Lindström–Bhat Matrices and Prime Factorizations of Integers

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Abstract

We study the arithmetic Möbius weights $g(n)$ attached to the factorization-poset meet kernel

$$K(m,n) = m \wedge n,$$

where the meet is taken componentwise on ordered prime-factor lists. The underlying paper establishes that these coefficients are strictly positive, admits an explicit closed formula for $g(n)$ on the natural numbers, and shows that the inverse Gram matrices stabilize entrywise to a positive symmetric infinite operator

$$A = G_{\infty}^{-1},$$

which is closable and possesses a positive Friedrichs extension. On the prime layer, the same framework identifies the inverse Gram matrix with a weighted grounded path Laplacian whose edge resistances are exactly the prime gaps, and yields determinant identities and geometric-mean bounds for products of prime gaps.

Building on this structure, we introduce and analyze the Dirichlet series

$$D_g(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Using the explicit block formula for $g(n)$, we derive an exact non-Eulerian prime cascade representation for $D_g(s)$, in which the local factors are governed directly by the prime gaps Δ_p. We then prove that $g(n)$ admits a multiplicative majorant

$$g(n) \le \widetilde{g}(n) := n \prod_{\substack{p \mid n \\ p \text{ odd}}} \frac{\Delta_p}{p},$$

whose Euler product can be controlled by an elementary interval-integral argument over consecutive prime gaps. This yields an unconditional absolute convergence theorem:

$$D_g(s) \text{ converges absolutely for } \Re(s) > 1, \qquad \sigma_a(D_g) = 1.$$

As a consequence, the summatory function

$$G(x) := \sum_{n \le x} g(n)$$

satisfies

$$G(x) \ll_{\varepsilon} x^{1+\varepsilon} \qquad (\varepsilon > 0).$$

We also show that if one had an asymptotic

$$G(x) \sim Cx(\log x)^{\beta},$$

then Abelian theory would force

$$D_g(s) \sim \frac{C\,\Gamma(\beta+1)}{(s-1)^{\beta+1}} \qquad (s \to 1^+).$$

In particular, the numerically suggested law

$$G(x) \sim C_g x (\log x)^2$$

would correspond to a cubic pole of D_g at $s = 1$. Finally, we prove that the multiplicative-majorant method alone yields the subexponential bound

$$G(x) \ll x \exp\bigl(C\sqrt{\log x \, \log\log x}\bigr),$$

thereby isolating precisely where a sharper polylogarithmic theory must exploit the terminal-block asymmetry of the exact formula rather than the multiplicative envelope alone.

Contents

1 Introduction

The paper begins from a nonclassical kernel on the natural numbers: if the ordered prime-factor lists of

$$m = p_1 \cdots p_r, \qquad n = q_1 \cdots q_s$$

are written in nondecreasing order, then the meet kernel is defined by

$$K(m,n)=\prod_{i=1}^{\min(r,s)}\min(p_i,q_i).$$

Equivalently,

$$K(m,n)=f(m\wedge n),\qquad f(x)=x,$$

where the meet is taken in the factorization poset on $\mathbb{N}$. This places the arithmetic kernel into the general Lindström–Bhat theory of meet matrices and yields a canonical Möbius transform

$$g(n)=\sum_{d\preceq n}\mu(d,n)\,d$$

such that

$$K(m,n)=\sum_{d\preceq m,\ d\preceq n}g(d).$$

The associated Gram matrices factor as

$$G_N=E_ND_NE_N^T,\qquad D_N=\operatorname{diag}(g(1),\dots,g(N)),$$

and hence

$$\det(G_N)=\prod_{n\le N}g(n).$$

A key structural fact proved in the paper is that all coefficients $g(n)$ are strictly positive integers.

