Practical Chamber Logic on Lattices: A2 Visualizations, Lorentzian Implementations, and the Engineering Value of Dense Lattices

Orges Leka

April 2026

Abstract

We develop a practical logic whose atomic predicates are implemented on lattice points by means of chambers. The hard predicate attached to a lattice point a is the upper-set test

$$P_a^C(x) = \mathbf{1}[x - a \in C],$$

where C is a chosen chamber. Its fuzzy version is obtained from a chamber margin and therefore has a direct geometric and implementable meaning. The guiding principle is not merely to produce another representation of an abstract preorder, but to obtain a logic whose semantics can be physically or numerically realized on a lattice, with controllable noise margins and explicit quantization behavior.

The paper has four goals. First, we define hard and fuzzy chamber logics on integral lattices and prove their basic properties in detail. Second, we explain the entire construction visually in the model case of the triangular lattice A_2, using several figures. Third, we insert a concrete kernel-to-logic pipeline based on centered Gram matrices, kernel PCA, and dense-lattice vector quantization. This turns a bare positive semidefinite kernel on finitely many objects into an explicit chamber logic implementation. Fourth, we compare this chamber viewpoint with the product-algebra/RKHS program of the paper *Logical algebras and reproducing kernel Hilbert spaces*: our logic becomes coordinatewise after a chamber choice, but its definition is intrinsically geometric and its implementation-theoretic content is visible from the outset. Fifth, we formulate a Lorentzian implementation principle which places E_8 and the Leech lattice into the same conceptual picture as A_2: the chamber is chosen upstairs in a Lorentzian model, and the quotient lattice inherits the logic.

The final theme is practical. The KPCA layer is explicitly treated as an approximation layer: one first compresses the RKHS geometry to a low-dimensional Euclidean model and then quantizes to a dense lattice. This is not exact logical equivalence to the original RKHS, but a structured approximation whose distortion can be estimated. We prove that, at fixed nearest-neighbor safety margin, a denser lattice stores more implementable logical states per unit volume. We then make this explicit for three canonical lattices: A_2, E_8, and the Leech lattice. In this sense the densest known lattices do not merely look beautiful; they support chamber logic with a concrete engineering advantage.

Contents

1 Introduction

The point of departure of this note is simple: a logic may be viewed not only as a symbolic calculus or as an abstract order, but also as an *implemented geometry.* A discrete implementation needs states, margins, robustness under perturbation, and a way to avoid arbitrary coordinate choices. Lattices provide a natural state space, chambers provide entailment directions, and dense lattices provide strong packing and storage properties.

This note is intentionally close in style to the recent arithmetic/Lorentzian note of Orges Leka. There, the basic mechanism is the Conway quotient

$$w^{\perp}/\mathbb{Z}w,$$

attached to a primitive isotropic vector in an integral Lorentzian lattice, and this mechanism is used to realize concrete Euclidean lattices such as the Leech lattice and, in the rank-8 case, E_8. The same philosophy will be used here, but with a different target: we use the Lorentzian framework to motivate and organize a *logic* on the resulting lattice rather than only the lattice itself.

The paper *Semantic Space of Logic* may be read as a one-direction version of the present story: there one fixes a perspective vector and obtains a logic from projection to a single semantic axis. The later paper *Logical algebras and reproducing kernel Hilbert spaces* then proceeds from an arbitrary countable preorder $(X, \preceq)$ and proves that it embeds into a coordinatewise product algebra $[0,1]^I$ and further into an RKHS, with implication of truth value 1 agreeing exactly with coordinatewise order. Our present note moves in the opposite direction. We begin with concrete geometric data — a lattice and a chamber family — and obtain from them a preorder, a hard Heyting logic, a fuzzy logic, and then a product/RKHS realization. In the second part of the paper we also run the construction in the opposite practical direction: starting from a finite kernel matrix, we perform kernel PCA, quantize into a dense lattice in dimensions 2, 3, 8, or 24, and then apply chamber logic on the quantized states. Thus the comparison with *Logical algebras and reproducing kernel Hilbert spaces* is not accidental: on finite windows our chamber logic becomes a special case of its coordinatewise order theorem, but the route to that coordinatewise form is geometric rather than axiomatic.

The leitidee is the following.

> *A practical logic should be implementable on a lattice, stable under perturbation, and independent of arbitrary chamber choices up to symmetry. Dense lattices are especially attractive because they maximize the number of robustly separated states per unit volume.*

The rest of the paper develops this statement carefully. The core mathematical tasks are:

(1) define hard and fuzzy chamber predicates and prove their basic properties;

(2) prove a chamber-independence theorem under symmetry;

(3) explain the entire mechanism concretely on A_2;

(4) compare the construction with the coordinatewise product-algebra/RKHS framework;

(5) lift the picture to Lorentzian implementations, so that E_8 and the Leech lattice fit the same general pattern;

(6) isolate and prove the practical benefit of dense lattices.

We will work in a level of generality broad enough for A_2, E_8, and the Leech lattice, but narrow enough that every claim used in the main line of the argument is either proved explicitly or reduced to a standard structural fact.

2 Hard chamber logic on lattices

2.1 Lattices, chambers, and atomic predicates

Definition 2.1. Let V be a finite-dimensional Euclidean real vector space. An *integral lattice* in V is a discrete subgroup $\Lambda \subset V$ of full rank such that $\langle x, y \rangle \in \mathbb{Z}$ for all $x, y \in \Lambda$.

Definition 2.2. A *chamber presentation* on V is a family of continuous affine-linear functionals

$$\ell_i : V \to \mathbb{R}, \qquad i \in I,$$

where I is either finite or locally finite, such that the set

$$C := \bigcap_{i \in I} \{v \in V : \ell_i(v) \geq 0\}$$

has nonempty interior. We call C the associated *chamber*. If all ℓ_i are linear, then C is a cone; if some are affine, then C is an affine chamber.

Remark 2.3. The root-lattice examples A_2 and E_8 are of the linear type. The Leech example arises most naturally from an affine chamber obtained from a Lorentzian cusp construction.

Definition 2.4. Fix a lattice $\Lambda \subset V$ and a chamber $C \subset V$. For $a, x \in \Lambda$ define

$$a \preceq_C x \quad :\Longleftrightarrow \quad x - a \in C.$$

Equivalently, in terms of the chamber presentation,

$$a \preceq_C x \quad \Longleftrightarrow \quad \ell_i(x - a) \geq 0 \quad \text{for all } i \in I.$$

The associated *hard atomic predicate* is

$$P_a^C(x) := \mathbf{1}[a \preceq_C x].$$

Proposition 2.5. *For every chamber C, the relation $\preceq_C$ is a preorder on Λ. If $C \cap (-C) = \{0\}$, then $\preceq_C$ is antisymmetric and therefore a partial order.*

Proof. Reflexivity is immediate because $x - x = 0 \in C$: every defining inequality gives $\ell_i(0) = 0$. For transitivity, assume $a \preceq_C x$ and $x \preceq_C y$. Then

$$y - a = (y - x) + (x - a).$$

Since each ℓ_i is affine-linear on differences, more precisely linear on vectors, we obtain

$$\ell_i(y - a) = \ell_i(y - x) + \ell_i(x - a) \geq 0 + 0 = 0$$

for all i. Hence $a \preceq_C y$. Thus $\preceq_C$ is a preorder.

If additionally $C \cap (-C) = \{0\}$ and both $a \preceq_C x$ and $x \preceq_C a$, then $x - a \in C$ and $a - x = -(x - a) \in C$. Hence $x - a \in C \cap (-C) = \{0\}$, so $x = a$. Therefore the preorder is antisymmetric. □

Definition 2.6. A subset $U \subseteq \Lambda$ is called *C-upper* if

$$x \in U \text{ and } x \preceq_C y \Longrightarrow y \in U.$$

We denote by $\mathcal{U}_C(\Lambda)$ the family of all C-upper subsets of Λ.

Proposition 2.7. *The family $\mathcal{U}_C(\Lambda)$ is closed under arbitrary unions and arbitrary intersections. In particular, with meet $\cap$ and join $\cup$, it is a complete distributive lattice. Every atomic predicate set*

$$\{x \in \Lambda : P_a^C(x) = 1\} = a + C \cap \Lambda$$

is C-upper.

Proof. Let $\{U_j\}_{j \in J}$ be a family of C-upper sets. If $x \in \bigcup_j U_j$ and $x \preceq_C y$, then $x \in U_{j_0}$ for some j_0, hence $y \in U_{j_0} \subseteq \bigcup_j U_j$. Thus the union is C-upper.

Likewise, if $x \in \bigcap_j U_j$ and $x \preceq_C y$, then for every j we have $x \in U_j$, hence $y \in U_j$. Therefore $y \in \bigcap_j U_j$. So intersections are also C-upper. Distributivity follows from ordinary set-theoretic distributivity.

Finally, the set of x satisfying $P_a^C(x) = 1$ is by definition

$$\{x \in \Lambda : x - a \in C\} = (a + C) \cap \Lambda.$$

If $x - a \in C$ and $x \preceq_C y$, then $y - x \in C$, and since C is closed under addition of chamber vectors, we get

$$y - a = (y - x) + (x - a) \in C.$$

Hence the set is C-upper. □

Remark 2.8. Thus the chamber does not merely define a preorder; it defines a full propositional universe of upper sets. The atomic predicates P_a^C are the simplest upper sets and serve as building blocks.

Theorem 2.9 (Native logical structure)**.** *For every chamber C, the complete lattice $\mathcal{U}_C(\Lambda)$ of C-upper sets is a complete Heyting algebra. The Heyting implication is*

$$U \Rightarrow_H V := \{x \in \Lambda : \forall y \in \Lambda,\ x \preceq_C y \text{ and } y \in U \Longrightarrow y \in V\},$$

and the Heyting negation is

$$\neg_H U := U \Rightarrow_H \varnothing.$$

Proof. It is standard that the upper sets of a preorder form a complete lattice under union and intersection; this was proved in proposition 2.7. It remains to verify the adjunction property for the displayed implication.

Let $W \in \mathcal{U}_C(\Lambda)$. We claim that

$$W \cap U \subseteq V \quad \Longleftrightarrow \quad W \subseteq U \Rightarrow_H V.$$

Assume first that $W \cap U \subseteq V$, and let $x \in W$. Since W is upper and $x \preceq_C y$ implies $y \in W$, every $y \succeq_C x$ belongs to W. Hence if moreover $y \in U$, then $y \in W \cap U \subseteq V$. Therefore $x \in U \Rightarrow_H V$. This proves $W \subseteq U \Rightarrow_H V$.

Conversely, suppose $W \subseteq U \Rightarrow_H V$ and let $x \in W \cap U$. Since $x \in W \subseteq U \Rightarrow_H V$, the defining condition of $U \Rightarrow_H V$ applied to $y = x$ yields $x \in V$ because $x \preceq_C x$ and $x \in U$. Thus $W \cap U \subseteq V$.

So $U \Rightarrow_H V$ is right adjoint to intersection with U, which is exactly the Heyting condition. Completeness follows from the completeness of the underlying lattice of upper sets. □

Remark 2.10. This theorem is logically important: the hard chamber logic is natively intuitionistic/Heyting in general. We do not need to force a Boolean structure on top of the lattice geometry.

