Goldbach and the Prime Counting Function

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1 Goldbach and the prime counting function, 06.01.2022

In this report we summarize definitions and results concerning the Hilbert–Poincaré series approach to the prime counting function, following [1, 2, 3]. We denote by

$$V := \bigoplus_{n\in\mathbb{N}} V_n, \qquad V_n := \langle \log(1), \ldots, \log(n)\rangle_{\mathbb{Q}},$$

and use that:

1. $\log(n) = \sum_{p|n} v_p(n)\log(p)$,
2. the family $\{\log(p) : p \text{ prime}\}$ is linearly independent over $\mathbb{Q}$.

It follows that

$$\pi(n) = \dim_{\mathbb{Q}}(V_n),$$

and the Hilbert–Poincaré series

$$H(t) = \sum_{n=1}^{\infty} \pi(n)t^n$$

has radius of convergence 1 by the Prime Number Theorem. Writing $H(t) = f(t)/(1-t)$ with

$$f(t) = \sum_{p \text{ prime}} t^p,$$

one defines the sequence $(b_n)_{n\geq -1}$ by

$$\frac{f'(t)}{f(t)} = \sum_{n=-1}^{\infty} b_n t^n,$$

so that for each prime p,

$$p = 2 + \sum_{q<p} b_{p-1-q}.$$

1.1 Examples

The first coefficients b_n are given by

$$2t^{-1} + 1 - t + 4t^2 - 5t^3 + 11t^4 - 16t^5 + 22t^6 - 37t^7 + 67t^8 - \cdots$$

hence $b_{-1} = 2, b_0 = 1, b_1 = -1, b_2 = 4, \ldots$, and for instance:

$$3 = 2 + b_0, \quad 5 = 2 + b_2 + b_1, \quad 7 = 2 + b_4 + b_3 + b_1.$$

Recurrence for $a_{n,k}$

Let $a_{n,k}$ be the number of ordered representations of n as a sum of k primes. Then one shows

$$n\, a_{n,k} = k\sum_{v=0}^{n} a_{v,k} b_{n-1-v},$$

leading to

$$a_{n,k} = \frac{k}{n-2k}\sum_{v=0}^{n-1} a_{v,k} b_{n-1-v}.$$

Moreover if $\{\alpha_m\}$ are the nonzero roots of $f(t)$, then for $n \geq 0$

$$b_n = -\sum_m \frac{1}{\alpha_m^{n+1}}.$$

One real root is

$$\gamma = -0.629233\cdots$$

(see [4]). Numerical evidence suggests

$$\lim_{n\to\infty} \frac{b_n}{b_{n+1}} = \gamma,$$

which could characterize γ once existence of the limit and $f(\gamma) = 0$ are shown.

Question 1.1 (Q1). *Is there an approach to prove existence of the above limit and that $f(\gamma) = 0$?*

1.2 Goldbach and convolutions

Define

$$\pi^*(n) = \sum_{k=0}^{n} \pi(k)\pi(n-k).$$

Using series multiplication yields

$$a_{n,2} = \pi^*(n) - 2\pi^*(n-1) + \pi^*(n-2),$$

so Goldbach can be phrased as

$$\forall n \geq 2 : \ \tfrac{\pi^*(2n)+\pi^*(2n-2)}{2} > \pi^*(2n-1).$$

A short computation shows this holds for $n \leq 39$. One then relates π^* and b_n to express the nth prime via π^*.

Question 1.2 (Q2). *Can one derive an explicit formula for the nth prime in terms of the values of π^* using the above relations?*

Convexity conjecture

In general, for $x_1, \ldots, x_n \in \mathbb{N}$ with integer mean,

$$\frac{1}{n}\sum_{i=1}^{n} \pi^*(x_i) \geq \pi^*\big(\tfrac{1}{n}\sum_i x_i\big)$$

is conjectured.

Question 1.3 (Q3). *Is there a proof of the above convexity-type inequality for π^*? (Note that for $x_1 = 2n, x_2 = 2n-2$ a strict version gives Goldbach.)*

1.3 Appendix: Derivation of recurrence

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For k>=1:
log(f(t)^k)' = k f'(t)/f(t) = sum_{n=-1}^oo k b_n t^n.
Also f(t)^k = sum_{n=0}oo a_{n,k} t^n, so
log(f(t)^k)' = (sum n a_{n,k} t^{n-1})/(sum a_{n,k} t^n).
Multiplying and comparing coefficients yields
n a_{n,k} = k sum_{v=0}^n a_{v,k} b_{n-1-v}.
Hence a_{n,k} = k/(n-2k) sum_{v=0}^{n-1} a_{v,k} b_{n-1-v}.
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References

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[3] Hilbert–Poincaré series and Serre theorem, *Wikipedia*, accessed 2025. https://en.wikipedia.org/wiki/Hilbert%E2%80%93Poincar%C3%A9_series#Hilbert%E2%80%93Serre_theorem

[4] OEIS A078756: Real root of f(t). http://oeis.org/A078756