Several striking arithmetic and spectral consequences are then developed. On the prime layer, the restricted Gram matrix becomes

$$M_k=(\min(p_i,p_j))_{1\le i,j\le k},$$

its determinant is the product of the prime gaps,

$$\det(M_k)=2\prod_{i=2}^{k}(p_i-p_{i-1}),$$

and its inverse is the grounded Laplacian of a weighted path whose edge resistances are exactly the prime gaps. This turns the prime-gap sequence into a one-dimensional electrical medium and leads to geometric, spectral, and random-medium interpretations. In particular, the paper proves a geometric-mean bound

$$\Big(\prod_{i=1}^{k}\Delta_i\Big)^{1/k}\le\frac{p_k}{k},$$

and hence, by the Prime Number Theorem,

$$\Big(\prod_{i=1}^{k}\Delta_i\Big)^{1/k}=O(\log k).$$

On the full integer side, the inverse Gram matrices stabilize entrywise, defining an infinite positive symmetric operator

$$A=G_\infty^{-1}$$

on $c_{00}(\mathbb{N})$; the paper shows that A is closable and admits a positive selfadjoint Friedrichs extension.

The present part of the manuscript adds a new analytic layer to this picture. Its starting point is the explicit formula for $g(n)$ proved earlier in the paper. That formula exhibits a pronounced asymmetry: internal odd prime blocks contribute factors of the form

$$\Delta_p \, p^{\alpha-1},$$

whereas the largest odd prime factor contributes a different terminal factor. Together with the separate 2-scaling law, this leads not to a classical Euler product, but to an ordered prime cascade. From that block structure we construct the Dirichlet series

$$D_g(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

and derive an exact representation in which each prime gap appears explicitly in the local factors

$$H_q(s) = 1 + \frac{\Delta_q}{q^s - q}, \qquad T_p(s) = \frac{\Delta_p(p^s - 1)}{p^s(p^s - p)}.$$

Thus $D_g(s)$ packages the full prime-gap sequence into a single analytic object, but in a fundamentally non-Eulerian way: the largest prime factor orders the cascade, and terminal-block asymmetry replaces multiplicativity.

The first main goal of the new analysis is to place this Dirichlet series on a rigorous analytic footing. A priori, the positivity and trivial bound $g(n) \leq n$ only imply absolute convergence for $\Re(s) > 2$. The key observation is that the explicit block formula also yields a multiplicative majorant

$$g(n) \leq \widetilde{g}(n) := n \prod_{\substack{p|n \\ p \text{ odd}}} \frac{\Delta_p}{p}.$$

The Dirichlet series of $\widetilde{g}$ has a genuine Euler product, and its convergence reduces to the elementary estimate

$$\sum_{p \geq 3} \frac{\Delta_p}{p^\sigma} < \infty \qquad (\sigma > 1),$$

obtained by integrating $x^{-\sigma}$ over successive prime intervals. This proves unconditionally that

$$D_g(s) \text{ converges absolutely for } \Re(s) > 1, \qquad \sigma_a(D_g) = 1.$$

In other words, the true absolute convergence boundary of the prime-gap-weighted Möbius series is the critical line $\Re(s) = 1$, not the crude line $\Re(s) = 2$ suggested by the trivial bound.

The second goal is to understand the finer behaviour near $s = 1$. Writing

$$G(x) := \sum_{n \leq x} g(n),$$

Abel summation gives

$$D_g(s) = s \int_1^\infty \frac{G(x)}{x^{s+1}} \, dx \qquad (\Re(s) > 1),$$

so the singularity of D_g at $s = 1$ is controlled by the growth of $G(x)$. This leads to an exact Abelian transfer principle: if

$$G(x) \sim Cx(\log x)^\beta,$$

then

$$D_g(s) \sim \frac{C\,\Gamma(\beta+1)}{(s-1)^{\beta+1}} \qquad (s \to 1^+).$$

In particular, the numerically suggested law

$$G(x) \sim C_g x(\log x)^2$$

would force a cubic pole

$$D_g(s) \sim \frac{2C_g}{(s-1)^3}.$$

This identifies the precise singular model to be tested in future work. At present, however, this remains conjectural. What is proved rigorously is only the weaker summatory bound

$$G(x) \ll_\varepsilon x^{1+\varepsilon},$$

and, via a sharper analysis of the majorant Euler product, the subexponential improvement

$$G(x) \ll x \exp\bigl(C\sqrt{\log x\, \log\log x}\bigr).$$

The same argument also shows why the multiplicative majorant alone does not seem capable of reaching a polylogarithmic estimate of the form

$$G(x) \ll x(\log x)^A :$$

the blow-up of the majorant near $s = 1$ is still too large by an extra factor of $1/\eta$. This isolates the exact point where one must go beyond the multiplicative envelope and use the terminal-block asymmetry of the true formula for $g(n)$.