2.2 Simplicial chambers and coordinatewise order

The comparison with coordinatewise logical algebras becomes transparent when the chamber is simplicial.

Definition 2.11. A chamber C is called *simplicial* if there exist linearly independent linear functionals

$$\lambda_1, \dots, \lambda_d \in V^*$$

with

$$C = \{v \in V : \lambda_i(v) \geq 0 \text{ for all } i = 1, \dots, d\}.$$

We call the λ_i the defining wall coordinates of C.

Proposition 2.12. *Let C be a simplicial chamber with wall coordinates $\lambda_1, \dots, \lambda_d$. Then*

$$a \preceq_C x \iff \lambda_i(x-a) \geq 0 \text{ for all } i.$$

Equivalently, the map

$$\eta_C(x) := (\lambda_1(x), \dots, \lambda_d(x)) \in \mathbb{R}^d$$

transports the chamber order to the ordinary coordinatewise order on $\mathbb{R}^d$:

$$a \preceq_C x \iff \eta_C(a) \leq \eta_C(x) \text{ coordinatewise.}$$

Proof. The first equivalence is exactly the definition of the chamber. For the second, observe that

$$\lambda_i(x-a) = \lambda_i(x) - \lambda_i(a).$$

Therefore $\lambda_i(x-a) \geq 0$ for all i if and only if $\lambda_i(a) \leq \lambda_i(x)$ for all i, which is exactly the coordinatewise inequality for $\eta_C(a)$ and $\eta_C(x)$. □

Remark 2.13. This proposition is the first bridge to *Logical algebras and reproducing kernel Hilbert spaces*. There the coordinatewise order is built from an abstract preorder by embedding into $[0,1]^I$; here the coordinatewise order appears automatically after choosing a chamber.

3 Fuzzy chamber logic

3.1 The chamber margin

We now soften the hard predicate. The idea is not to use an ad hoc scalar score but the signed margin to the chamber walls.

Definition 3.1. Let C be given by normalized wall functionals

$$\ell_i : V \to \mathbb{R}, \qquad \|\ell_i\|_* = 1,$$

with

$$C = \bigcap_{i \in I} \{v : \ell_i(v) \geq 0\}.$$

The associated *chamber margin* is

$$m_C(v) := \inf_{i \in I} \ell_i(v).$$

Proposition 3.2. *For every $v \in V$ one has*

$$v \in C \iff m_C(v) \geq 0.$$

Moreover, if each ℓ_i has dual norm 1, then m_C is 1-Lipschitz:

$$|m_C(v) - m_C(w)| \leq \|v - w\|.$$

Proof. By definition,

$$m_C(v) = \inf_i \ell_i(v).$$

Hence $m_C(v) \geq 0$ if and only if every $\ell_i(v) \geq 0$, which is equivalent to $v \in C$.

For the Lipschitz statement, each ℓ_i individually is 1-Lipschitz because

$$|\ell_i(v) - \ell_i(w)| = |\ell_i(v - w)| \leq \|\ell_i\|_* \cdot \|v - w\| = \|v - w\|.$$

Now fix $v, w \in V$. For every i we have

$$\ell_i(v) \leq \ell_i(w) + \|v - w\|.$$

Taking infima over i yields

$$m_C(v) \leq m_C(w) + \|v - w\|.$$

Exchanging v and w gives

$$m_C(w) \leq m_C(v) + \|v - w\|.$$

Combining the two inequalities proves

$$|m_C(v) - m_C(w)| \leq \|v - w\|.$$

□

3.2 Sigmoid truth values

Definition 3.3. The *sigmoid chamber truth value* attached to $a \in \Lambda$ is

$$\mu_a^C(x) := \sigma(m_C(x - a)), \qquad \sigma(t) := \frac{1}{1 + e^{-t}}.$$

Remark 3.4. The hard predicate P_a^C is the sign-threshold version of the same chamber margin, while μ_a^C is its canonical smooth relaxation. No additional sharpness parameter is built into the logic: the only numerical quantity is the chamber margin itself.

Proposition 3.5. *For every $a \in \Lambda$ and $x \in V$ one has*

$$\mu_a^C(x) > \frac{1}{2} \iff x - a \in \operatorname{int}(C), \qquad \mu_a^C(x) = \frac{1}{2} \iff m_C(x-a) = 0, \qquad \mu_a^C(x) < \frac{1}{2} \iff x - a \notin C.$$

In particular, the value 1/2 is the chamber wall value.

Proof. The sigmoid is strictly increasing and satisfies $\sigma(0) = 1/2$. Hence

$$\sigma(t) > 1/2 \iff t > 0, \qquad \sigma(t) = 1/2 \iff t = 0, \qquad \sigma(t) < 1/2 \iff t < 0.$$

Applying this to $t = m_C(x - a)$ and using proposition 3.2 gives the stated implications. □

Theorem 3.6 (Noise stability for the canonical fuzzy logic)**.** *The sigmoid chamber predicate is 1/4-Lipschitz in the state variable:*

$$|\mu_a^C(x) - \mu_a^C(y)| \leq \frac{1}{4} \|x - y\|$$

for all $x, y \in V$. In particular, additive perturbations of size at most δ change the truth value by at most $\delta/4$.

Proof. The derivative of the sigmoid is

$$\sigma'(t) = \sigma(t)(1 - \sigma(t)),$$

and the elementary inequality $u(1-u) \leq 1/4$ for $u \in [0,1]$ implies

$$|\sigma'(t)| \leq 1/4$$

for all $t \in \mathbb{R}$. Therefore σ is 1/4-Lipschitz. By definition,

$$\mu_a^C(x) = \sigma(m_C(x - a)).$$

Hence

$$\begin{aligned} |\mu_a^C(x) - \mu_a^C(y)| &\leq \frac{1}{4}\, |m_C(x-a) - m_C(y-a)| \\ &\leq \frac{1}{4}\, \|(x-a) - (y-a)\| \\ &= \frac{1}{4}\, \|x - y\|, \end{aligned}$$

where the second inequality is proposition 3.2. □

Remark 3.7. This theorem is one of the main practical motivations for chamber logic. The truth value is not only geometrically meaningful; it is *stable*. The margin measures robustness, and the sigmoid converts this robustness into a canonical fuzzy truth scale.

4 Applications to practical logic implementation via KPCA and dense lattice quantization

The previous sections defined the logic once a lattice and a chamber had already been chosen. We now explain how to reach that situation in practice when one starts only with finitely many objects and a positive semidefinite kernel.

4.1 Finite kernel data and kernel PCA

Let

$$X = \{x_1, \ldots, x_n\}$$

be a finite set, and let

$$k : X \times X \to \mathbb{R}$$

be a positive semidefinite kernel. Its Gram matrix is

$$K = (k(x_i, x_j))_{1 \leq i,j \leq n}.$$

Write

$$H_n := I_n - \frac{1}{n}\mathbf{1}\mathbf{1}^\top, \qquad K_c := H_n K H_n$$

for the centered kernel matrix.

Definition 4.1. The *kernel principal component coordinates* of the finite kernel datum (X, k) are obtained from an orthogonal diagonalization

$$K_c = U\Lambda U^\top, \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq 0.$$

For a target dimension $d \leq n$, the d-dimensional KPCA coordinate of x_j is

$$z_d(x_j) := \left(\sqrt{\lambda_1}u_1(j), \ldots, \sqrt{\lambda_d}u_d(j)\right) \in \mathbb{R}^d,$$

where u_r denotes the rth column of U.

Proposition 4.2. *For all* $1 \leq i, j \leq n$ *one has*

$$\langle z_d(x_i), z_d(x_j)\rangle = \sum_{r=1}^{d} \lambda_r u_r(i) u_r(j).$$

Consequently,

$$(K_c)_{ij} = \langle z_d(x_i), z_d(x_j)\rangle + \sum_{r>d} \lambda_r u_r(i) u_r(j).$$

In particular, z_d *is the best rank-*d *spectral approximation to the centered kernel geometry.*

Proof. The first identity is the definition of the Euclidean inner product in the coordinate vector $z_d(x_j)$. The second identity follows from the spectral decomposition

$$K_c = \sum_{r=1}^{n} \lambda_r u_r u_r^\top.$$

The optimality statement is exactly the Eckart–Young theorem applied to the symmetric positive semidefinite matrix K_c. □

Remark 4.3. The KPCA layer is therefore an approximation layer. It preserves only the leading spectral directions of the RKHS geometry. This is deliberate: the practical logic is meant to be implementable in dimensions such as 2, 3, 8, or 24, not to reproduce the full ambient Hilbert space exactly.

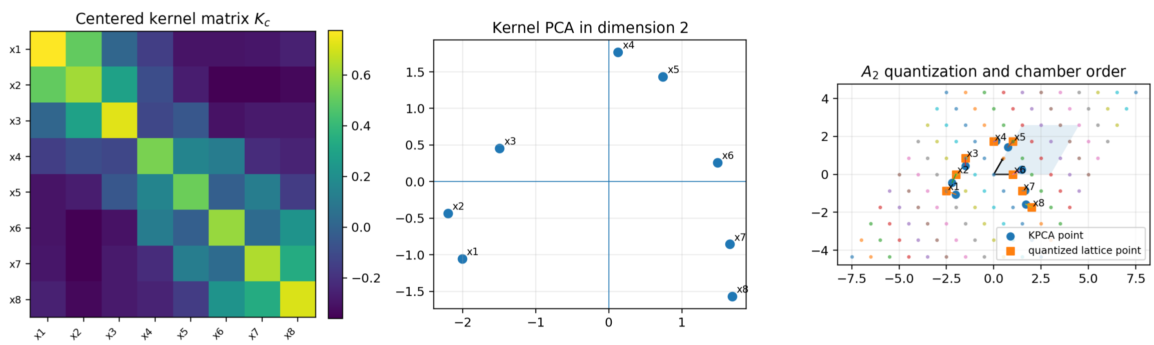

Figure 1: A finite kernel datum, its centered Gram matrix, the resulting two-dimensional KPCA embedding, and quantization to the triangular lattice A_2. This is the practical bridge from a kernel to a chamber logic.

4.2 Quantization to a dense lattice

Fix now a target lattice $\Lambda_d \subset \mathbb{R}^d$ and a nearest-neighbor vector quantizer

$$Q_{\Lambda_d} : \mathbb{R}^d \to \Lambda_d.$$

Set

$$\lambda(x_j) := Q_{\Lambda_d}(z_d(x_j)).$$

This replaces the original finite kernel datum by a finite cloud of lattice states.

