, x_1, \dots, x_\ell\} \subseteq \downarrow_P x.$$

Now use the Pratt–Möbius expansion

$$x^n = \sum_{d \le_P x} C_n(d).$$

Since every factor

$$p^n - (p-1)^n$$

is positive, one has $C_n(d) \ge 0$ for all d. Therefore restricting the sum to the T-chain gives

$$x^n = \sum_{d \le_P x} C_n(d) \ge \sum_{j=0}^{\ell} C_n(x_j).$$

It remains to prove the pointwise bound

$$C_n(d) \ge \frac{d^n}{\operatorname{rad}(d)}.$$

If

$$d = \prod_{p^e \| d} p^e,$$

then by definition

$$C_n(d) = \prod_{p^e \| d} \Big(p^n - (p-1)^n\Big) p^{n(e-1)}.$$

Factor out p^n from each bracket:

$$p^n - (p-1)^n = p^n \left(1 - \left(1 - \frac{1}{p}\right)^n\right).$$

Hence

$$C_n(d) = \prod_{p^e \| d} p^{ne} \left(1 - \left(1 - \frac{1}{p}\right)^n\right) = d^n \prod_{p \mid d} \left(1 - \left(1 - \frac{1}{p}\right)^n\right).$$

For $u \in [0,1]$ and $n \ge 1$ we have

$$1 - (1-u)^n \ge u.$$

Applying this with $u = 1/p$ gives

$$1 - \left(1 - \frac{1}{p}\right)^n \ge \frac{1}{p}.$$

Therefore
$$C_n(d) \geq d^n \prod_{p|d} \frac{1}{p} = \frac{d^n}{\mathrm{rad}(d)}.$$

Applying this to each $d = x_j$ yields

$$C_n(x_j) \geq \frac{x_j^n}{\mathrm{rad}(x_j)}.$$

Combining with the previous inequality, we obtain

$$x^n \geq \sum_{j=0}^{\ell} C_n(x_j) \geq \sum_{j=0}^{\ell} \frac{x_j^n}{\mathrm{rad}(x_j)}.$$

This proves the claim. □

Proposition 37. *For all $r \geq 2$ and all $a_1, \ldots, a_r \in \mathbb{N}$, one has*

$$m_p(a_1 + \cdots + a_r) \geq m_p(\gcd(a_1, \ldots, a_r)) \qquad \text{for every prime } p.$$

Equivalently,
$$\gcd(a_1, \ldots, a_r) \leq_P a_1 + \cdots + a_r.$$

Proof. We argue by induction on r.

The case $r = 2$ is exactly the two-variable inequality

$$m_p(x + y) \geq m_p(\gcd(x, y)).$$

Now assume the claim holds for some $r \geq 2$, and set

$$s_r := a_1 + \cdots + a_r.$$

Applying the two-variable case to s_r and a_{r+1} gives

$$m_p(s_r + a_{r+1}) \geq m_p(\gcd(s_r, a_{r+1})).$$

Let
$$d := \gcd(a_1, \ldots, a_r, a_{r+1}).$$

Then $d \mid a_i$ for all $1 \leq i \leq r + 1$, hence $d \mid s_r$ and $d \mid a_{r+1}$. Therefore

$$d \mid \gcd(s_r, a_{r+1}).$$

Since ordinary divisibility implies Pratt order, we get

$$d \leq_P \gcd(s_r, a_{r+1}),$$

and thus

$$m_p(\gcd(s_r, a_{r+1})) \geq m_p(d).$$

Combining the inequalities, we obtain

$$m_p(a_1 + \cdots + a_r + a_{r+1}) = m_p(s_r + a_{r+1}) \geq m_p(\gcd(s_r, a_{r+1})) \geq m_p(\gcd(a_1, \ldots, a_r, a_{r+1})).$$

This proves the induction step. $\square$

Definition 8 (Pratt height)**.** For each prime q, define the q-th Pratt height of $x \in \mathbb{Q}_{>0}$ by

$$|x|_{P,q} := q^{-m_q(x)},$$

where

$$m_q(x) := \sum_p v_p(x)\, m_q(p).$$

In particular, for $n \in \mathbb{N}$,

$$|n|_{P,q} = q^{-m_q(n)}.$$

Proposition 38. *For every prime q and all $x, y \in \mathbb{Q}_{>0}$, one has*

$$|xy|_{P,q} = |x|_{P,q}\, |y|_{P,q},$$

and

$$|x + y|_{P,q} \leq |\overset{*}{\gcd}(x, y)|_{P,q}.$$

Moreover, if $a \leq_P b$, then

$$|a|_{P,q} \geq |b|_{P,q}.$$

Equivalently,

$$a \leq_P b \iff |a|_{P,q} \geq |b|_{P,q} \quad \textit{for all primes } q.$$

Proof. The multiplicativity follows from

$$m_q(xy) = m_q(x) + m_q(y).$$

Also, from

$$m_q(x + y) \geq m_q(\overset{*}{\gcd}(x, y))$$

we obtain

$$|x + y|_{P,q} = q^{-m_q(x+y)} \leq q^{-m_q(\gcd^*(x,y))} = |\overset{*}{\gcd}(x, y)|_{P,q}.$$