The guiding philosophy of this work is that the same arithmetic object now appears in three mutually reinforcing forms:

1. as Möbius weights in the factorization-poset meet-kernel decomposition,
2. as conductances and bottlenecks in the prime-layer inverse Gram operator,
3. as coefficients of a non-Eulerian Dirichlet cascade that encodes prime-gap geometry analytically.

From this perspective, the Dirichlet series $D_g(s)$ is the first explicit analytic continuation of the block structure of Proposition 63, while the operator $A = G_\infty^{-1}$ provides the spectral completion of the same arithmetic data. The central open problem is now clear: determine the true asymptotic growth of $G(x)$, and hence the exact singular order of $D_g(s)$ at $s = 1$.

2 Prime factorizations and a minimum kernel

Every integer $n \geq 1$ has a unique prime factorization

$$n = p_1 p_2 \cdots p_r,$$

where the primes are written in nondecreasing order and multiplicities are included. We denote the list of primes by

$$\pi(n) = (p_1, \ldots, p_r), \qquad r = \Omega(n).$$

For two integers

$$m = p_1 \cdots p_r, \qquad n = q_1 \cdots q_s$$

(with both lists sorted) define the kernel

$$K(m,n) = \prod_{i=1}^{\min(r,s)} \min(p_i, q_i).$$

Equivalently,

$$K(m,n) = f(m \wedge n)$$

where

$$f(x) = x$$

and the operation $m \wedge n$ is defined by

$$\pi(m \wedge n) = (\min(p_1, q_1), \ldots, \min(p_{\min(r,s)}, q_{\min(r,s)})).$$

3 A poset structure on the integers

Define a partial order $\preceq$ on the positive integers by

$$a \preceq b \iff \Omega(a) \leq \Omega(b) \text{ and } p_i(a) \leq p_i(b) \text{ for } i = 1, \ldots, \Omega(a).$$

Then every pair of elements has a meet

$$a \wedge b$$

defined by componentwise minima of the prime factor lists.

Thus $(\mathbb{N}, \preceq)$ forms a meet-semilattice.

4 Meet matrices

Let P be a finite meet-semilattice and let

$$x_1, \ldots, x_n \in P.$$

For a function $f : P \to \mathbb{R}$ define the matrix

$$M_{ij} = f(x_i \wedge x_j).$$

Such matrices are called *meet matrices*. They were introduced by Bhat [2] and studied extensively in the linear algebra literature.

For integers $1, \ldots, n$ the Gram matrix of the kernel above is

$$G_n = (K(i,j))_{1 \leq i,j \leq n}.$$

5 Möbius inversion on posets

Let $(P, \leq)$ be a finite poset. The *zeta function* is

$$\zeta(x, y) = \begin{cases} 1 & x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

The *Möbius function* μ is defined by

$$\sum_{x \leq z \leq y} \mu(x, z) = \delta_{x,y}.$$

For a function $f : P \to \mathbb{R}$ define its Möbius transform

$$g(x) = \sum_{y \leq x} \mu(y, x) f(y).$$

Then

$$f(x) = \sum_{y \leq x} g(y).$$

This theory is classical; see [5, 6].

6 Factorization of meet matrices

Assume the elements $x_1, \ldots, x_n$ are ordered such that

$$x_i \leq x_j \implies i \leq j.$$

Define the incidence matrix

$$E_{ik} = \begin{cases} 1 & x_k \leq x_i \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$D = \operatorname{diag}(g(x_1), \ldots, g(x_n)).$$

Proposition 1. *The meet matrix satisfies*

$$M = EDE^T.$$

Proof. The (i, j) entry of EDE^T equals

$$\sum_{k=1}^{n} E_{ik} g(x_k) E_{jk}.$$

Since $E_{ik}E_{jk} = 1$ exactly when $x_k \leq x_i$ and $x_k \leq x_j$, this equals

$$\sum_{x_k \leq x_i \wedge x_j} g(x_k).$$

Using the inversion identity

$$f(x) = \sum_{y \leq x} g(y),$$

we obtain

$$(EDE^T)_{ij} = f(x_i \wedge x_j).$$

Thus $M = EDE^T$. □

This decomposition appears in the meet-matrix literature, see for example [4, **?**].