Definition 4.4. Let C be an admissible chamber of Λ_d. The *KPCA–lattice chamber logic* of (X, k) in dimension d is the chamber logic on the quantized states $\lambda(x_j)$:

$$P_a^C(x_j) := \mathbf{1}[\lambda(x_j) - \lambda(a) \in C], \qquad \mu_a^C(x_j) := \sigma(m_C(\lambda(x_j) - \lambda(a))).$$

Proposition 4.5. *Suppose that the chamber margin on $\mathbb{R}^d$ is 1-Lipschitz. Let*

$$\widetilde{\mu}_a^C(x_j) := \sigma(m_C(z_d(x_j) - z_d(a)))$$

be the pre-quantization sigmoid truth value. Then

$$|\mu_a^C(x_j) - \widetilde{\mu}_a^C(x_j)| \le \frac{1}{4}\Big(\|\lambda(x_j) - z_d(x_j)\| + \|\lambda(a) - z_d(a)\|\Big).$$

Proof. By the 1/4-Lipschitz property of the sigmoid and the 1-Lipschitz property of the chamber margin,

$$\begin{aligned}
|\mu_a^C(x_j) - \widetilde{\mu}_a^C(x_j)| &= \Big|\sigma(m_C(\lambda(x_j) - \lambda(a))) - \sigma(m_C(z_d(x_j) - z_d(a)))\Big| \\
&\le \frac{1}{4}\Big|m_C(\lambda(x_j) - \lambda(a)) - m_C(z_d(x_j) - z_d(a))\Big| \\
&\le \frac{1}{4}\Big\|(\lambda(x_j) - \lambda(a)) - (z_d(x_j) - z_d(a))\Big\| \\
&\le \frac{1}{4}\Big(\|\lambda(x_j) - z_d(x_j)\| + \|\lambda(a) - z_d(a)\|\Big),
\end{aligned}$$

which is the claimed estimate. □

Remark 4.6. Proposition 4.5 is the main correctness estimate for the practical pipeline. The KPCA stage creates a low-dimensional semantic approximation, and the lattice quantizer creates a robust discrete approximation. The chamber logic then operates on this robust approximation, and its truth values differ from the continuous KPCA truth values by at most the explicit quantization error bound above.

4.3 Canonical target dimensions and canonical chamber families

The dimensions 2, 3, 8, and 24 are distinguished because they support highly symmetric dense lattices:

$$A_2, \qquad A_3 \cong D_3 \text{ (FCC)}, \qquad E_8, \qquad \Lambda_{24}.$$

In the first three cases one uses the natural Weyl chamber family. In dimension 24 one uses Conway's Lorentzian cusp chamber family upstairs in $II_{25,1}$ and then transports the logic to the quotient lattice.

Definition 4.7. The *default practical targets* are the following.

(1) $d = 2$: the triangular lattice A_2 with its six Weyl chambers;

(2) $d = 3$: the face-centered cubic lattice $A_3 \cong D_3$ with its Weyl chambers;

(3) $d = 8$: the lattice E_8 with a simple-root chamber, equivalently the cone of nonnegative simple-root coordinates;

(4) $d = 24$: the Leech lattice, implemented through Conway's chamber in $II_{25,1}$.

Remark 4.8. For E_8 one may implement the chamber explicitly by choosing a simple-root basis $\alpha_1, \ldots, \alpha_8$ with Gram matrix equal to the E_8 Cartan matrix. Then

$$C_{E_8} = \Big\{ \sum_{i=1}^{8} c_i \alpha_i : c_i \geq 0 \Big\}$$

is the standard dominant chamber, and the chamber margin is simply $m_{C_{E_8}}(\sum c_i \alpha_i) = \min_i c_i$.

Remark 4.9. For the Leech lattice one does not force an intrinsic finite root system. Instead one works in the Lorentzian lattice

$$II_{25,1} = \Lambda_{24} \oplus II_{1,1},$$

chooses Conway's isotropic Weyl vector

$$\rho = (0, 0, 1),$$

and uses the standard simple roots

$$r_\lambda = (\lambda, 1, \lambda^2/2 - 1), \qquad \lambda \in \Lambda_{24}.$$

The practical chamber is then implemented upstairs by the inequalities determined by these simple roots, and the resulting logic is pushed down to the quotient $\rho^\perp/\mathbb{Z}\rho \cong \Lambda_{24}$. This is exactly the right 24-dimensional analogue of a Weyl chamber for implementation purposes.

4.4 Voting and learned overlays

Once the kernel has been chamberized, one may aggregate several truth assignments without changing the underlying logic. If $x^{(1)}, \ldots, x^{(m)} \in X$ represent several observations or several agents, then the averaged chamber truth value

$$\overline{\mu}_a^C := \frac{1}{m} \sum_{j=1}^{m} \mu_a^C(x^{(j)})$$

defines a natural chamber version of belief voting or truth voting. This is an optional overlay rather than part of the native logic. Likewise one may attach a learned layer, such as a support-vector machine on the KPCA coordinates, but this is again an outer learning layer rather than the core chamber logic itself.

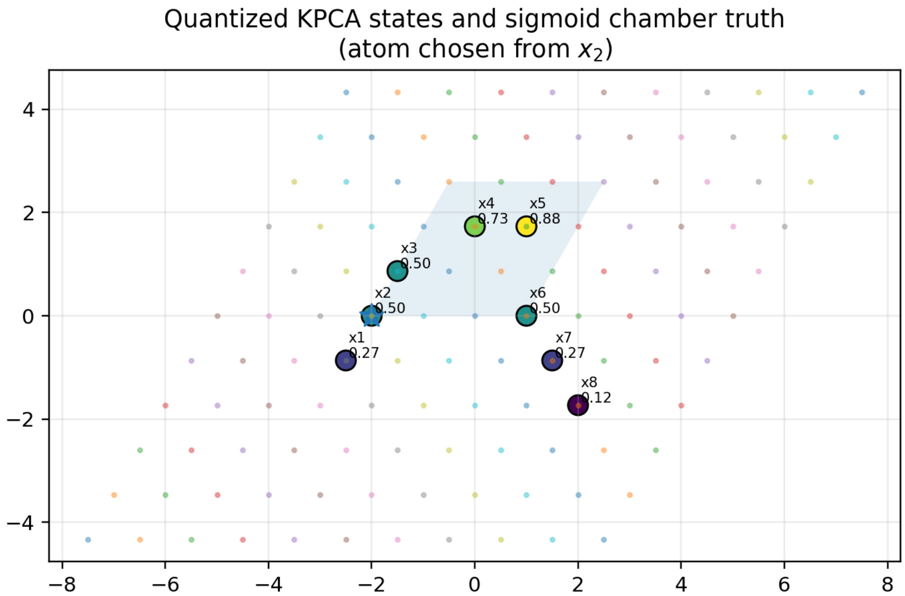

Figure 2: A toy finite kernel dataset after KPCA and A_2 quantization. The colors display a sigmoid chamber truth value for one chosen atomic predicate.

5 The model example A_2

5.1 The lattice and its Gram matrix

We now explain everything explicitly in dimension 2. Let

$$b_1 = (1, 0), \qquad b_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Then

$$A_2 = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2$$

is the triangular lattice. Its Gram matrix in the basis (b_1, b_2) is

$$G_{A_2} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix},$$

so

$$\det(G_{A_2}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Hence the covolume of A_2 is

$$\det(A_2) = \sqrt{\det(G_{A_2})} = \frac{\sqrt{3}}{2}.$$

Proposition 5.1. *The nearest-neighbor distance in A_2 is 1. Consequently the packing radius is 1/2, and the lattice packing density of A_2 equals*

$$\Delta(A_2) = \frac{\pi(1/2)^2}{\det(A_2)} = \frac{\pi}{2\sqrt{3}}.$$

Proof. Every nonzero lattice vector has the form $mb_1 + nb_2$. Its squared norm is

$$\|mb_1 + nb_2\|^2 = m^2 + mn + n^2.$$

The smallest positive value of the quadratic form $m^2 + mn + n^2$ on $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ is 1, attained for example at $(1, 0)$, $(0, 1)$, and $(1, -1)$. Thus the shortest nonzero vectors have length 1, so the packing radius is $1/2$.

The area of a radius-$1/2$ disk is $\pi/4$, while the fundamental cell area equals $\sqrt{3}/2$. Therefore

$$\Delta(A_2) = \frac{\pi/4}{\sqrt{3}/2} = \frac{\pi}{2\sqrt{3}}.$$

□

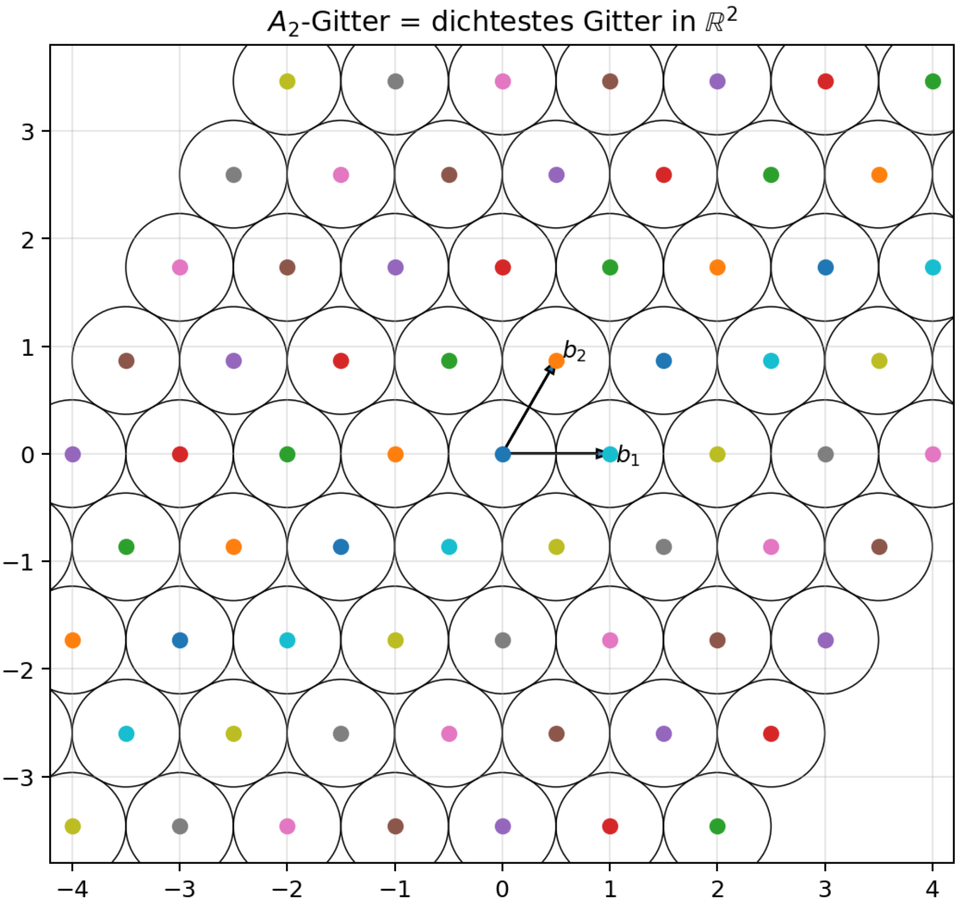

Figure 3: The triangular lattice A_2 with its densest circle packing. This is the basic state space for the visual part of the paper.