Finally, if $a \leq_P b$, then $m_q(a) \leq m_q(b)$ for every q, hence

$$|a|_{P,q} = q^{-m_q(a)} \geq q^{-m_q(b)} = |b|_{P,q}.$$

$\square$

Proposition 39. *For every prime q, the Pratt height*

$$|x|_{P,q} := q^{-m_q(x)}$$

transforms Pratt meets into maxima:

$$|a \wedge_P b|_{P,q} = \max\{|a|_{P,q}, |b|_{P,q}\}.$$

More generally,

$$|a_1 \wedge_P \cdots \wedge_P a_r|_{P,q} = \max_{1\le i\le r} |a_i|_{P,q}.$$

Proof. By definition of the Pratt meet,

$$m_q(a \wedge_P b) = \min(m_q(a), m_q(b)).$$

Hence

$$|a \wedge_P b|_{P,q} = q^{-m_q(a\wedge_P b)} = q^{-\min(m_q(a),m_q(b))} = \max(q^{-m_q(a)}, q^{-m_q(b)}),$$

which is exactly

$$|a \wedge_P b|_{P,q} = \max\{|a|_{P,q}, |b|_{P,q}\}.$$

The multi-variable case is identical. □

40 TODOs: the matrix $A = (m_p(q))$

A central linear object attached to the Pratt coordinates is the infinite matrix

$$A = (a_{pq})_{p,q\in\mathbb{P}}, \qquad a_{pq} := m_p(q).$$

Its q-th column is precisely the Pratt vector of the prime q:

$$\phi(q) = (m_p(q))_p.$$

If $n = \prod_q q^{v_q(n)}$ and $v(n) = (v_q(n))_q$ denotes the usual prime-exponent vector, then complete additivity gives

$$\phi(n) = A\, v(n).$$

Thus A is the transition matrix from the ordinary prime-factor coordinates to the Pratt coordinates.

Basic structure

When the primes are ordered increasingly as $2, 3, 5, 7, \ldots$, the matrix A is lower triangular with ones on the diagonal:

$$a_{pq} = 0 \quad (p > q), \qquad a_{pp} = 1.$$

Indeed, the Pratt tree of a prime q only contains labels at most q, and the root contributes exactly one occurrence of q. Consequently, every finite principal truncation A_N is invertible and satisfies

$$\det(A_N) = 1.$$

In particular, A is invertible on the finitely supported sequence space c_{00}, so the ordinary valuations $v_p(n)$ and the Pratt coordinates $m_p(n)$ determine each other uniquely.

Column recursion

The columns satisfy a direct recursive description. For $q = 2$ one has

$$\phi(2) = e_2.$$

If $q > 2$ is prime and

$$q - 1 = \prod_r r^{v_r(q-1)},$$

then the Pratt tree of q consists of a root labeled q together with the Pratt forests of the prime divisors of $q - 1$, counted with multiplicity. Hence

$$\phi(q) = e_q + \sum_{r|q-1} v_r(q-1)\,\phi(r),$$

and coordinatewise

$$m_p(q) = \delta_{p,q} + \sum_{r|q-1} v_r(q-1)\,m_p(r).$$

This recursion shows that A is built by repeatedly propagating the local factorization data of $q - 1$ through smaller primes.

Combinatorial interpretation

The entry $a_{pq} = m_p(q)$ counts the number of vertices labeled p in the Pratt tree of q, with multiplicity. In particular,

$$a_{pq} > 0$$

if and only if there exists a chain of primes

$$q = q_0, \qquad q_1 \mid q_0 - 1, \qquad q_2 \mid q_1 - 1, \ \ldots, \ q_k = p.$$

Thus A may be regarded as a weighted reachability matrix for the directed Pratt graph.

