A Generalized Krieg-Paley Construction of Even Unimodular Lattices From Dimension 24 to an Infinite Family

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April 4, 2026

Abstract

In [1], Aloys Krieg and Max Koecher describe a concrete 24-dimensional even unimodular positive definite lattice via a Paley-type block construction; this matrix is one of the classical realizations of the Leech lattice. In this note we show that the same idea admits a two-parameter generalization. For every prime

$$p \equiv 3, 11, 15 \pmod{16}$$

we construct an explicit even unimodular positive definite lattice of dimension

$$d = 2(p + 1),$$

and prove this by a Schur-complement calculation. The original 24-dimensional Krieg construction is recovered from $p = 11$, while $p = 43$ yields an explicit lattice in dimension 88. By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many such primes, hence infinitely many dimensions produced by this explicit matrix construction. We also provide a SymPy verification script and tabulate the first ten admissible primes and dimensions.

Contents

1 Introduction

Theta series and modular forms provide a natural bridge between explicit Gram matrices and arithmetic structure. In the book *Elliptische Funktionen und Modulformen*, Koecher and Krieg discuss theta series of even unimodular lattices and present a concrete 24-dimensional matrix construction of the Leech lattice using quadratic residues modulo 11 [1]. In the notation of that construction one starts with a skew-symmetric Paley-type matrix A satisfying

$$A^t A = 11 I_{12},$$

and then forms the symmetric block matrix

$$L_{24} = \begin{pmatrix} 4I_{12} & A - 2I_{12} \\ A^t - 2I_{12} & 4I_{12} \end{pmatrix},$$

which is even, unimodular and positive definite.

The purpose of the present note is to show that the same underlying idea extends beyond dimension 24. The correct generalization is not obtained by keeping the constants 4 and 2 fixed. Instead one introduces two parameters and proves that the resulting block matrix is even, unimodular and positive definite exactly when a simple quadratic Diophantine condition is satisfied.

The resulting family has two especially attractive features:

(i) it recovers Krieg's 24-dimensional matrix as the special case $(p, u, v) = (11, 2, 1)$;

(ii) it yields a new explicit matrix realization in dimension 88 from the prime $p = 43$ via $(u, v) = (6, 5)$.

Moreover, Dirichlet's theorem implies that there are infinitely many admissible primes, hence infinitely many explicit even unimodular positive definite lattices arising from this construction.

2 The Paley-type input matrix

Let p be an odd prime with

$$p \equiv 3 \pmod 4.$$

Set

$$n := p + 1, \qquad e = (1, \dots, 1)^t \in \mathbb{Z}^p.$$

Let $\chi(a) = \left(\frac{a}{p}\right)$ denote the quadratic character modulo p, extended by $\chi(0) = 0$. Define

$$B = (\chi(i - j))_{1 \le i,j \le p} \in M_p(\mathbb{Z}),$$

and then

$$A := \begin{pmatrix} 0 & e^t \\ -e & B \end{pmatrix} \in M_n(\mathbb{Z}).$$

Lemma 2.1. *For $p \equiv 3 \pmod 4$, the matrix A satisfies*

$$A^t = -A, \qquad A^t A = pI_n.$$

Proof. Since $p \equiv 3 \pmod 4$, one has $\chi(-x) = -\chi(x)$ for $x \not\equiv 0 \pmod p$, hence $B^t = -B$. Thus $A^t = -A$.

The standard quadratic-character identities give

$$Be = 0, \qquad B^tB = pI_p - J_p,$$

where $J_p = ee^t$ is the all-ones matrix. Therefore

$$A^tA = \begin{pmatrix} e^te & -e^tB \\ -B^te & ee^t + B^tB \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & J_p + (pI_p - J_p) \end{pmatrix} = pI_n.$$

□

3 The generalized block construction

Let $a, b \in \mathbb{Z}$ and define the symmetric block matrix

$$G_{a,b}(p) := \begin{pmatrix} aI_n & A - bI_n \\ A^t - bI_n & aI_n \end{pmatrix}.$$

Its size is $2n \times 2n$, hence the corresponding dimension is

$$d = 2n = 2(p+1).$$

Proposition 3.1. *For every odd prime $p \equiv 3 \pmod 4$ one has*

$$\det G_{a,b}(p) = (a^2 - p - b^2)^n.$$

Moreover, $G_{a,b}(p)$ is positive definite if and only if

$$a > 0, \qquad a^2 - p - b^2 > 0.$$

Proof. Because $A^t = -A$ and $A^tA = pI_n$,

$$(A^t - bI_n)(A - bI_n) = A^tA - b(A^t + A) + b^2 I_n = (p + b^2)I_n.$$

The Schur complement of the upper-left block aI_n is therefore

$$aI_n - \frac{1}{a}(A^t - bI_n)(A - bI_n) = \frac{a^2 - p - b^2}{a} I_n.$$

Hence $G_{a,b}(p)$ is congruent over $\mathbb{R}$ to

$$\operatorname{diag}\left(aI_n, \frac{a^2 - p - b^2}{a} I_n\right),$$

which proves the criterion for positive definiteness.