5.2 The chamber and the induced logic

Choose the standard chamber

$$C_0 := \{\alpha b_1 + \beta b_2 : \alpha \geq 0,\ \beta \geq 0\}.$$

Thus for lattice points

$$a = a_1 b_1 + a_2 b_2, \qquad x = x_1 b_1 + x_2 b_2,$$

we have

$$a \preceq_{C_0} x \quad \Longleftrightarrow \quad a_1 \leq x_1 \text{ and } a_2 \leq x_2.$$

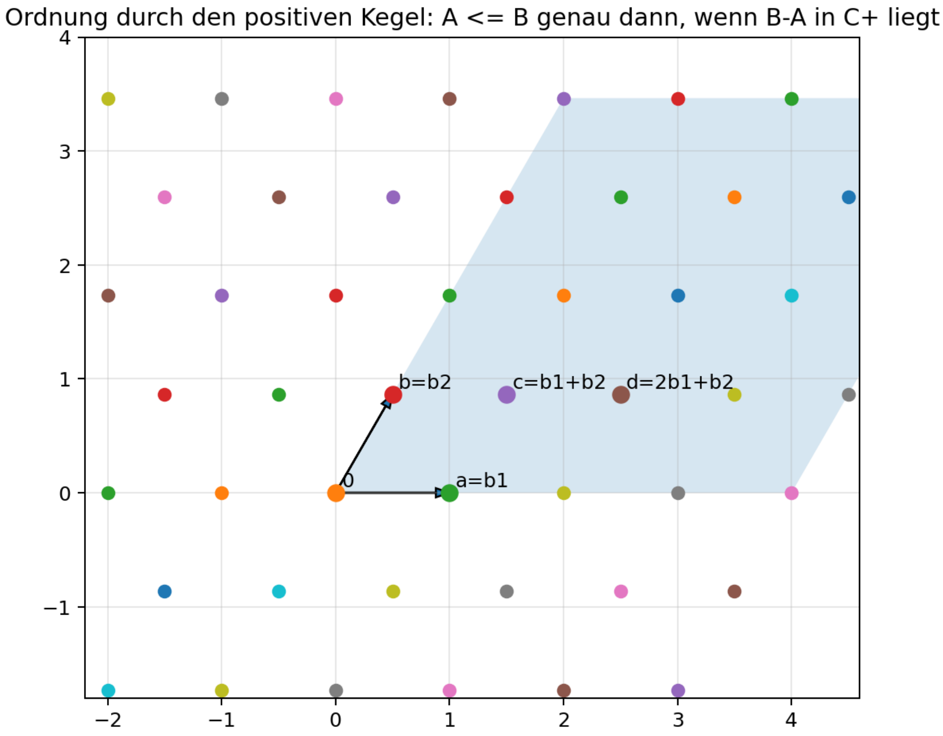

Figure 4: The positive chamber C_0 in A_2. The chamber order is the positive-cone order: $a \preceq_{C_0} x$ if and only if $x - a \in C_0$.

Proposition 5.2. *For $a = b_1$, the atomic predicate set*

$$\{x \in A_2 : P_a^{C_0}(x) = 1\}$$

consists exactly of the lattice points with basis coordinates (m, n) satisfying $m \geq 1$ and $n \geq 0$.

Proof. Write $x = mb_1 + nb_2$. Then

$$x - a = (m-1)b_1 + nb_2.$$

By definition, $x - a \in C_0$ if and only if the coefficients of b_1 and b_2 are both nonnegative, namely if and only if $m - 1 \geq 0$ and $n \geq 0$. This is exactly the stated condition. $\square$

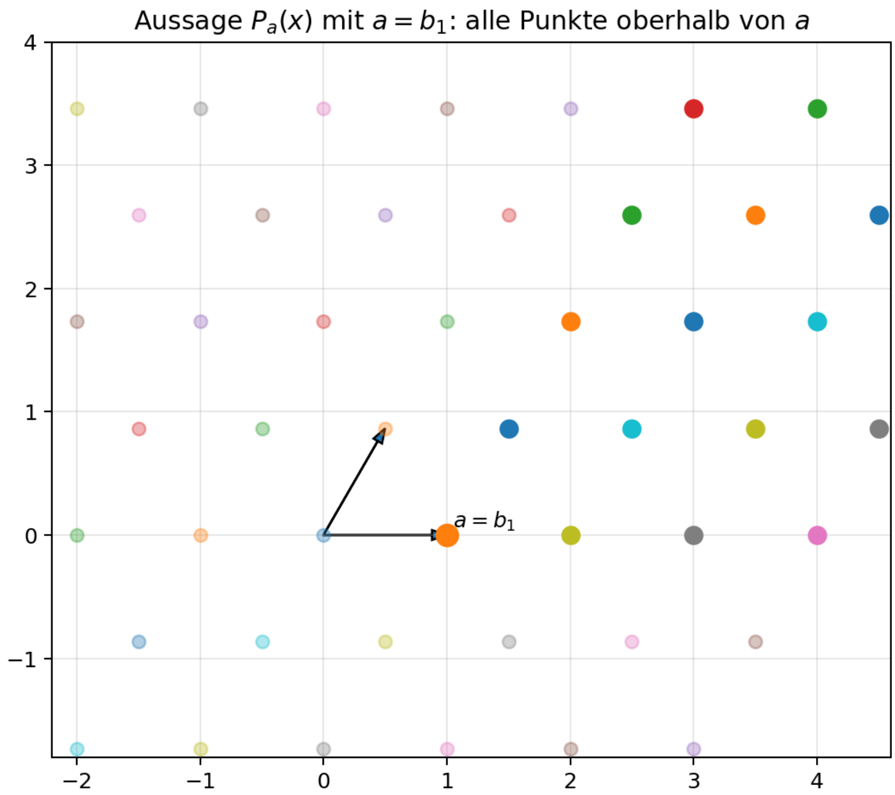

Figure 5: The upper set defined by the atomic predicate $P_{b_1}^{C_0}$. Dark points satisfy the predicate; lighter points do not.

Proposition 5.3. *Let $a = b_1$ and $b = b_2$. Then*

$$P_a^{C_0} \wedge P_b^{C_0} = P_{a+b}^{C_0}$$

as predicates on A_2.

Proof. Let $x = mb_1 + nb_2$. By proposition 5.2, the truth of $P_a^{C_0}(x)$ is equivalent to $m \geq 1$ and $n \geq 0$, while the truth of $P_b^{C_0}(x)$ is equivalent to $m \geq 0$ and $n \geq 1$. Both hold simultaneously if and only if

$$m \geq 1 \quad \text{and} \quad n \geq 1.$$

But this is exactly the condition that

$$x - (a + b) = (m - 1)b_1 + (n - 1)b_2$$

lies in C_0. Hence

$$P_a^{C_0}(x) = 1 \text{ and } P_b^{C_0}(x) = 1 \quad \Longleftrightarrow \quad P_{a+b}^{C_0}(x) = 1.$$

Therefore the conjunction of the two predicates equals $P_{a+b}^{C_0}$. □

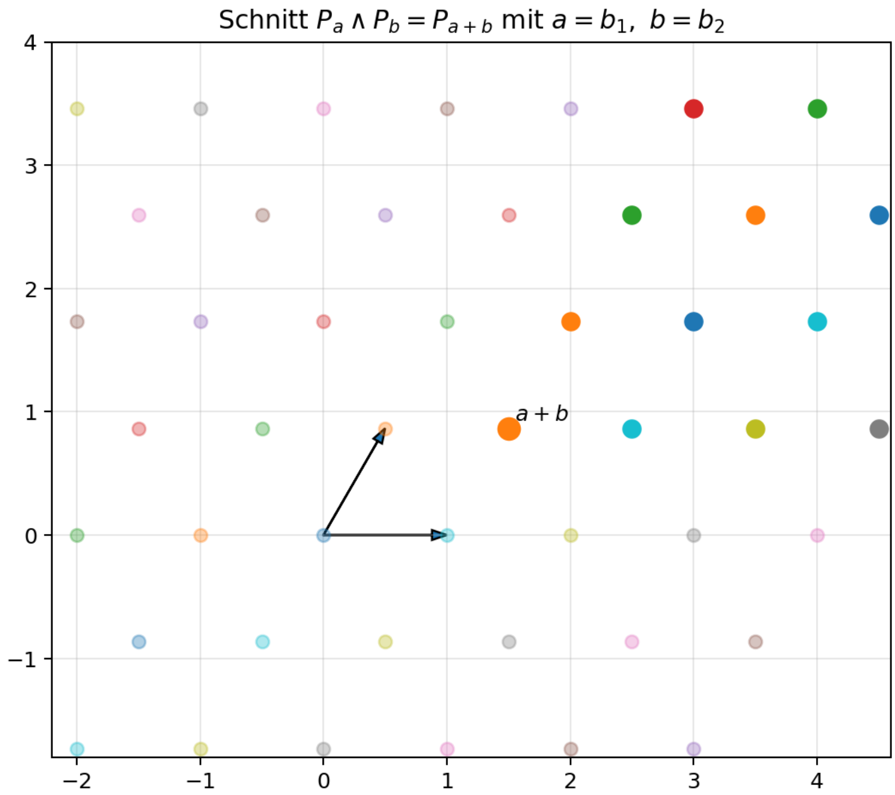

Figure 6: For $a = b_1$ and $b = b_2$, the conjunction $P_a^{C_0} \wedge P_b^{C_0}$ is the upper set generated by $a + b$.

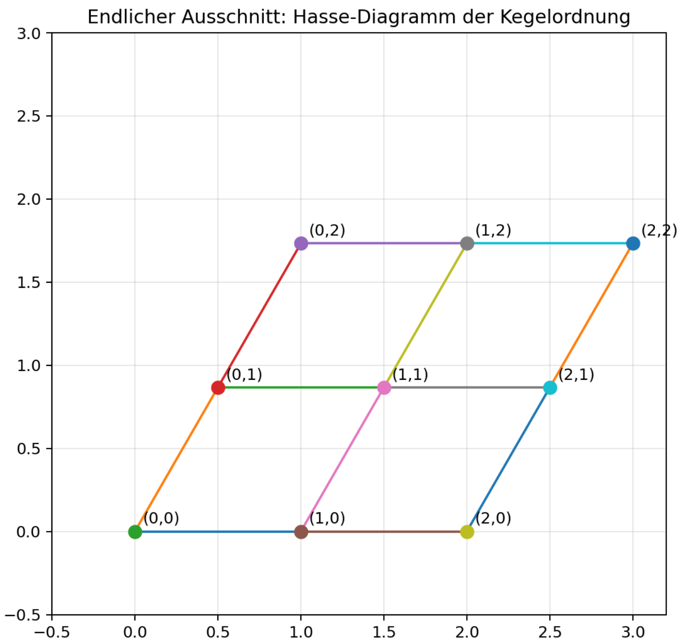

Figure 7: A finite window of the A_2 chamber order, viewed as a Hasse diagram. This is the discrete logical skeleton underlying the pictures above.

5.3 A four-state hard logic on a finite A_2 window

Let

$$W := \{0, a, b, c\}, \qquad a := b_1, \qquad b := b_2, \qquad c := b_1 + b_2.$$

Inside this finite window the chamber order is the diamond

$$0 < a < c, \qquad 0 < b < c, \qquad a \parallel b.$$

Define the atomic predicates

$$p := P_a^{C_0}, \qquad q := P_b^{C_0}, \qquad r := P_c^{C_0}.$$

Proposition 5.4. *On the finite window W the truth table of the atomic predicates is*

state x	$p(x)$	$q(x)$	$r(x)$
0	0	0	0
a	1	0	0
b	0	1	0
c	1	1	1

Proof. By definition, $p(x) = 1$ precisely when $x - a \in C_0$, and similarly for q and r. The four rows are then checked directly:

- at $x = 0$ none of $-a, -b, -c$ lies in C_0;
- at $x = a$ one has $a - a = 0 \in C_0$ but $a - b$ and $a - c$ fail to lie in C_0;
- at $x = b$ the analogous statement holds with a and b interchanged;
- at $x = c$ one has $c - a = b \in C_0$, $c - b = a \in C_0$, and $c - c = 0 \in C_0$.