First examples

The first few columns are

$$\phi(2) = (1, 0, 0, 0, \dots),$$

$$\phi(3) = (1, 1, 0, 0, \dots),$$
$$\phi(5) = (2, 0, 1, 0, \dots),$$
$$\phi(7) = (2, 1, 0, 1, \dots),$$
$$\phi(11) = (3, 0, 1, 0, 1, \dots),$$
$$\phi(19) = (3, 2, 0, 0, 0, 0, 0, 1, \dots).$$

For example,

$$5 - 1 = 2^2 \implies \phi(5) = e_5 + 2\phi(2),$$
$$7 - 1 = 2 \cdot 3 \implies \phi(7) = e_7 + \phi(2) + \phi(3),$$
$$19 - 1 = 2 \cdot 3^2 \implies \phi(19) = e_{19} + \phi(2) + 2\phi(3).$$

These examples make visible the recursive nature of the columns.

The first row

The row indexed by $p = 2$ appears especially significant. The number $m_2(q)$ counts the number of 2-labels in the Pratt tree of q. By the recursion above,

$$m_2(q) = \sum_{r \mid q-1} v_r(q-1)\, m_2(r) \qquad (q > 2).$$

This row often dominates the others numerically and seems to measure a kind of Pratt height or combinatorial complexity. It is closely tied to iterated totient phenomena and therefore naturally relates to the framework of Erdős–Granville–Pomerance–Spiro.

Inverse matrix

Since A is unitriangular, it has an inverse $B = A^{-1}$, again unitriangular. Then

$$v(n) = B\,\phi(n).$$

So the ordinary prime-exponent vector can be recovered linearly from the Pratt coordinates. It is plausible that the coefficients of B encode a Pratt version of Möbius correction or inclusion–exclusion, and a systematic study of these entries may provide a useful linear approach to the Pratt Möbius theory developed elsewhere in this note.

Natural questions

The matrix $A = (m_p(q))$ raises several structural questions.

- Determine the growth of $m_p(q)$ for fixed small p as $q \to \infty$.
- Study the arithmetic and combinatorial structure of the inverse matrix A^{-1}.

- Express A more explicitly in terms of a one-step matrix encoding the factorization of $q-1$.
- Investigate spectra, row sums, and column sums of finite truncations.
- Clarify the relation between the first row, iterated totients, and Pratt heights.

A particularly suggestive auxiliary matrix is

$$N_{pq} := v_p(q-1).$$

Informally, N records the direct parent relations in the Pratt recursion, whereas A should be viewed as the weighted transitive closure of this one-step data. Making this relation precise would likely sharpen the linear-algebraic side of the theory.

41 Conclusion

The main message of this paper is that the label counts in Pratt forests form a coherent arithmetic system. Starting from the completely additive coordinates $m_p(n)$, one obtains a new partial order on the positive integers, a natural meet operation, explicit Euler-type product formulae, Möbius inversions on the associated poset, positive kernels and feature maps, and a family of canonical decompositions for polynomial values. What begins as a recursive combinatorial definition turns out to support algebraic, analytic, probabilistic, and Hilbert-space interpretations.

A second point is conceptual. The Pratt decomposition has appeared before, most notably in the work of Erdős, Granville, Pomerance, and Spiro on iterates of arithmetic functions [1]. There it serves as part of the background structure behind iterative phenomena. The present paper takes the opposite viewpoint: the decomposition itself is placed at the center and developed as an independent object. This shift reveals new questions about the Pratt poset, its Möbius function, its kernels, and the rigidity of the associated coefficient systems.

Several directions remain open. On the structural side, one would like a sharper understanding of joins, lower sets, and Möbius inversion on the Pratt poset. On the analytic side, the Dirichlet series attached to the functions C_n and to related transforms deserve a more systematic treatment. On the algebraic side, the Pratt–Möbius operator suggests a general calculus for polynomial and multivariate polynomial values over Pratt lower sets; its interaction with positivity, supports, and special polynomial families remains largely unexplored.

More broadly, the paper suggests that Pratt trees should be viewed not only as primality certificates, but as a source of arithmetic coordinates and canonical expansions. If that viewpoint is correct, then the Pratt poset may deserve a place alongside divisibility, additive, and valuation-based structures as a useful organizing framework in multiplicative number theory.

References

[1] P. Erdős, A. Granville, C. Pomerance, and C. Spiro, *On the Normal Behavior of the Iterates of Some Arithmetic Functions*, in *Analytic Number Theory (Allerton Park, IL, 1989)*, Progress in Mathematics **85**, Birkhäuser Boston, Boston, MA, 1990, pp. 165–204.

[2] S. Plouffe, *Numbers in the base* e^{π}, preprint, 2025.