The determinant is obtained from the same Schur complement formula:

$$\det G_{a,b}(p) = \det(aI_n) \cdot \det\left(\frac{a^2 - p - b^2}{a} I_n\right) = a^n \left(\frac{a^2 - p - b^2}{a}\right)^n = (a^2 - p - b^2)^n.$$

□

Theorem 3.2. *The matrix $G_{a,b}(p)$ is an even, unimodular, positive definite integral Gram matrix if and only if*

$$a \in 2\mathbb{Z}_{>0}, \qquad a^2 - p - b^2 = 1.$$

Equivalently, writing

$$a = 2u, \qquad b = 2v,$$

this is equivalent to

$$p = 4(u^2 - v^2) - 1.$$

In that case the dimension is

$$d = 2(p+1) = 8(u^2 - v^2),$$

so in particular d is divisible by 8.

Proof. Integrality is automatic because $A \in M_n(\mathbb{Z})$ and $a, b \in \mathbb{Z}$. The diagonal entries of $G_{a,b}(p)$ are all equal to a, so the corresponding quadratic form is even if and only if a is even.

By Proposition 3.1, unimodularity is equivalent to

$$|a^2 - p - b^2| = 1.$$

Under positive definiteness we must have $a^2 - p - b^2 > 0$, so the only possible sign is

$$a^2 - p - b^2 = 1.$$

Thus $G_{a,b}(p)$ is even, unimodular and positive definite exactly when

$$a \in 2\mathbb{Z}_{>0}, \qquad a^2 - p - b^2 = 1.$$

Now write $a = 2u$ and $b = 2v$. Then

$$4u^2 - p - 4v^2 = 1,$$

which is equivalent to

$$p = 4(u^2 - v^2) - 1.$$

Finally,

$$d = 2(p+1) = 2(4(u^2 - v^2)) = 8(u^2 - v^2).$$

□

4 Examples and the recovery of dimensions 24 and 88

The original Krieg construction is recovered immediately.

Corollary 4.1. *For $u = 2$ and $v = 1$ one gets*

$$p = 11, \qquad d = 24, \qquad a = 4, \qquad b = 2,$$

and $G_{4,2}(11)$ is precisely Krieg's 24-dimensional matrix construction.

Proof. Indeed,

$$p = 4(u^2 - v^2) - 1 = 4(4-1) - 1 = 11,$$

so $n = p + 1 = 12$ and therefore $d = 2n = 24$. The corresponding parameters are $a = 2u = 4$ and $b = 2v = 2$. □

The dimension-88 case also appears naturally.

Corollary 4.2. *For $u = 6$ and $v = 5$ one gets*

$$p = 43, \qquad d = 88, \qquad a = 12, \qquad b = 10,$$

and the matrix

$$G_{12,10}(43) = \begin{pmatrix} 12I_{44} & A - 10I_{44} \\ A^t - 10I_{44} & 12I_{44} \end{pmatrix}$$

is an even unimodular positive definite Gram matrix in dimension 88.

Proof. Here

$$p = 4(u^2 - v^2) - 1 = 4(36 - 25) - 1 = 43,$$

so $n = 44$ and $d = 88$. Theorem 3.2 applies. □

5 Infinitely many admissible primes

We now show that there are infinitely many primes for which the construction works.

Lemma 5.1. *Let $m \in \mathbb{Z}_{>0}$. Then m is a difference of two squares,*

$$m = u^2 - v^2,$$

if and only if

$$m \not\equiv 2 \pmod 4.$$

Proof. Since

$$u^2 - v^2 = (u - v)(u + v),$$

the two factors have the same parity. Thus a difference of two squares is either odd or divisible by 4, hence never congruent to 2 modulo 4.

Conversely, if m is odd, then

$$m = \left(\frac{m+1}{2}\right)^2 - \left(\frac{m-1}{2}\right)^2.$$

If $m \equiv 0 \pmod 4$, write $m = 4t$; then

$$m = (t + 1)^2 - (t - 1)^2$$

whenever $t \geq 1$, and the trivial case $m = 0$ does not occur here. Thus every positive integer $m \not\equiv 2 \pmod 4$ is a difference of two squares. □

Theorem 5.2. *There exist infinitely many primes p for which the generalized Krieg construction produces an even unimodular positive definite lattice. Equivalently, there exist infinitely many dimensions*

$$d = 2(p + 1)$$

of this form.

Proof. By Theorem 3.2, we need primes of the form

$$p = 4(u^2 - v^2) - 1.$$

Set

$$m = \frac{p+1}{4}.$$

Then this condition is equivalent to saying that m is a difference of two squares. By the preceding lemma, this is equivalent to

$$\frac{p+1}{4} \not\equiv 2 \pmod 4.$$

A short congruence check shows that, among primes $p \equiv 3 \pmod 4$, this is equivalent to

$$p \equiv 3, 11, 15 \pmod{16}.$$

Indeed, the only excluded class is $p \equiv 7 \pmod{16}$.

Dirichlet's theorem on primes in arithmetic progressions implies that each residue class

$$3,\ 11,\ 15 \pmod{16}$$

contains infinitely many primes. Therefore there are infinitely many admissible primes p, and hence infinitely many admissible dimensions $d = 2(p+1)$. □

This gives a second explicit construction mechanism for even unimodular positive definite lattices in dimensions divisible by 8: instead of starting from theta-series existence theorems or from Conway's Lorentzian quotient construction, one obtains them directly from block matrices built out of quadratic residue symbols.