This gives exactly the displayed table. □

Proposition 5.5. *In the Heyting algebra of C_0-upper subsets of W one has*

$$p \wedge q = r, \qquad p \vee q = \{a, b, c\}, \qquad p \Rightarrow_H q = q, \qquad q \Rightarrow_H p = p, \qquad p \Rightarrow_H r = r, \qquad r \Rightarrow_H p = W.$$

Moreover,

$$\neg_H p = \neg_H q = \neg_H r = \varnothing.$$

Proof. The conjunction and join statements are direct set computations using the truth table in proposition 5.4. For implication we use the Heyting definition from theorem 2.9.

For instance, $x \in p \Rightarrow_H q$ means that every $y \succeq_{C_0} x$ that satisfies p also satisfies q. If $x = b$ or $x = c$, this is true because the only $y \succeq_{C_0} x$ in W are b, c or just c, and whenever such a y lies in p it is in fact c, which also lies in q. Hence $b, c \in p \Rightarrow_H q$. On the other hand, $a \notin p \Rightarrow_H q$ because $a \succeq_{C_0} a$, and $a \in p$ but $a \notin q$; likewise $0 \notin p \Rightarrow_H q$ because $a \succeq_{C_0} 0$ and again $a \in p \setminus q$. Therefore

$$p \Rightarrow_H q = \{b, c\} = q.$$

The identity $q \Rightarrow_H p = p$ is symmetric.

For $p \Rightarrow_H r$, the same argument shows that only c forces the implication, hence $p \Rightarrow_H r = \{c\} = r$. Finally $r \Rightarrow_H p = W$ because every point of W has the property that any extension y with $y \in r$ must be c, and $c \in p$.

The negation statement follows because $\neg_H U = U \Rightarrow_H \varnothing$. For p, if $x \in W$ then there exists $y \succeq_{C_0} x$ with $y \in p$ (namely a for $x = 0, a$ and c for $x = b, c$), so no state satisfies the defining implication to the empty set. Hence $\neg_H p = \varnothing$, and the same argument applies to q and r. □

5.4 Fuzzy truth on A_2

For the chamber C_0 the defining wall coordinates are simply

$$\lambda_1(\alpha b_1 + \beta b_2) = \alpha, \qquad \lambda_2(\alpha b_1 + \beta b_2) = \beta.$$

Hence

$$m_{C_0}(\alpha b_1 + \beta b_2) = \min(\alpha, \beta).$$

This is an especially transparent formula: the fuzzy truth depends on the smaller of the two chamber coordinates.

Proposition 5.6. *For $a = b_1$ and $x = \alpha b_1 + \beta b_2$, one has*

$$\mu_a^{C_0}(x) = \sigma\big(\min(\alpha - 1, \beta)\big).$$

In particular, the truth value is close to 1 exactly when x lies well inside the chamber translated by a.

Proof. Since

$$x - a = (\alpha - 1)b_1 + \beta b_2,$$

its chamber coordinates are $(\alpha - 1, \beta)$. The chamber margin is the minimum of these two numbers, so the definition of the sigmoid chamber truth value yields the formula. □

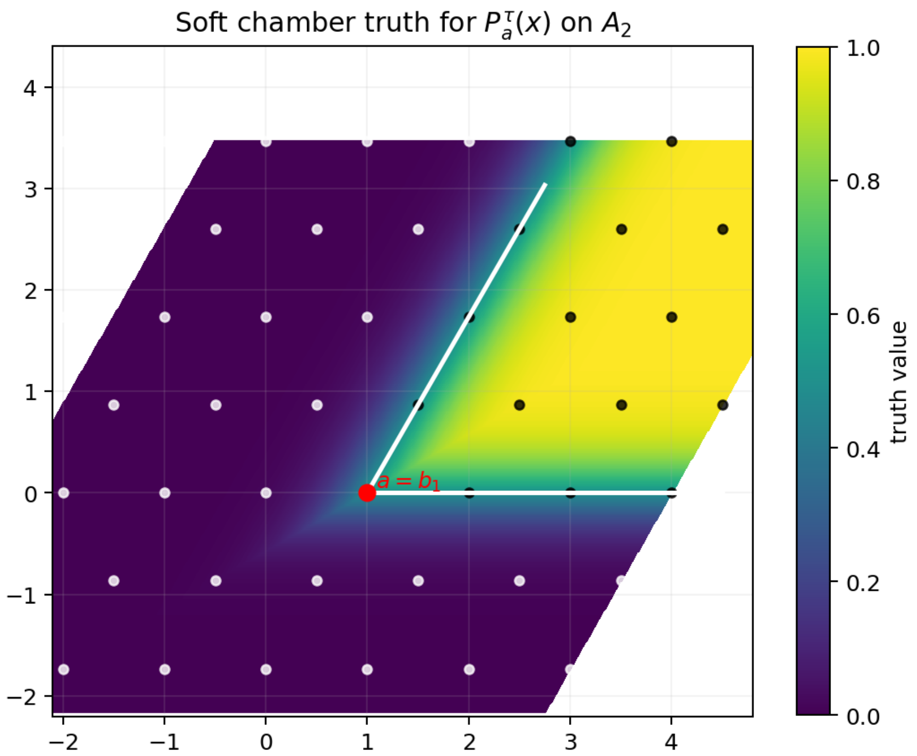

Figure 8: A soft predicate on A_2. The value is near 1 deep inside the translated chamber, near 0 outside it, and intermediate near the chamber walls.

5.5 Explicit many-valued logics on the same A_2 semantics

The same chamber truth values may be fed into standard many-valued calculi. This is the point where the geometric semantics becomes a directly executable logic.

Proposition 5.7. *Let*

$$x = 0.8b_1 + 0.6b_2.$$

For the atomic predicates

$$p := P_0^{C_0}, \qquad q := P_{b_1}^{C_0},$$

one has

$$\mu_p^{C_0}(x) = \sigma(0.6) \approx 0.646, \qquad \mu_q^{C_0}(x) = \sigma(-0.2) \approx 0.450.$$

Proof. For $p = P_0^{C_0}$ one has $x - 0 = 0.8b_1 + 0.6b_2$, so the chamber margin is

$$m_{C_0}(x) = \min(0.8, 0.6) = 0.6.$$

Hence

$$\mu_p^{C_0}(x) = \sigma(0.6) = \frac{1}{1 + e^{-0.6}} \approx 0.646.$$

For $q = P_{b_1}^{C_0}$ one has

$$x - b_1 = -0.2b_1 + 0.6b_2,$$

so

$$m_{C_0}(x - b_1) = \min(-0.2, 0.6) = -0.2.$$

Therefore

$$\mu_q^{C_0}(x) = \sigma(-0.2) = \frac{1}{1 + e^{0.2}} \approx 0.450.$$

□

Definition 5.8. For $u, v \in [0, 1]$, the *Gödel operations* are

$$u \wedge_G v := \min(u, v), \qquad u \vee_G v := \max(u, v), \qquad u \to_G v := \begin{cases} 1, & u \leq v, \\ v, & u > v. \end{cases}$$

The *Łukasiewicz operations* are

$$u \otimes_L v := \max(0, u + v - 1), \qquad u \oplus_L v := \min(1, u + v), \qquad u \to_L v := \min(1, 1 - u + v).$$

Proposition 5.9. *With the values from proposition 5.7, one obtains*

operation	*Gödel value*	*Łukasiewicz value*
$p \wedge q$ / $p \otimes q$	0.450	0.096
$p \vee q$ / $p \oplus q$	0.646	1.000
$p \to q$	0.450	0.804
$q \to p$	1.000	1.000

Proof. Substitute

$$u = 0.646, \qquad v = 0.450$$

into the formulas of the preceding definition. Thus

$$u \wedge_G v = \min(0.646, 0.450) = 0.450, \qquad u \vee_G v = \max(0.646, 0.450) = 0.646,$$

and because $u > v$ one gets $u \to_G v = v = 0.450$, whereas $v \to_G u = 1$.

For Łukasiewicz logic,

$$u \otimes_L v = \max(0, 0.646 + 0.450 - 1) = 0.096,$$

$$u \oplus_L v = \min(1, 0.646 + 0.450) = 1,$$

and

$$u \to_L v = \min(1, 1 - 0.646 + 0.450) = 0.804, \qquad v \to_L u = \min(1, 1 - 0.450 + 0.646) = 1.$$

This gives exactly the stated table. □

Remark 5.10. The same chamber semantics therefore supports different many-valued calculi. The geometric part of the theory produces the truth values; Gödel and Łukasiewicz logic are two different logical engines acting on the same chamber-generated numbers.

5.6 Kegel-unabhangigkeit on A_2

The triangular lattice is the root lattice of type A_2, and its Weyl chambers are exactly the six sectors obtained by rotating C_0 by multiples of $\pi/3$.

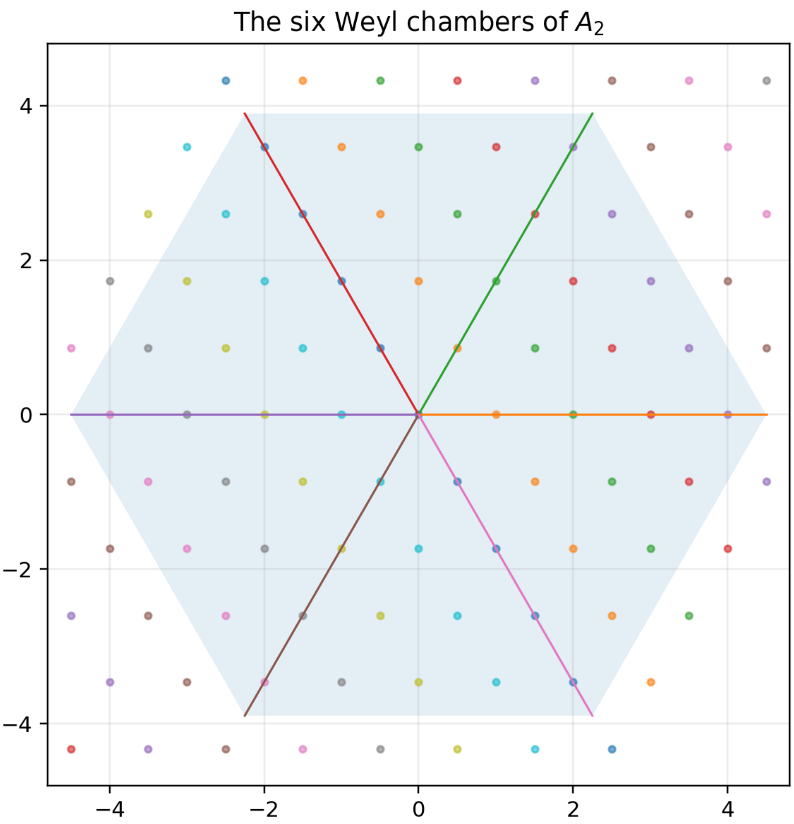

Figure 9: The six Weyl chambers of A_2. Different chamber choices yield isomorphic logics because the dihedral symmetry group acts transitively on the chambers.