7 Determinant theorem

Theorem 1 (Lindström–Bhat determinant formula)**.** *Let*

$$M_{ij} = f(x_i \wedge x_j)$$

be a meet matrix. Let

$$g(x) = \sum_{y \le x} \mu(y, x) f(y).$$

Then

$$\det(M) = \prod_{i=1}^{n} g(x_i).$$

Proof. From the factorization

$$M = EDE^T$$

we obtain

$$\det(M) = \det(E)\det(D)\det(E^T).$$

Because the elements were ordered according to the poset, E is triangular with ones on the diagonal. Hence

$$\det(E) = 1.$$

Therefore

$$\det(M) = \det(D) = \prod_{i=1}^{n} g(x_i).$$

For detailed treatments of this argument see [4, **?**]. □

8 Feature vectors and RKHS representation

The kernel satisfies

$$K(x, y) = \sum_{d \le x,\, d \le y} g(d).$$

Define feature coordinates

$$\phi_d(x) = \sqrt{g(d)}\, \mathbf{1}_{\{d \le x\}}.$$

Then

$$K(x, y) = \sum_{d} \phi_d(x)\phi_d(y),$$

so

$$K(x, y) = \langle \phi(x), \phi(y) \rangle.$$

Thus the kernel admits an explicit feature map indexed by the elements of the poset.

9 Application to integers

For the integer poset defined by prime factorizations we obtain

$$K(m,n) = m \wedge n$$

in the componentwise prime-factor sense.

The Gram matrix

$$G_n = (K(i,j))_{1 \le i,j \le n}$$

is therefore a meet matrix, and the determinant formula above yields

$$\det(G_n) = \prod_{k=1}^{n} g(k),$$

where g is the Möbius transform of $f(k) = k$ with respect to the poset $\preceq$ defined by prime factor lists.

10 Properties of the kernel K

Recall that every integer $m \ge 1$ is written in its prime-factor list form

$$m = p_1 p_2 \cdots p_r, \qquad 1 < p_1 \le p_2 \le \cdots \le p_r,$$

and similarly

$$n = q_1 q_2 \cdots q_s, \qquad 1 < q_1 \le q_2 \le \cdots \le q_s.$$

We define

$$K(m,n) = \prod_{i=1}^{\min(r,s)} \min(p_i, q_i).$$

We also adopt the convention that the empty product equals 1, so that $K(1,n) = 1$ for all n.

It is convenient to write

$$\pi(m) = (p_1, \ldots, p_r), \qquad \Omega(m) = r.$$

10.1 Basic properties

Proposition 2. *For all $m, n \ge 1$ the kernel K satisfies:*

$$\begin{aligned} K(m,n) &= K(n,m), \\ K(m,m) &= m, \\ K(1,n) &= 1. \end{aligned}$$

Proof. Symmetry is immediate from $\min(p_i, q_i) = \min(q_i, p_i)$. If $m = n$, then each factor is $\min(p_i, p_i) = p_i$, hence

$$K(m,m) = \prod_{i=1}^{r} p_i = m.$$

If $m = 1$, then $\Omega(1) = 0$, so the product is empty and equals 1. □

Proposition 3. *For all $m, n \geq 1$,*

$$K(m,n) \geq 1.$$

If moreover $m, n > 1$, then

$$K(m,n) \geq 2^{\min(\Omega(m),\Omega(n))}.$$

Proof. Every factor $\min(p_i, q_i)$ is at least 2. □

10.2 The meet interpretation

Define a partial order $\preceq$ on $\mathbb{N}$ by

$$a \preceq b \iff \Omega(a) \leq \Omega(b) \text{ and } p_i(a) \leq p_i(b) \text{ for all } i = 1, \ldots, \Omega(a).$$

Then $(\mathbb{N}, \preceq)$ is a meet-semilattice, with meet

$$\pi(m \wedge n) = (\min(p_1, q_1), \ldots, \min(p_{\min(r,s)}, q_{\min(r,s)})).$$