6 The first ten admissible primes and dimensions

The first ten admissible primes are listed in Table 1. For each such prime we record one choice of integers (u, v) satisfying

$$p = 4(u^2 - v^2) - 1,$$

and the associated parameters

$$a = 2u, \qquad b = 2v, \qquad d = 2(p+1).$$

Table 1: The first ten admissible primes and dimensions

p	$d = 2(p+1)$	$(p+1)/4$	u	v	$(a,b) = (2u, 2v)$
3	8	1	1	0	(2,0)
11	24	3	2	1	(4,2)
19	40	5	3	2	(6,4)
31	64	8	3	1	(6,2)
43	88	11	6	5	(12,10)
47	96	12	4	2	(8,4)
59	120	15	8	7	(16,14)
67	136	17	9	8	(18,16)
79	160	20	6	4	(12,8)
83	168	21	11	10	(22,20)

7 A SymPy verification script

A short SymPy script can be used to construct the matrices $G_{a,b}(p)$ for the first ten admissible primes and verify the key identities

$$A^t = -A, \qquad A^t A = pI_n,$$

as well as integrality, evenness and the determinant formula

$$\det G_{a,b}(p) = (a^2 - p - b^2)^n = 1.$$

The full script is supplied as a companion file. Its main output lists the first ten admissible primes, the associated dimensions, and confirms the symbolic checks case by case.

8 A shifted self-dual code attached to the generalized Krieg–Paley matrix

The block matrix construction

$$G_{a,b}(p) = \begin{pmatrix} aI_n & A - bI_n \\ A^t - bI_n & aI_n \end{pmatrix}, \qquad n = p+1,$$

suggests a natural "shifted" code construction. The key point is that the off-diagonal block is not A itself, but rather the shifted matrix $A - bI_n$. This leads to a canonical self-dual code over every finite field whose characteristic divides a.

Proposition 8.1. *Let $p \equiv 3 \pmod 4$ be prime, let $A \in M_n(\mathbb{Z})$ be the Paley-type matrix from Section 2, and let $a, b \in \mathbb{Z}$ satisfy*

$$a > 0, \qquad a^2 - p - b^2 = 1.$$

Let q be a prime dividing a, and let $\mathbb{F}_q$ denote the finite field with q elements. Define the length-$2n$ linear code

$$C_q(a,b,p) := \langle\, (I_n \mid A - bI_n)\, \rangle \subset \mathbb{F}_q^{2n},$$

where the matrix $I_n \mid A - bI_n$ is reduced modulo q.

Then $C_q(a,b,p)$ is self-dual.

Proof. Set

$$M := (I_n \mid A - bI_n) \in M_{n,2n}(\mathbb{F}_q).$$

We compute

$$MM^t = I_n + (A - bI_n)(A^t - bI_n).$$

Since $A^t = -A$ and $A^tA = pI_n$, we have

$$(A - bI_n)(A^t - bI_n) = AA^t - b(A + A^t) + b^2 I_n = pI_n + b^2 I_n.$$

Hence

$$MM^t = (1 + p + b^2)I_n.$$

By the defining relation $a^2 - p - b^2 = 1$, this becomes

$$MM^t = a^2 I_n.$$

Now $q \mid a$, so modulo q we obtain

$$MM^t = 0.$$

Therefore the code generated by the rows of M is self-orthogonal.

On the other hand, because the left block of M is the identity matrix I_n, the rows of M are linearly independent over $\mathbb{F}_q$. Thus

$$\dim_{\mathbb{F}_q} C_q(a,b,p) = n.$$

Since $C_q(a,b,p) \subset \mathbb{F}_q^{2n}$ is self-orthogonal and has dimension $n = \frac{1}{2}(2n)$, it must be self-dual. □

Remark 8.2. *The shift by $-bI_n$ is essential. If one instead considers the naive generator matrix $(I_n \mid A)$, then in general*

$$(I_n \mid A)(I_n \mid A)^t = I_n + AA^t = (1+p)I_n,$$

which need not vanish modulo a given prime. By contrast, the shifted matrix $(I_n \mid A - bI_n)$ satisfies

$$(I_n \mid A - bI_n)(I_n \mid A - bI_n)^t = a^2 I_n,$$

so orthogonality is automatic modulo every prime divisor of a. Thus the shifted code is the natural finite-field object attached to the generalized Krieg–Paley construction.

Remark 8.3. *For the admissible family*

$$a = 2u, \qquad b = 2v, \qquad p = 4(u^2 - v^2) - 1,$$

the proposition yields a self-dual code over every prime field $\mathbb{F}_q$ with $q \mid 2u$. In particular:

- *for the* 24*-dimensional case $(u, v) = (2, 1)$ one gets a binary self-dual code;*
- *for the* 40*-dimensional case $(u, v) = (3, 2)$ one gets a ternary self-dual code;*
- *for the* 88*-dimensional case $(u, v) = (6, 5)$ one gets both binary and ternary self-dual codes;*
- *for the* 96*-dimensional case $(u, v) = (4, 2)$ one again gets a binary self-dual code.*

This provides a systematic code-theoretic companion to the explicit Gram matrix construction.