Definition 5.11. Let $\mathcal{C}(\Lambda)$ be a family of admissible chambers on a lattice Λ. We say that the resulting chamber logic is *chamber-independent up to symmetry* if for any $C, C' \in \mathcal{C}(\Lambda)$ there exists a lattice isometry $g \in \mathrm{Aut}(\Lambda)$ such that $g(C) = C'$ and therefore

$$P_a^C(x) = P_{g(a)}^{C'}(g(x))$$

for all $a, x \in \Lambda$.

Theorem 5.12. *Assume a group $\Gamma \le \mathrm{Aut}(\Lambda)$ acts transitively on an admissible chamber family $\mathcal{C}(\Lambda)$. Then the induced chamber logic is chamber-independent up to symmetry.*

Proof. Let $C, C' \in \mathcal{C}(\Lambda)$. By transitivity there exists $g \in \Gamma$ with $g(C) = C'$. Then for any $a, x \in \Lambda$ we have

$$\begin{aligned} P_a^C(x) = 1 &\iff x - a \in C \\ &\iff g(x-a) \in g(C) = C' \\ &\iff g(x) - g(a) \in C' \\ &\iff P_{g(a)}^{C'}(g(x)) = 1. \end{aligned}$$

Hence the truth assignments are transported by g, which is exactly chamber-independence up to symmetry. □

Corollary 5.13. *The A_2 chamber logic for the Weyl chamber family is chamber-independent up to symmetry.*

Proof. The automorphism group of A_2 contains the dihedral group of order 12, generated by a rotation through $\pi/3$ and a reflection across a wall. This group acts transitively on the six Weyl chambers. Hence the claim follows from theorem 5.12. □

Remark 5.14. This is already a practical statement. The chamber is no longer a fragile hyperparameter. Any two chamber choices produce the same logic after relabeling by a symmetry of the lattice.

6 Relation with logical algebras and RKHS

We now compare the chamber picture with the framework of *Semantic Space of Logic* and *Logical algebras and reproducing kernel Hilbert spaces*. The comparison has three levels.

(i) In *Semantic Space of Logic*, one fixes a perspective vector and obtains a one-direction semantic logic by projection to that axis.

(ii) In the logical-algebra paper, one starts from an abstract countable preorder and constructs a coordinatewise product-algebra representation, then an RKHS realization.

(iii) In the present note, one starts from concrete geometric data — lattice plus chamber — and obtains a preorder whose coordinatewise description emerges from the chamber walls.

6.1 Finite windows and product algebras

Because lattices are infinite, a literal embedding into $[0,1]^d$ on all of Λ requires a monotone rescaling. In practical implementations one always works on a finite window, so that issue disappears.

Definition 6.1. Let $W \subset \Lambda$ be a finite implementation window and let C be simplicial with wall coordinates $\lambda_1, \ldots, \lambda_d$. For each i define the rescaled coordinate

$$\eta_i^W(x) := \frac{\lambda_i(x) - m_i}{M_i - m_i},$$

where

$$m_i := \min_{x \in W} \lambda_i(x), \qquad M_i := \max_{x \in W} \lambda_i(x).$$

Then

$$\eta_C^W : W \to [0,1]^d, \qquad x \mapsto (\eta_1^W(x), \ldots, \eta_d^W(x)).$$

Proposition 6.2. *Let $W \subset \Lambda$ be finite and let C be simplicial. Then for $a, x \in W$ one has*

$$a \preceq_C x \quad \Longleftrightarrow \quad \eta_C^W(a) \leq \eta_C^W(x) \text{ coordinatewise.}$$

In particular, on every finite implementation window the chamber logic is represented by a coordinatewise product logic.

Proof. Each rescaling function

$$t \mapsto \frac{t - m_i}{M_i - m_i}$$

is strictly increasing on the interval containing the $\lambda_i(W)$-values. Hence

$$\lambda_i(a) \leq \lambda_i(x) \quad \Longleftrightarrow \quad \eta_i^W(a) \leq \eta_i^W(x)$$

for each i. The conclusion now follows from proposition 2.12. □

Remark 6.3. This is precisely where the present paper meets the logical-algebra paper. Theorem 1 there states that any countable preorder embeds into a coordinatewise product algebra; in our situation the preorder is not arbitrary but geometrically produced by a chamber, and on finite windows the embedding is immediate. In comparison, *Semantic Space of Logic* may be viewed as a one-dimensional special case in which one keeps only a single semantic direction; the chamber logic keeps several walls simultaneously and is therefore genuinely multi-directional even before one passes to the product coordinates.

6.2 The RKHS side

Definition 6.4. Let $\Lambda \subset V$ be an integral lattice with Gram matrix G in a chosen basis. The corresponding *Gram kernel* is

$$k(x, y) := \langle x, y \rangle.$$

More generally one may use any positive semidefinite function of the Gram data, such as a Gaussian lattice kernel

$$k_\sigma(x, y) := \exp\left(-\frac{\|x - y\|^2}{2\sigma^2}\right).$$

Proposition 6.5. *Every positive semidefinite lattice kernel determines an RKHS. In particular, the Gram kernel and the Gaussian lattice kernel determine RKHS realizations of the chamber logic.*

Proof. This is the standard Moore-Aronszajn theorem: every positive semidefinite kernel on a set X defines a unique RKHS of functions on X in which evaluation is continuous and the kernel is reproduced by inner products. Since both kernels displayed above are positive semidefinite, each yields an RKHS on the underlying lattice. □

Remark 6.6. The novelty of the present note is not the existence of an RKHS — that is standard, and in fact central in the logical-algebra paper — but the fact that the preorder and the fuzzy truth scale already come with an implementation semantics before the RKHS is introduced.

Proposition 6.7. *On every finite window the chamber logic of a simplicial chamber is a special case of the product-algebra/RKHS program of* Logical algebras and reproducing kernel Hilbert spaces*. The extra content of the chamber viewpoint is:*

(1) a geometric meaning of the coordinates as wall distances,

(2) a direct fuzzy truth value from the chamber margin,

(3) a symmetry criterion for chamber independence,

(4) and a direct engineering interpretation in terms of lattice state packing.

Proof. The first sentence is exactly proposition 6.2. The four extra items are not present in the bare order-embedding theorem because they depend on having a distinguished geometric realization. Item (1) follows from the definition of the wall coordinates. Item (2) is proposition 3.2 together with the fuzzy construction. Item (3) is theorem 5.12. Item (4) will be proved in section 8. □

7 Lorentzian implementations and the examples E_8 and Λ_{24}

7.1 The Lorentzian implementation principle

The arithmetic/Lorentzian note repeatedly uses the construction

$$\Lambda_w := w^\perp/\mathbb{Z}w,$$

where w is a primitive isotropic vector in an integral Lorentzian lattice. We now attach logic to the same mechanism.

Definition 7.1. A *Lorentz implementation datum* consists of

(1) an integral Lorentzian lattice L of signature $(n, 1)$,

(2) a primitive isotropic vector $w \in L$,

(3) an admissible family $\mathcal{C}_w$ of linear or affine chambers in a neighborhood of the cusp determined by w,

(4) and a symmetry group $\Gamma_w \leq \mathrm{O}(L)$ preserving the cusp $[w]$ and acting on $\mathcal{C}_w$.

The resulting quotient lattice is

$$\Lambda_w := w^\perp/\mathbb{Z}w.$$

Each chamber $C \in \mathcal{C}_w$ induces a chamber logic on Λ_w by transporting the corresponding half-space inequalities to the quotient.

Proposition 7.2. *If Γ_w acts transitively on $\mathcal{C}_w$, then the induced logic on Λ_w is chamber-independent up to symmetry.*

Proof. The proof is formally identical to that of theorem 5.12. If $g \in \Gamma_w$ carries one chamber to another, then g preserves $w^\perp$ and the line $\mathbb{Z}w$, hence descends to an automorphism of Λ_w. The truth of chamber predicates is therefore transported by the descended automorphism. □

Remark 7.3. This is the right way to include the Leech lattice without forcing it to behave like an ordinary finite root lattice. In the Leech case the chamber is most naturally chosen *upstairs* in the Lorentzian space and only then pushed down to the Euclidean quotient.

7.2 E_8 as a Lorentz quotient

The arithmetic/Lorentzian note proves that in signature $(9,1)$ every primitive isotropic vector produces the same quotient lattice, namely E_8.

Proposition 7.4. *Let $w \in II_{9,1}$ be primitive isotropic. Then*

$$\Lambda_w = w^\perp/\mathbb{Z}w$$

is positive definite, even, unimodular, and of rank 8. Hence $\Lambda_w \cong E_8$.

Proof. Because $II_{9,1}$ is even and unimodular, the standard Lorentzian quotient construction implies that $w^\perp/\mathbb{Z}w$ is positive definite, even, and unimodular. Its rank is 8 because quotienting by the isotropic line decreases the rank from 10 to 8. There is only one positive definite even unimodular lattice of rank 8, namely E_8. Therefore the quotient is isometric to E_8. □

Remark 7.5. The same note also emphasizes that the orthogonal group of $II_{9,1}$ transports primitive isotropic vectors among themselves, so all such quotients are isometric to the same lattice. This is the rigid rank-8 version of the general Lorentzian philosophy.

Proposition 7.6. *The Weyl chamber logic on E_8 is chamber-independent up to symmetry.*

Proof. The root system of E_8 has Weyl group $W(E_8)$, and the standard theory of finite root systems says that the Weyl group acts transitively on Weyl chambers. Therefore theorem 5.12 applies. □

Remark 7.7. For implementation one may choose any simple-root basis of E_8. A convenient purely algebraic choice is to specify the basis by its Gram matrix, namely the E_8 Cartan matrix

$$G_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

If $x - a = \sum_{i=1}^{8} c_i\alpha_i$ in this basis, then the standard dominant chamber is given by $c_i \geq 0$ for all i, and the canonical chamber margin is $\min_i c_i$. This is the exact 8-dimensional analogue of the A_2 chamber formulas.

7.3 The Leech lattice from Conway's Lorentzian construction

In the classical Conway construction, the primitive isotropic vector

$$w_0 = (1, 2, \ldots, 24; 70)$$

in the even unimodular Lorentzian lattice $II_{25,1}$ produces the Leech lattice as quotient.

Proposition 7.8. *Let*

$$w_0 = (1, 2, \ldots, 24; 70) \in II_{25,1}.$$

Then w_0 is primitive isotropic and

$$w_0^\perp/\mathbb{Z}w_0 \cong \Lambda_{24}.$$

Proof. The identity

$$1^2 + 2^2 + \cdots + 24^2 = 70^2$$

shows that w_0 is isotropic. Since the spatial coordinates have greatest common divisor 1, the vector is primitive. Conway's quotient construction therefore applies. The resulting rank-24 positive definite even unimodular lattice is the classical Leech lattice. □

Remark 7.9. The arithmetic/Lorentzian note makes a conceptual point that is important for logic as well: the meaningful symmetry action is not a direct action of $\mathrm{Aut}(\Lambda_{24})$ on isotropic vectors, but the action of an appropriate subgroup of the ambient Lorentzian orthogonal group on the primitive isotropic data. This is exactly what one wants for chamber logic: the symmetry controlling chamber choices lives naturally upstairs.