Hence

$$K(m,n) = f(m \wedge n), \qquad f(x) = x.$$

Proposition 4. *For all $m, n \geq 1$,*

$$K(m,n) = m \iff m \preceq n, \qquad K(m,n) = n \iff n \preceq m.$$

Proof. Since $K(m,n) = f(m \wedge n) = m \wedge n$ as an integer, we have

$$K(m,n) = m \iff m \wedge n = m \iff m \preceq n.$$

The second statement is symmetric. □

Proposition 5. *If $m \preceq m'$, then for every n,*

$$K(m,n) \leq K(m',n).$$

Similarly, if $n \preceq n'$, then

$$K(m,n) \leq K(m,n').$$

Proof. Assume $m \preceq m'$. Then $\Omega(m) \leq \Omega(m')$ and

$$p_i(m) \leq p_i(m')$$

for every relevant index i. Hence

$$\min(p_i(m), q_i) \leq \min(p_i(m'), q_i)$$

for all $i \leq \min(\Omega(m), \Omega(n))$, and multiplying yields

$$K(m,n) \leq K(m',n).$$

The second claim is analogous. □

10.3 Upper bounds

Proposition 6. *For all $m, n \geq 1$,*

$$K(m,n) \leq \min(m,n).$$

Proof. Let $t = \min(\Omega(m), \Omega(n))$. Then

$$\min(p_i, q_i) \leq p_i \qquad (1 \leq i \leq t),$$

so

$$K(m,n) \leq \prod_{i=1}^{t} p_i \leq \prod_{i=1}^{\Omega(m)} p_i = m.$$

By symmetry also $K(m,n) \leq n$. □

Proposition 7. *For all $m, n \geq 1$,*

$$K(m,n)^2 \leq mn, \qquad \textit{equivalently} \qquad K(m,n) \leq \sqrt{mn}.$$

Proof. Let $t = \min(\Omega(m), \Omega(n))$. Since

$$\min(p_i, q_i) \leq \sqrt{p_i q_i},$$

we obtain

$$K(m,n) = \prod_{i=1}^{t} \min(p_i, q_i) \leq \prod_{i=1}^{t} \sqrt{p_i q_i} = \sqrt{\left(\prod_{i=1}^{t} p_i\right)\left(\prod_{i=1}^{t} q_i\right)}.$$

Now

$$\prod_{i=1}^{t} p_i \leq m, \qquad \prod_{i=1}^{t} q_i \leq n,$$

hence

$$K(m,n) \leq \sqrt{mn}.$$

□

Remark 1. The inequality

$$K(m,n)^2 \leq K(m,m)\,K(n,n)$$

is formally identical to the Cauchy–Schwarz inequality for a positive semidefinite kernel, since $K(m,m) = m$.

10.4 Special cases

Proposition 8. *If p, q are primes, then*

$$K(p,q) = \min(p,q).$$

More generally, for primes p, q and integers $a, b \geq 0$,

$$K(p^a, q^b) = \min(p,q)^{\min(a,b)}.$$

In particular,

$$K(p^a, p^b) = p^{\min(a,b)}.$$

Proof. The prime-factor list of p^a consists of a copies of p, and similarly for q^b. The result follows directly from the definition. □

10.5 Comparison with divisibility and with gcd

The kernel K is not the gcd-kernel and is not governed by ordinary divisibility. It depends on the *positions* of the ordered prime factors.

Proposition 9. *In general, $K(m,n) \neq \gcd(m,n)$. Moreover, neither inequality*

$$K(m,n) \leq \gcd(m,n) \qquad \textit{nor} \qquad K(m,n) \geq \gcd(m,n)$$

holds universally.

Proof. For

$$m = 6 = (2,3), \qquad n = 10 = (2,5),$$

we have

$$K(6,10) = 2 \cdot 3 = 6, \qquad \gcd(6,10) = 2.$$

Thus $K(m,n)$ can be strictly larger than $\gcd(m,n)$.