Remark 8.4. *Once the shifted code $C_q(a, b, p)$ is available, one may ask whether the lattice defined by the Gram matrix $G_{a,b}(p)$ is isometric to a Construction A lattice attached to $C_q(a, b, p)$. If such an isometry can be established, then the theta series of the lattice can be recovered from the weight enumerator of the code. The proposition above does not by itself prove such an isometry, but it shows that the generalized Krieg–Paley matrices come equipped with a canonical family of self-dual codes.*

9 Conclusion

The Leech-lattice matrix described by Krieg is not an isolated curiosity. After introducing the two parameters (a, b), the same Paley-type block ansatz yields an infinite family of even unimodular positive definite lattices in dimensions

$$d = 8(u^2 - v^2),$$

provided that the associated prime

$$p = 4(u^2 - v^2) - 1$$

exists. Dirichlet's theorem shows that this happens infinitely often, because all primes in the congruence classes

$$p \equiv 3, 11, 15 \pmod{16}$$

are admissible.

Thus the generalized Krieg-Paley construction furnishes a second explicit route to positive definite, even, unimodular lattices in higher dimensions, complementing other constructions based on theta series, codes, or Lorentzian quotients. In particular, the dimensions 24 and 88 arise naturally inside the same family.

10 Theta series in dimension 88 and recursive formulas for Ramanujan's τ-function

In this section we explain in detail how the theta series of an even unimodular lattice of rank 88 gives rise to a recursive determination of Ramanujan's tau-function. The main point is that once the theta series is written in the standard basis of $M_{44}(\mathbb{S}L_2(\mathbb{Z}))$, the coefficient of Δ appears linearly, and this linearity leads to a recursion for $\tau(n)$ in terms of the representation numbers of the lattice.

We first develop the general theory for an arbitrary even unimodular lattice of rank 88. After that we specialize to the four rank-88 lattices occurring in our computations.

10.1 General modular-form decomposition in weight 44

Let L be an even unimodular positive definite lattice of rank 88, and write

$$\Theta_L(\tau) = \sum_{v\in L} q^{(v,v)/2} = \sum_{n\geq 0} a_n q^n, \qquad q = e^{2\pi i\tau}.$$

Thus

$$a_n = \#\{v \in L : (v, v) = 2n\}.$$

Since L is even unimodular of rank 88, Θ_L is a modular form of weight 44 for $\mathbb{S}L_2(\mathbb{Z})$.

We recall the standard modular forms

$$E_4(\tau) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \qquad \Delta(\tau) = \sum_{n\geq 1} \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + \cdots .$$

Here

$$\sigma_3(n) = \sum_{d|n} d^3$$

is the divisor sum, and $\tau(n)$ is Ramanujan's tau-function.

Proposition 10.1. *The space $M_{44}(\mathbb{S}L_2(\mathbb{Z}))$ has dimension 4, and*

$$E_4^{11}, \qquad E_4^8\Delta, \qquad E_4^5\Delta^2, \qquad E_4^2\Delta^3$$

form a basis of this space. Consequently, for every even unimodular lattice L of rank 88, there exist uniquely determined constants $A_L, B_L, C_L \in \mathbb{C}$ such that

$$\Theta_L = E_4^{11} + A_L E_4^8\Delta + B_L E_4^5\Delta^2 + C_L E_4^2\Delta^3. \tag{1}$$

Proof. The graded ring of modular forms for $\mathbb{S}L_2(\mathbb{Z})$ is

$$M_*(\mathbb{S}L_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6] = \mathbb{C}[E_4, \Delta],$$

where E_4 has weight 4 and Δ has weight 12. Therefore $M_{44}(\mathbb{S}L_2(\mathbb{Z}))$ is spanned by all monomials

$$E_4^a\Delta^b$$

with

$$4a + 12b = 44, \qquad a, b \geq 0.$$

Dividing by 4, this becomes

$$a + 3b = 11.$$

The nonnegative integer solutions are

$$(a, b) = (11, 0), (8, 1), (5, 2), (2, 3).$$

Hence the four monomials

$$E_4^{11}, \qquad E_4^8\Delta, \qquad E_4^5\Delta^2, \qquad E_4^2\Delta^3$$

span $M_{44}(\mathbb{S}L_2(\mathbb{Z}))$. Since $\dim M_{44} = 4$, they form a basis. The decomposition (1) follows immediately. □

10.2 The first coefficients and the general formulas for A_L, B_L, C_L

We now determine the coefficients A_L, B_L, C_L in terms of the first Fourier coefficients a_1, a_2, a_3 of Θ_L.

First note that

$$E_4 = 1 + 240q + 2160q^2 + 6720q^3 + \cdots, \qquad \Delta = q - 24q^2 + 252q^3 + \cdots.$$

Substituting these into (1) and comparing coefficients of q, q^2, q^3 yields explicit linear equations.

Theorem 10.2. *Let*

$$\Theta_L(\tau) = \sum_{n\geq 0} a_n q^n$$

be the theta series of an even unimodular lattice of rank 88*, and write*

$$\Theta_L = E_4^{11} + A_L E_4^8 \Delta + B_L E_4^5 \Delta^2 + C_L E_4^2 \Delta^3.$$

Then

$$A_L = a_1 - 2640, \tag{2}$$

$$B_L = a_2 - 1896a_1 + 1813680, \tag{3}$$

$$C_L = a_3 + 599940a_1 - 1152a_2 - 244992000. \tag{4}$$

Proof. We compute the first coefficients of the basis elements.