Proposition 7.10. *Let $\mathcal{C}_{w_0}$ be the natural family of affine cusp chambers in the Conway/Lorentz model of the Leech lattice at the isotropic cusp $[w_0]$. If the cusp stabilizer acts transitively on this chamber family, then the induced Leech chamber logic is chamber-independent up to symmetry.*

Proof. This is an immediate application of proposition 7.2. □

Remark 7.11. The point of proposition 7.10 is methodological. For the Leech lattice one should not force an intrinsic finite root-chamber structure that is not really there. The chamber is naturally chosen in the Lorentzian model, and the quotient inherits the logic. In Conway coordinates

$$II_{25,1} = \Lambda_{24} \oplus II_{1,1}, \qquad \rho = (0, 0, 1),$$

there is a standard family of simple roots

$$r_\lambda = (\lambda, 1, \lambda^2/2 - 1), \qquad \lambda \in \Lambda_{24},$$

all satisfying $\langle r_\lambda, r_\lambda \rangle = 2$ and $\langle r_\lambda, \rho \rangle = -1$. Thus a practical 24-dimensional implementation may keep the chamber upstairs and evaluate the Conway simple-root inequalities there, instead of seeking a finite root basis downstairs in Λ_{24} itself. This keeps the definition aligned with Conway's construction instead of bending the Leech lattice to fit the root-lattice cases.

8 Practical advantages of dense lattices

We now prove the point that motivates the whole note: dense lattices support chamber logic with concrete implementation advantages.

8.1 State density at fixed safety margin

Definition 8.1. For a lattice $\Lambda \subset \mathbb{R}^n$ let

$$\mu(\Lambda) := \min\{\|x\| : x \in \Lambda \setminus \{0\}\}$$

be the minimum nonzero vector length, and let

$$\rho(\Lambda) := \frac{\mu(\Lambda)}{2}$$

be the packing radius. The lattice-state density is

$$\delta_{\text{state}}(\Lambda) := \frac{1}{\det(\Lambda)}.$$

Theorem 8.2. *Fix a radius $\rho > 0$ and consider all lattices scaled so that their packing radius equals ρ. Then*

$$\delta_{\text{state}}(\Lambda) = \frac{\Delta(\Lambda)}{\text{vol}(B_n(\rho))},$$

where $\Delta(\Lambda)$ is the sphere-packing density. Consequently, among lattices with the same nearest-neighbor safety margin, those with larger packing density realize more lattice states per unit volume.

Proof. By definition of packing density,

$$\Delta(\Lambda) = \frac{\text{vol}(B_n(\rho(\Lambda)))}{\det(\Lambda)}.$$

If $\rho(\Lambda) = \rho$, then

$$\Delta(\Lambda) = \frac{\text{vol}(B_n(\rho))}{\det(\Lambda)}.$$

Rearranging gives

$$\frac{1}{\det(\Lambda)} = \frac{\Delta(\Lambda)}{\text{vol}(B_n(\rho))}.$$

But the left-hand side is exactly the number of lattice points per unit volume, i.e. the implementable state density. Therefore larger packing density is equivalent to larger state density at fixed safety margin. □

Remark 8.3. This theorem is the main practical corollary of dense lattices. It says that density is not merely an aesthetic or asymptotic concept: at a fixed robustness threshold, a denser lattice stores more logically distinguishable states.

8.2 A2 versus the square lattice

Figure 10: At equal nearest-neighbor spacing, the triangular lattice stores more lattice points per area than the square lattice. This is the simplest visual manifestation of theorem 8.2.

Proposition 8.4. *Normalize both A_2 and $\mathbb{Z}^2$ so that the nearest-neighbor distance is 1. Then the ratio of their state densities is*

$$\frac{\delta_{\text{state}}(A_2)}{\delta_{\text{state}}(\mathbb{Z}^2)} = \frac{2}{\sqrt{3}} \approx 1.1547.$$

Equivalently, for the same safety margin, A_2 stores about 15.47% more lattice states per unit area than the square lattice.

Proof. For $\mathbb{Z}^2$ with nearest-neighbor distance 1, the determinant is 1. For A_2 with nearest-neighbor distance 1, proposition 5.1 gives determinant $\sqrt{3}/2$. Therefore

$$\delta_{\text{state}}(\mathbb{Z}^2) = 1, \qquad \delta_{\text{state}}(A_2) = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}.$$

Taking the ratio proves the claim. □

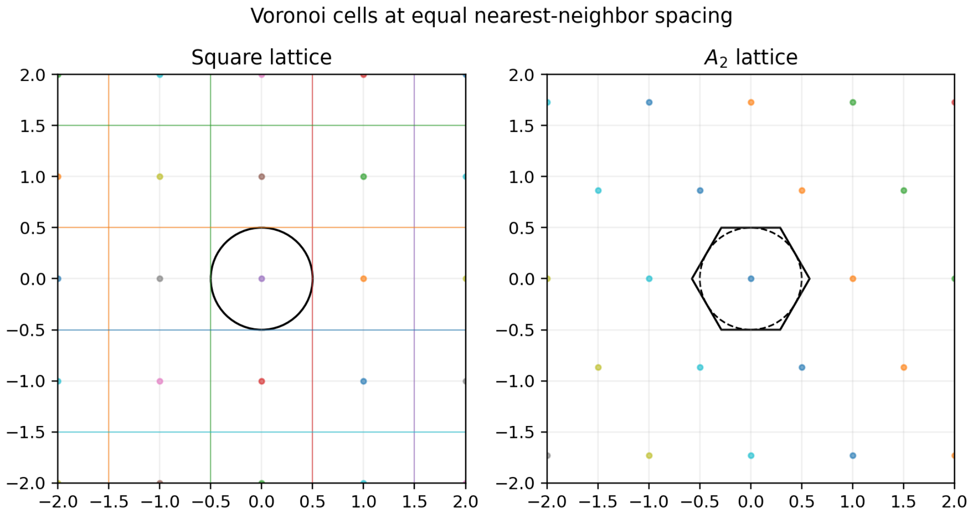

Figure 11: Voronoi cells at the same nearest-neighbor spacing. The hexagonal cell of A_2 expresses the same phenomenon as the density computation: more isotropic geometry, less wasted area.

8.3 The canonical higher-dimensional examples

The same argument becomes striking in dimensions 8 and 24.

Proposition 8.5. *Normalize E_8 and an orthogonal lattice so that both have minimum norm 2. Then*

$$\frac{\delta_{\text{state}}(E_8)}{\delta_{\text{state}}(\sqrt{2}\,\mathbb{Z}^8)} = 16.$$

Thus, at the same nearest-neighbor safety margin, E_8 realizes 16 times as many lattice states per unit volume as the orthogonal implementation.

Proof. The lattice E_8 is even unimodular, so $\det(E_8) = 1$, and its minimum norm is 2. The orthogonal comparison lattice with the same minimum norm is $\sqrt{2}\,\mathbb{Z}^8$, whose determinant is

$$(\sqrt{2})^8 = 2^4 = 16.$$

Hence

$$\delta_{\text{state}}(E_8) = 1, \qquad \delta_{\text{state}}(\sqrt{2}\,\mathbb{Z}^8) = \frac{1}{16}.$$

The ratio is therefore 16. □

Proposition 8.6. *Normalize the Leech lattice and an orthogonal lattice so that both have minimum norm 4. Then*

$$\frac{\delta_{\text{state}}(\Lambda_{24})}{\delta_{\text{state}}(2\mathbb{Z}^{24})} = 2^{24} = 16{,}777{,}216.$$

Thus, at the same nearest-neighbor safety margin, the Leech lattice realizes 2^{24} *times as many lattice states per unit volume as the orthogonal implementation.*

Proof. The Leech lattice is unimodular, hence has determinant 1, and its minimum norm equals 4. The orthogonal comparison lattice with minimum norm 4 is $2\mathbb{Z}^{24}$, whose determinant is

$$2^{24}.$$

Therefore

$$\delta_{\text{state}}(\Lambda_{24}) = 1, \qquad \delta_{\text{state}}(2\mathbb{Z}^{24}) = 2^{-24}.$$

The ratio is exactly 2^{24}. □

Table 1: Three canonical lattices for practical chamber logic. The last column compares each lattice with the orthogonal implementation having the same minimum norm.

Lattice	Dimension	Minimum norm	State-density ratio to orthogonal baseline
A_2	2	1	$\frac{2}{\sqrt{3}} \approx 1.1547$
E_8	8	2	16
Λ_{24}	24	4	$2^{24} = 16{,}777{,}216$

Remark 8.7. The table should be read exactly in the spirit of this paper: not as a claim that dense lattices solve every logical problem, but as a proof that in chamber-based implementations they offer a tangible geometric advantage. They support more robustly separated logical states in the same volume.

8.4 The practical meaning of chamber-independence

Proposition 8.8. *Assume the chamber logic is chamber-independent up to symmetry. Then changing the chamber does not change the implementation class of the logic; it only relabels the lattice states by a lattice automorphism.*

Proof. This is simply a reformulation of theorem 5.12. If g transports C to C', then every truth table computed from C is conjugate by g to the truth table computed from C'. Thus chamber choice does not change the logic as an implementation class. □

Remark 8.9. This is practically important. A chamber-dependent logic that changes substantially when one rotates the coordinate system has a hyperparameter problem. Chamber-independent lattices such as A_2 and E_8 avoid this: the chamber may be chosen for convenience without changing the logical content.

9 Discussion and outlook

The model example A_2 makes the whole theory visible. One sees the lattice states, the chamber, the hard upper sets, the fuzzy truth values, and the symmetry that removes the arbitrariness of chamber choice. The higher-dimensional examples then reveal why one should care: E_8 and the Leech lattice show that the same idea scales to exceptional lattices with extraordinary packing performance.

The comparison with *Logical algebras and reproducing kernel Hilbert spaces* is especially useful here. That paper shows how to pass from a preorder to product logic and then to an RKHS. The present note supplies a complementary source of preorders — chambers on lattices — and explains why these preorders are particularly meaningful for implementation. A chamber is not merely a set of inequalities; it is a physical or numerical design choice. Dense lattices make that design choice economically efficient, and chamber symmetry makes it stable.

A natural next step is to push the Lorentzian viewpoint further. In the Leech case the correct chamber family is affine and cusp-based rather than finite and root-based. That suggests a broader theory of *Lorentz-induced chamber logics* in which Euclidean lattice logic is always the shadow of a higher-dimensional isotropic geometry. The arithmetic/Lorentzian note points strongly in that direction.

Another natural next step is experimental. One can compare lattice-based chamber logic implementations under synthetic noise, under quantization constraints, and under chamber perturbations. Theorems such as theorems 3.6 and 8.2 already predict what should happen: the soft truth values should be stable, and dense lattices should deliver more robust state capacity at fixed margin.