On the other hand, there is no divisibility interpretation forcing a uniform opposite inequality either, since the two constructions measure different structures. □

Proposition 10. *Ordinary divisibility does not imply $K(m,n) = m$.*

Proof. Take

$$m = 6 = (2,3), \qquad n = 12 = (2,2,3).$$

Then $6 \mid 12$, but

$$K(6,12) = 2 \cdot 2 = 4 \neq 6.$$

Hence $K(m,n) = m$ is controlled by the order relation $\preceq$, not by ordinary divisibility. □

10.6 A characterization of the value 1

Proposition 11. *For $m,n \geq 1$,*

$$K(m,n) = 1 \iff m = 1 \textit{ or } n = 1.$$

Proof. If $m = 1$ or $n = 1$, then the product is empty and equals 1. Conversely, if $m,n > 1$, then at least one factor occurs and every factor is at least 2, so $K(m,n) \geq 2$. □

10.7 Interpretation

The kernel K does not measure common divisors in the classical sense. Rather, it measures the similarity of the *ordered prime-factor lists* of two integers. Large values of $K(m,n)$ occur when the early entries of the sorted prime-factor lists of m and n are close in the componentwise minimum sense.

For example,

$$30 = (2,3,5), \qquad 42 = (2,3,7),$$

so

$$K(30,42) = 2 \cdot 3 \cdot 5 = 30,$$

whereas

$$\gcd(30,42) = 6.$$

Thus K is sensitive to the entire prime-factor profile rather than only to the overlap of prime divisors.

10.8 Feature-theoretic reformulation

Whenever K is realized as a positive semidefinite meet kernel on a finite meet-closed subset, one has a Möbius inversion formula

$$K(m,n) = \sum_{d \preceq m,\ d \preceq n} g(d),$$

where g is the Möbius transform of $f(x) = x$ on the corresponding poset. If all $g(d) \geq 0$, then one obtains explicit feature coordinates

$$\phi_d(n) = \sqrt{g(d)}\, 1_{\{d \preceq n\}},$$

and hence

$$K(m,n) = \langle \phi(m), \phi(n) \rangle.$$

In that setting, the inequality

$$K(m,n)^2 \leq K(m,m)\, K(n,n) = mn$$

is precisely the Cauchy–Schwarz inequality in the associated feature space.

11 Integral lattice embedding

We now use the positive definiteness of K to obtain an integral feature embedding.

Theorem 2. *We know that the kernel*

$$K(m,n) = \prod_{i \geq 1} \min(p_i(m), p_i(n))$$

is strictly *positive definite on every finite set of distinct integers. Let g be the Möbius transform in the meet-matrix expansion*

$$K(m,n) = \sum_{d \preceq m,\ d \preceq n} g(d).$$

Then

$$g(d) > 0 \qquad \textit{for every } d \geq 1.$$

In particular,

$$g(d) \in \mathbb{Z}_{>0} \qquad \textit{for every } d \geq 1.$$

Proof. Fix $N \geq 1$ and let

$$X_N = \{x_1, \ldots, x_N\}$$

be the first N elements in a linear extension of the order $\preceq$. Let

$$G_N = (K(x_i, x_j))_{1 \leq i,j \leq N}$$

be the corresponding Gram matrix.

By the meet-matrix factorization,

$$G_N = E_N D_N E_N^T,$$

where

$$(E_N)_{ij} = 1_{\{x_j \preceq x_i\}}, \qquad D_N = \operatorname{diag}(g(x_1), \ldots, g(x_N)).$$

Since E_N is unitriangular, it is invertible. Hence

$$D_N = E_N^{-1} G_N E_N^{-T}.$$

Because G_N is positive definite and congruence by an invertible matrix preserves positive definiteness, D_N is positive definite. Since D_N is diagonal, every diagonal entry is strictly positive:

$$g(x_i) > 0 \qquad (1 \le i \le N).$$

As N was arbitrary, this holds for all $d \in \mathbb{N}$.