Coefficient of q. Since

$$E_4^{11} = 1 + 11 \cdot 240\, q + \cdots = 1 + 2640q + \cdots$$

and $\Delta = q + \cdots$, while Δ^2 and Δ^3 begin with q^2 and q^3, respectively, the coefficient of q in (1) is

$$a_1 = 2640 + A_L.$$

Hence

$$A_L = a_1 - 2640.$$

Coefficient of q^2. The coefficient of q^2 in E_4^{11} is

$$11 \cdot 2160 + \binom{11}{2} 240^2 = 23760 + 55 \cdot 57600 = 3191760.$$

The coefficient of q^2 in $E_4^8\Delta$ is

$$[q^1]E_4^8 \cdot [q^1]\Delta + [q^0]E_4^8 \cdot [q^2]\Delta = (8 \cdot 240) \cdot 1 + 1 \cdot (-24) = 1920 - 24 = 1896.$$

The coefficient of q^2 in $E_4^5\Delta^2$ is 1, because $\Delta^2 = q^2 + \cdots$. Finally, $E_4^2\Delta^3$ contributes nothing to q^2. Therefore

$$a_2 = 3191760 + 1896A_L + B_L.$$

Substituting $A_L = a_1 - 2640$, we obtain

$$B_L = a_2 - 1896a_1 + 1813680.$$

Coefficient of q^3. The coefficient of q^3 in E_4^{11} is

$$11 \cdot 6720 + 11 \cdot 10 \cdot 240 \cdot 2160 + \binom{11}{3} 240^3 = 2338057920.$$

The coefficient of q^3 in $E_4^8\Delta$ is

$$[q^2]E_4^8 \cdot [q^1]\Delta + [q^1]E_4^8 \cdot [q^2]\Delta + [q^0]E_4^8 \cdot [q^3]\Delta.$$

Now

$$[q^1]E_4^8 = 8 \cdot 240 = 1920,$$

and

$$[q^2]E_4^8 = 8 \cdot 2160 + \binom{8}{2} 240^2 = 17280 + 28 \cdot 57600 = 1630080.$$

Hence

$$[q^3](E_4^8\Delta) = 1630080 \cdot 1 + 1920 \cdot (-24) + 1 \cdot 252 = 1584252.$$

Similarly,

$$[q^3](E_4^5\Delta^2) = [q^1]E_4^5 \cdot [q^2]\Delta^2 + [q^0]E_4^5 \cdot [q^3]\Delta^2.$$

Since

$$[q^1]E_4^5 = 5 \cdot 240 = 1200, \qquad \Delta^2 = q^2 - 48q^3 + \cdots,$$

we get

$$[q^3](E_4^5\Delta^2) = 1200 \cdot 1 + 1 \cdot (-48) = 1152.$$

Finally, since $\Delta^3 = q^3 + \cdots$, we have

$$[q^3](E_4^2\Delta^3) = 1.$$

Therefore

$$a_3 = 2338057920 + 1584252A_L + 1152B_L + C_L.$$

Substituting the formulas already found for A_L and B_L gives

$$C_L = a_3 + 599940a_1 - 1152a_2 - 244992000.$$

This proves the claim. □

Corollary 10.3. *If L has minimum 4, then $a_1 = 0$, and the formulas simplify to*

$$A_L = -2640, \tag{5}$$
$$B_L = a_2 + 1813680, \tag{6}$$
$$C_L = a_3 - 1152a_2 - 244992000. \tag{7}$$

Proof. If $\min(L) = 4$, then there are no vectors of norm 2, so $a_1 = 0$. Substitute $a_1 = 0$ into (2)–(4). □

10.3 General coefficient formulas and the recursion for $\tau(n)$

We now rewrite the decomposition in a form that isolates $\tau(n)$.

Define

$$s_0 = 1, \qquad s_n = 240\,\sigma_3(n) \quad (n \geq 1),$$

so that

$$E_4(q) = \sum_{n\geq 0} s_n q^n.$$

For each integer $r \geq 1$, write

$$E_4(q)^r = \sum_{n\geq 0} e_r(n) q^n.$$

Then

$$e_r(n) = \sum_{\substack{n_1+\cdots+n_r=n \\ n_i\geq 0}} s_{n_1}\cdots s_{n_r}. \tag{8}$$

Thus each $e_r(n)$ is an explicit polynomial in the divisor sums $\sigma_3(m)$.

Next define the standard convolution sums

$$T_2(n) = \sum_{\substack{i+j=n \\ i,j\geq 1}} \tau(i)\tau(j), \qquad (n\geq 2), \tag{9}$$

$$T_3(n) = \sum_{\substack{i+j+k=n \\ i,j,k\geq 1}} \tau(i)\tau(j)\tau(k), \qquad (n\geq 3). \tag{10}$$

Equivalently,

$$\Delta(q)^2 = \sum_{n\geq 2} T_2(n)q^n, \qquad \Delta(q)^3 = \sum_{n\geq 3} T_3(n)q^n.$$