Finally, dimension 3 deserves to be mentioned. The densest lattice there is the face-centered cubic lattice, i.e. the root lattice $A_3 \cong D_3$. It fits the same chamber picture and suggests that the sequence

$$A_2,\ A_3,\ E_8,\ \Lambda_{24}$$

should be viewed as a family of increasingly powerful platforms for practical chamber logic: first visible, then spatial, then exceptional, then cusp-exceptional.

10 Conclusion

We summarize the paper in one sentence.

> *A practical lattice logic is obtained by reading entailment from a chamber, fuzzifying it by a chamber margin, and exploiting dense lattice geometry to maximize the number of robustly separated logical states.*

The hard predicates are upper sets. Their fuzzy versions are stable under perturbation. On finite windows they become coordinatewise product logics of exactly the kind studied in *Logical algebras and reproducing kernel Hilbert spaces*, while the KPCA–quantization layer makes the whole construction usable starting from a plain finite kernel matrix. The chamber viewpoint contributes additional geometric meaning and practical consequences. The lattice examples show the spectrum of the theory clearly:

- A_2 makes everything visible and computable.
- E_8 shows that the same chamber logic admits a canonical Lorentzian implementation and an exceptional state-density advantage.

- The Leech lattice shows that even the cusp case can be included without distorting the basic principle.

The conceptual message is therefore the following. Dense lattices are not only efficient packings of spheres. They are also promising carriers of implementable logic.

11 Appendix: A kernelized chamber-learning implementation layer

This section gives a practical implementation layer that sits between a semantic kernel and the chamber–lattice logic developed in the main text. The methods shown here WORK AS PRESENTED ONLY FOR Lattices with a root system! The point of departure is a situation that occurs naturally in applications: the user does *not* begin with a lattice, but rather with a finite collection of semantic objects together with a positive semidefinite kernel and a small amount of expert entailment knowledge.

Input data

Let

$$X = \{x_1, \dots, x_n\}$$

be a finite family of semantic objects. Assume that a symmetric positive semidefinite kernel

$$k : X \times X \to \mathbb{R}$$

is given, with Gram matrix

$$K = (k(x_i, x_j))_{1 \le i,j \le n}.$$

In addition, assume that expert knowledge is given in the form of a directed entailment graph

$$E \subseteq X \times X,$$

where $(x_i, x_j) \in E$ means that the expert asserts that $x_i \Rightarrow x_j$ should hold at least at the level of the chosen axioms.

The goal is to construct from (K, E) a family of chamber coordinates, then a chamber logic, and only *afterwards* a lattice implementation layer. This reverses the naive order "kernel $\to$ KPCA $\to$ quantization $\to$ logic" and makes the logical structure primary rather than accidental.

Kernelized chamber coordinates

Let $d \in \{2, 3, 8, 24\}$ be the target dimension. We seek a map

$$F : X \to \mathbb{R}^d, \qquad x_i \mapsto F_i,$$

with the property that for every expert entailment edge $(x_i, x_j) \in E$, the difference vector $F_j - F_i$ points into the positive chamber. In the simplest implementation the positive chamber is the positive orthant $\mathbb{R}^d_{\ge 0}$.

By the representer principle, each coordinate function may be written in the form

$$f_r(x) = \sum_{\ell=1}^{n} \alpha_{\ell r} k(x_\ell, x), \qquad r = 1, \dots, d.$$

If $A = (\alpha_{\ell r}) \in \mathbb{R}^{n\times d}$, then the coordinate matrix is simply

$$F = KA \in \mathbb{R}^{n\times d}.$$

Thus the chamber-learning problem is reduced to choosing A.

Definition 11.1 (softplus large-margin chamber objective)**.** Fix a margin parameter $m > 0$, regularization weight $\lambda > 0$, and negative-edge weight $\mu \geq 0$. Let $N \subseteq X \times X$ be an optional family of negative or reverse edges. Define

$$\mathcal{L}(A) := \frac{1}{2}\sum_{r=1}^{d} \alpha_r^\top K \alpha_r + \lambda \sum_{(i,j)\in E} \sum_{r=1}^{d} \mathrm{sp}\big(m - (F_{jr} - F_{ir})\big) + \mu \sum_{(i,j)\in N} \mathrm{sp}\Big(m + \frac{1}{d}\sum_{r=1}^{d}(F_{jr} - F_{ir})\Big),$$

where $\mathrm{sp}(t) = \log(1 + e^t)$ is the softplus function and $F = KA$.

The first term is the RKHS regularizer, while the second term enforces the desired chamber orientation for the expert edges. The third term weakly penalizes reverse edges or sampled negative pairs.

Proposition 11.2. *The objective $\mathcal{L}(A)$ is convex in A.*

Proof. The map $A \mapsto KA$ is linear. The regularizer

$$\frac{1}{2}\sum_{r=1}^{d} \alpha_r^\top K \alpha_r$$

is convex because K is positive semidefinite. The arguments of all softplus terms are affine functions of A, and softplus is convex. A nonnegative sum of convex functions is convex. Therefore $\mathcal{L}(A)$ is convex. □

Remark 11.3. For the purposes of implementation, one may either solve this problem directly with a convex optimization package, or use a smooth gradient-based solver such as L-BFGS on the same convex objective. The accompanying Python script uses the latter strategy in order to avoid heavy external dependencies.

From learned chambers to a lattice implementation

Once the chamber coordinates $F_i \in \mathbb{R}^d$ have been learned, a lattice is used only as a *robust implementation layer*. Let $\Lambda \subset \mathbb{R}^d$ be one of the canonical dense lattices in the chosen dimension:

$$A_2\ (d = 2), \qquad A_3/\mathrm{FCC}\ (d = 3), \qquad E_8\ (d = 8).$$

Let $B \in \mathbb{R}^{d\times d}$ be a lattice basis and let

$$Q_\Lambda : \mathbb{R}^d \to \Lambda$$

be a nearest-lattice quantizer. In the implementation we use a Babai-type rounding step followed by a local coefficient search, which is strictly better than plain Babai nearest-plane while remaining lightweight.

Define the quantized chamber coordinates by

$$\lambda_i := Q_\Lambda(F_i) \in \Lambda.$$

The lattice is therefore not asked to invent the logic. It only discretizes a chamber structure that has already been learned from the kernel and the expert graph.

Hard and fuzzy chamber logic

Fix the positive chamber

$$C = \Big\{ \sum_{r=1}^{d} c_r b_r \ : \ c_r \geq 0 \Big\} \subseteq \Lambda \otimes \mathbb{R},$$

where $b_1, \dots, b_d$ is the chosen basis. For every $a \in X$, define the hard atomic proposition

$$P_a(x_i) := \mathbf{1}[\lambda_i - \lambda_a \in C].$$

This is the native hard chamber logic. In general, the algebra generated by the upper sets is a Heyting algebra rather than a Boolean algebra.

For a fuzzy relaxation we use the chamber margin

$$m_C(v) := \min_r c_r \qquad \text{if } v = \sum_{r=1}^{d} c_r b_r,$$

and define the fuzzy truth value

$$\mu_a(x_i) := \sigma\big(m_C(\lambda_i - \lambda_a)\big), \qquad \sigma(t) = \frac{1}{1+e^{-t}}.$$

The sigmoid appears here as the canonical smooth relaxation of the hard threshold at the chamber boundary.

Once the truth values $\mu_a(x_i)$ are available, one may apply standard many-valued implication calculi pointwise. Two useful examples are:

$$u \to_G v = \begin{cases} 1, & u \leq v, \\ v, & u > v, \end{cases}$$
$$u \to_L v = \min(1, 1-u+v),$$

that is, the G"odel and Łukasiewicz implications, respectively.

A concrete arithmetic kernel example

We now specialize to the arithmetic kernel

$$k(a,b) = \frac{\gcd(a,b)^2}{ab}$$

on the signed set

$$X_{100} = \{\pm 1, \pm 2, \dots, \pm 100\}.$$

The user-specified axioms are the smaller set

$$A_{10} = \{\pm 1, \pm 2, \dots, \pm 10\}.$$

Choose the perspective vector $w = 3$. For each axiom $a \in A_{10}$ define the normalized perspective truth value

$$\tau_w(a) := \frac{1 + k(w,a)}{2} \in [0,1].$$

Then define the perspective implication degree by the Łukasiewicz formula

$$a \Rightarrow_w b := \min(1, 1 - \tau_w(a) + \tau_w(b)).$$

The expert graph on the axioms is obtained by retaining exactly those edges for which the implication degree is maximal:

$$(a, b) \in E_w \iff a \neq b \text{ and } a \Rightarrow_w b = 1.$$

Equivalently, because of the Łukasiewicz formula,

$$(a, b) \in E_w \iff \tau_w(a) \leq \tau_w(b).$$

In this way the user supplies a small and interpretable family of axioms, while the chamber-learning layer is asked to generalize from these local entailments to the entire family X_{100}.

Remark 11.4. This example is particularly useful from the user point of view. The user does not have to define a full logic on 200 objects. Instead, the user provides a semantic kernel and a compact axiom graph on only 20 objects. The chamber learner then extends these entailment patterns to all other objects in X_{100}, and the lattice layer discretizes the result for practical use.

Why this is better than direct quantize-then-logic

The original unsupervised pipeline

$$K \to \text{KPCA} \to \text{lattice quantization} \to \text{logic}$$

is useful as a baseline, but it may introduce substantial quantization noise before any order has been learned. In contrast, the present chamber-learning pipeline first extracts a logic-compatible order from the kernel using expert supervision and only then adds a lattice implementation layer:

$$K + E \implies \text{learned chamber coordinates} \implies \text{dense lattice implementation} \implies \text{hard/fuzzy cham}$$

This is the mathematically cleaner formulation. It explains how a user can start from semantic similarity and a small amount of trusted entailment information and still obtain a practical lattice logic with hard entailment tables, fuzzy truth tables, and G"odel or Łukasiewicz implication matrices.

References

[1] O. Leka, *Arithmetic Progression, Primitive Isotropic Vectors, and Elliptic Curves in Conway's Lorentzian Construction*, 2026 draft note.

[2] O. Leka, *Logical algebras and reproducing kernel Hilbert spaces*, 2025, https://orges-leka.de/logical_algebras_and_reproducing_kernel_hilbert_spaces.pdf.

[3] B. Scholkopf, A. Smola, and K.-R. Muller, Kernel principal component analysis, *Lecture Notes in Computer Science* **1327** (1997), 583–588.

[4] R. E. Borcherds, Automorphic forms and Lie algebras, 1996 manuscript, https://math.berkeley.edu/~reb/papers/cdm/cdm.pdf.

[5] O. Leka, *Semantic Space of Logic*, 2025, https://www.orges-leka.de/semantic_space_of_logic.pdf.

[6] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, 1999.

[7] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337–404.

[8] M. S. Viazovska, The sphere packing problem in dimension 8, *Ann. of Math.* **185** (2017), 991–1015.

[9] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. S. Viazovska, The sphere packing problem in dimension 24, *Ann. of Math.* **185** (2017), 1017–1033.

[10] P. Hajek, *Metamathematics of Fuzzy Logic*, Kluwer, 1998.