Proposition 10.4. *Let L be an even unimodular lattice of rank 88, and let A_L, B_L, C_L be as in (1). Then for every $n\geq 1$,*

$$a_n = e_{11}(n) + A_L\sum_{m=1}^{n} e_8(n-m)\tau(m) + B_L\sum_{m=2}^{n} e_5(n-m)T_2(m) + C_L\sum_{m=3}^{n} e_2(n-m)T_3(m). \tag{11}$$

Proof. Expand each term in (1) as a power series:

$$E_4^{11} = \sum_{n\geq 0} e_{11}(n)q^n, \qquad E_4^8\Delta = \left(\sum_{r\geq 0} e_8(r)q^r\right)\left(\sum_{m\geq 1}\tau(m)q^m\right),$$

$$E_4^5\Delta^2 = \left(\sum_{r\geq 0} e_5(r)q^r\right)\left(\sum_{m\geq 2} T_2(m)q^m\right),$$

$$E_4^2\Delta^3 = \left(\sum_{r\geq 0} e_2(r)q^r\right)\left(\sum_{m\geq 3} T_3(m)q^m\right).$$

Now take the coefficient of q^n on both sides. This gives exactly (11). □

The crucial observation is that the $m=n$ term in the second sum contributes

$$A_L e_8(0)\tau(n) = A_L\tau(n),$$

because $e_8(0)=1$. By contrast, the sums involving $T_2(n)$ and $T_3(n)$ only depend on $\tau(1),\ldots,\tau(n-1)$, since in the definitions of $T_2(n)$ and $T_3(n)$ all indices are positive and strictly smaller than n.

This gives the desired recursion.

Theorem 10.5 (General recursion for $\tau(n)$ from a rank-88 lattice)**.** *Let L be an even unimodular lattice of rank 88, with theta coefficients*

$$\Theta_L(\tau) = \sum_{n\geq 0} a_n q^n,$$

and modular decomposition

$$\Theta_L = E_4^{11} + A_L E_4^8\Delta + B_L E_4^5\Delta^2 + C_L E_4^2\Delta^3.$$

Assume that

$$A_L \neq 0.$$

Then for every $n \geq 1$,

$$\tau(n) = \frac{1}{A_L}\left(a_n - e_{11}(n) - A_L \sum_{m=1}^{n-1} e_8(n-m)\tau(m) - B_L \sum_{m=2}^{n} e_5(n-m)T_2(m) - C_L \sum_{m=3}^{n} e_2(n-m)T_3(m)\right). \tag{12}$$

In particular, $\tau(n)$ is determined recursively by the lattice representation numbers $a_1, \dots, a_n$ and the divisor sums $\sigma_3(1), \dots, \sigma_3(n)$.

Proof. Start from (11). Isolate the term with $m = n$ in the second sum:

$$\sum_{m=1}^{n} e_8(n-m)\tau(m) = e_8(0)\tau(n) + \sum_{m=1}^{n-1} e_8(n-m)\tau(m) = \tau(n) + \sum_{m=1}^{n-1} e_8(n-m)\tau(m),$$

because $e_8(0) = 1$. Thus (11) becomes

$$a_n = e_{11}(n) + A_L\tau(n) + A_L \sum_{m=1}^{n-1} e_8(n-m)\tau(m) + B_L \sum_{m=2}^{n} e_5(n-m)T_2(m) + C_L \sum_{m=3}^{n} e_2(n-m)T_3(m).$$

Now solve for $\tau(n)$:

$$A_L\tau(n) = a_n - e_{11}(n) - A_L \sum_{m=1}^{n-1} e_8(n-m)\tau(m) - B_L \sum_{m=2}^{n} e_5(n-m)T_2(m) - C_L \sum_{m=3}^{n} e_2(n-m)T_3(m).$$

Since $A_L \neq 0$, division by A_L gives (12).

Finally, the coefficients $e_r(n)$ are explicit polynomials in the values $\sigma_3(m)$ by (8), while $T_2(n)$ and $T_3(n)$ only involve earlier values of τ. Hence the formula is indeed recursive. □

Remark 10.6. *The nonvanishing condition $A_L \neq 0$ is equivalent to*

$$a_1 \neq 2640$$

by (2). In all examples below this condition is satisfied.

10.4 The first steps of the recursion

For later use we record the first few consequences.

Proposition 10.7. *Under the assumptions of Theorem 10.5, the recursion recovers the initial values*

$$\tau(1) = 1, \qquad \tau(2) = -24, \qquad \tau(3) = 252.$$

Proof. We illustrate the first three steps.

Step $n = 1$. Since $T_2(1)$ and $T_3(1)$ do not occur, (11) gives

$$a_1 = e_{11}(1) + A_L\tau(1).$$

But $e_{11}(1) = 11 \cdot 240 = 2640$ and $A_L = a_1 - 2640$, so

$$a_1 = 2640 + (a_1 - 2640)\tau(1).$$

Because $A_L \neq 0$, this forces $\tau(1) = 1$.

Step $n = 2$. Now

$$T_2(2) = \tau(1)^2 = 1.$$

Also

$$e_{11}(2) = 3191760, \qquad e_8(1) = 8 \cdot 240 = 1920.$$

Thus

$$a_2 = e_{11}(2) + A_L(\tau(2) + e_8(1)\tau(1)) + B_L T_2(2).$$

Since $\tau(1) = 1$, this determines $\tau(2)$ uniquely. Substituting the formulas for A_L and B_L from Theorem 10.2 gives $\tau(2) = -24$.

Step $n = 3$. Here

$$T_2(3) = 2\tau(1)\tau(2), \qquad T_3(3) = \tau(1)^3.$$

Moreover

$$e_{11}(3) = 2338057920, \qquad e_8(2) = 8 \cdot 2160 + \binom{8}{2} 240^2 = 1630080, \qquad e_5(1) = 5 \cdot 240 = 1200.$$

Hence

$$a_3 = e_{11}(3) + A_L(\tau(3) + e_8(1)\tau(2) + e_8(2)\tau(1)) + B_L(T_2(3) + e_5(1)T_2(2)) + C_L T_3(3).$$

Since $\tau(1) = 1$ and $\tau(2) = -24$ are already known, this determines $\tau(3)$. After simplification one obtains $\tau(3) = 252$. □

10.5 Four explicit rank-88 lattices and the corresponding recursions

We now specialize the preceding theory to the four rank-88 lattices appearing in our computations. Among the rank-88 classes found in the search, there are four distinct initial theta profiles. We choose one representative lattice for each profile and denote them by

$$L_{88}^{(1)}, \qquad L_{88}^{(2)}, \qquad L_{88}^{(3)}, \qquad L_{88}^{(4)}.$$

Their initial theta coefficients are:

$$(a_1, a_2, a_3) = \begin{cases} (0, 125944, 98241758), & L_{88}^{(1)}, \\ (0, 125184, 99276008), & L_{88}^{(2)}, \\ (0, 160114, 143441218), & L_{88}^{(3)}, \\ (82, 295176, 235176678), & L_{88}^{(4)}. \end{cases}$$

The first three lattices have minimum 4, while the fourth has minimum 2.

Theorem 10.8. *The four theta series are given by*

$$\Theta_{L_{88}^{(1)}} = E_4^{11} - 2640\, E_4^8\Delta + 1939624\, E_4^5\Delta^2 - 291837730\, E_4^2\Delta^3, \tag{13}$$

$$\Theta_{L_{88}^{(2)}} = E_4^{11} - 2640\, E_4^8\Delta + 1938864\, E_4^5\Delta^2 - 289927960\, E_4^2\Delta^3, \tag{14}$$

$$\Theta_{L_{88}^{(3)}} = E_4^{11} - 2640\, E_4^8\Delta + 1973794\, E_4^5\Delta^2 - 286002110\, E_4^2\Delta^3, \tag{15}$$

$$\Theta_{L_{88}^{(4)}} = E_4^{11} - 2558\, E_4^8\Delta + 1953384\, E_4^5\Delta^2 - 300662994\, E_4^2\Delta^3. \tag{16}$$

Proof. For the first three lattices, $a_1 = 0$, so Corollary 10.3 applies.
For $L_{88}^{(1)}$ we have

$$a_2 = 125944, \qquad a_3 = 98241758,$$

hence

$$A = -2640, \qquad B = 125944 + 1813680 = 1939624,$$

$$C = 98241758 - 1152 \cdot 125944 - 244992000 = -291837730.$$

This gives (13).

For $L_{88}^{(2)}$ we have

$$a_2 = 125184, \qquad a_3 = 99276008,$$

so

$$A = -2640, \qquad B = 125184 + 1813680 = 1938864,$$

$$C = 99276008 - 1152 \cdot 125184 - 244992000 = -289927960.$$

This gives (14).

For $L_{88}^{(3)}$ we have

$$a_2 = 160114, \qquad a_3 = 143441218,$$

so

$$A = -2640, \qquad B = 160114 + 1813680 = 1973794,$$

$$C = 143441218 - 1152 \cdot 160114 - 244992000 = -286002110.$$

This gives (15).

For $L_{88}^{(4)}$ we use the general formulas (2)–(4), because here $a_1 = 82$. Thus

$$A = 82 - 2640 = -2558,$$

$$B = 295176 - 1896 \cdot 82 + 1813680 = 1953384,$$

$$C = 235176678 + 599940 \cdot 82 - 1152 \cdot 295176 - 244992000 = -300662994.$$

This gives (16). □

10.6 The four explicit recursive formulas for $\tau(n)$

For convenience, define

$$S_1(n) = \sum_{m=1}^{n-1} e_8(n-m)\tau(m),$$

$$S_2(n) = \sum_{m=2}^{n} e_5(n-m)T_2(m),$$

$$S_3(n) = \sum_{m=3}^{n} e_2(n-m)T_3(m).$$

Then Theorem 10.5 gives the following four recursive presentations of Ramanujan's tau-function.

Corollary 10.9 (Recursive formulas for $\tau(n)$ coming from the four lattices)**.** *For every $n \geq 1$ we have:*

$$\tau(n) = \frac{1}{2640}\Big(e_{11}(n) - a_n^{(1)} - 2640\, S_1(n) + 1939624\, S_2(n) - 291837730\, S_3(n)\Big), \tag{17}$$

$$\tau(n) = \frac{1}{2640}\Big(e_{11}(n) - a_n^{(2)} - 2640\, S_1(n) + 1938864\, S_2(n) - 289927960\, S_3(n)\Big), \tag{18}$$

$$\tau(n) = \frac{1}{2640}\Big(e_{11}(n) - a_n^{(3)} - 2640\, S_1(n) + 1973794\, S_2(n) - 286002110\, S_3(n)\Big), \tag{19}$$

$$\tau(n) = \frac{1}{2558}\Big(e_{11}(n) - a_n^{(4)} - 2558\, S_1(n) + 1953384\, S_2(n) - 300662994\, S_3(n)\Big), \tag{20}$$

where

$$\Theta_{L_{88}^{(j)}}(\tau) = \sum_{n\geq 0} a_n^{(j)} q^n.$$

Proof. This is obtained by substituting the values of (A_L, B_L, C_L) from Theorem 10.8 into the general recursion (12). For example, in the first case

$$a_n^{(1)} = e_{11}(n) - 2640(\tau(n) + S_1(n)) + 1939624\, S_2(n) - 291837730\, S_3(n),$$

hence

$$2640\,\tau(n) = e_{11}(n) - a_n^{(1)} - 2640\, S_1(n) + 1939624\, S_2(n) - 291837730\, S_3(n),$$

which is exactly (17). The other three formulas are identical in form. □

Remark 10.10. *These identities provide four lattice-specific recursive realizations of $\tau(n)$. The mechanism itself is classical: it comes from writing a theta series as a modular form and comparing coefficients. What is special here is that the lattices are explicit rank-88 even unimodular lattices, and the resulting recursions are tied directly to their representation numbers*

$$a_n^{(j)} = \#\{v \in L_{88}^{(j)} : (v, v) = 2n\}.$$

In this sense, the formulas above may be viewed as new recursive presentations of Ramanujan's tau-function arising from these explicit dimension-88 lattices.

Remark 10.11. *The first three lattices all have minimum 4, so in those cases the coefficient of $E_4^8\Delta$ is automatically -2640. Thus the differences between (17), (18), and (19) are entirely encoded in the coefficients of $E_4^5\Delta^2$ and $E_4^2\Delta^3$. The fourth lattice has roots, hence $a_1 = 82$, and this changes the linear coefficient from -2640 to -2558.*

Remark 10.12. *For practical computations, one proceeds inductively as follows. Assume that $\tau(1), \ldots, \tau(n-1)$ are known. Then:*

1. *compute $e_{11}(n)$, $e_8(n-m)$, $e_5(n-m)$, $e_2(n-m)$ from the divisor sums $\sigma_3(r)$ via (8);*
2. *compute $T_2(m)$ and $T_3(m)$ from the already known values $\tau(1), \ldots, \tau(n-1)$ using (9) and (10);*
3. *insert these quantities into one of the recursions (17)–(20).*

This yields $\tau(n)$.

10.7 Conway–Lorentz constructions in rank 88 and explicit (A, B, X)-data

We now record the Conway–Lorentz construction in rank 88 in the four explicit instances obtained in our search. In each case one starts from an identity of the form

$$A^2 + (A+1)^2 + \cdots + B^2 = X^2, \qquad B - A + 1 = 88,$$

and forms the primitive isotropic vector

$$w = (0, A, A+1, \ldots, B; X) \in II_{89,1}.$$

Then

$$(w, w) = 0,$$

so the quotient

$$L^{\mathrm{Con}}(A, B, X) := w^{\perp}/\mathbb{Z}w$$

is an even unimodular positive definite lattice of rank 88.[1]

[1] For a detailed account of this construction, including explicit bases and further examples, see O. Leka, *A Lorentzian Construction in Dimension 88 and Infinitely Many Further Ranks*, `https://www.orges-leka.de/88_lorentz_paper.pdf`.

In the four cases singled out here, the triples (A, B, X) are:

$$(A_1, B_1, X_1) = (192, 279, 2222),$$
$$(A_2, B_2, X_2) = (225, 312, 2520),$$
$$(A_3, B_3, X_3) = (4152, 4239, 39556),$$
$$(A_4, B_4, X_4) = (5925, 6012, 56078).$$

Equivalently, the corresponding square identities are

$$192^2 + 193^2 + \cdots + 279^2 = 2222^2, \tag{21}$$
$$225^2 + 226^2 + \cdots + 312^2 = 2520^2, \tag{22}$$
$$4152^2 + 4153^2 + \cdots + 4239^2 = 39556^2, \tag{23}$$
$$5925^2 + 5926^2 + \cdots + 6012^2 = 56078^2. \tag{24}$$

Thus we obtain four explicit Lorentzian quotient lattices

$$L_{88}^{\mathrm{Con},(j)} := L^{\mathrm{Con}}(A_j, B_j, X_j), \qquad j = 1, 2, 3, 4.$$

Remark 10.13. *The first four Conway-type rank-*88 *solutions found in the search naturally fall into distinct theta profiles. In particular, the first two examples have kissing numbers* 125944 *and* 125184*, while the third and fourth examples share the same minimum and kissing profile in the initial data but arise from different explicit Lorentzian parameters* (A, B, X)*. Thus the Conway construction already produces several explicit rank-*88 *lattices with different initial theta behavior.*

Acknowledgment

I would like to thank Aloys Krieg for the stimulating suggestion to view the problem through the lens of modular forms.

References

[1] M. Koecher and A. Krieg, *Elliptische Funktionen und Modulformen*, Springer-Lehrbuch Masterclass, Springer, Berlin-Heidelberg, 1998.