Finally, $g(d)$ is defined recursively from integer values of $f(d) = d$ by subtraction of previously computed integers, so $g(d) \in \mathbb{Z}$. Therefore

$$g(d) \in \mathbb{Z}_{>0}$$

for all d. □

Corollary 1. *The kernel K admits a feature embedding with integer, in fact binary, coordinates.*

Proof. For every $d \ge 1$ and every $1 \le j \le g(d)$ introduce a coordinate $e_{d,j}$. Define

$$\Phi(n) = \sum_{d \preceq n} \sum_{j=1}^{g(d)} e_{d,j}.$$

Equivalently,

$$\Phi_{d,j}(n) = 1_{\{d \preceq n\}}, \qquad 1 \le j \le g(d).$$

Thus every coordinate of $\Phi(n)$ is either 0 or 1, so

$$\Phi(n) \in \{0,1\}^{(\mathcal{D})} \subset \mathbb{Z}^{(\mathcal{D})}.$$

Moreover,

$$\langle \Phi(m), \Phi(n) \rangle = \sum_{d} \sum_{j=1}^{g(d)} 1_{\{d \preceq m\}} 1_{\{d \preceq n\}} = \sum_{d \preceq m,\ d \preceq n} g(d) = K(m,n).$$

Hence K is the Gram kernel of integer-valued feature vectors. □

Definition 1. *For $N \ge 1$, let*

$$L_N := \mathbb{Z}\Phi(1) + \cdots + \mathbb{Z}\Phi(N).$$

Then L_N is an integral lattice with Gram matrix

$$G_N = (K(i,j))_{1 \le i,j \le N}.$$

11.1 Basic properties of the lattice L_N

Proposition 12. *For every $N \ge 1$, the lattice L_N has the following properties:*

1. *L_N is integral.*

2. *$\operatorname{rank}(L_N) = N$.*

3. *The squared norm of the basis vector* $\Phi(n)$ *is*

$$\|\Phi(n)\|^2 = K(n,n) = n.$$

4. *The inner products are*

$$\langle \Phi(m), \Phi(n) \rangle = K(m,n) \in \mathbb{Z}_{>0}.$$

   Hence the basis is pairwise acute.

5. *The lattice determinant is*

$$\det(L_N) = \det(G_N) = \prod_{x \in X_N} g(x).$$

Proof. Integrality is immediate from the construction. Since K is strictly positive definite, the Gram matrix G_N is positive definite, hence invertible, so the vectors $\Phi(1), \ldots, \Phi(N)$ are linearly independent and therefore form a basis of L_N. This proves $\operatorname{rank}(L_N) = N$.

The norm identity follows from

$$\|\Phi(n)\|^2 = \langle \Phi(n), \Phi(n) \rangle = K(n,n) = n.$$

The inner product statement is exactly the reproducing property of the embedding.

Finally, since $\Phi(1), \ldots, \Phi(N)$ form a basis of L_N, the lattice determinant equals the determinant of its Gram matrix. By the Lindström–Bhat determinant formula,

$$\det(G_N) = \prod_{x \in X_N} g(x).$$

□

Remark 2. The lattice L_N is generally *not even*, since

$$\|\Phi(n)\|^2 = n$$

can be odd. It is also generally *not unimodular*, because $\det(G_N)$ is typically larger than 1.

11.2 The case $N = 20$

For the first twenty integers, the Möbius weights are

$$\big(g(1), \ldots, g(20)\big) = (1,1,1,2,2,2,2,4,2,4,4,4,2,4,2,8,4,4,2,8).$$

Thus the ambient binary feature space for $\{1, \ldots, 20\}$ has dimension

$$\sum_{d=1}^{20} g(d) = 63.$$

It is convenient to write the feature support of n in *block form*: the symbol $d^{g(d)}$ means that there are exactly $g(d)$ coordinates attached to d, all equal to 1 whenever $d \preceq n$.

n	$\pi(n)$	$\|\Phi(n)\|^2$	active blocks in $\Phi(n)$
1	1	1	1
2	2	2	$1,\ 2$
3	3	3	$1,\ 2,\ 3$
4	$2,2$	4	$1,\ 2,\ 4^2$
5	5	5	$1,\ 2,\ 3,\ 5^2$
6	$2,3$	6	$1,\ 2,\ 4^2,\ 6^2$
7	7	7	$1,\ 2,\ 3,\ 5^2,\ 7^2$
8	$2,2,2$	8	$1,\ 2,\ 4^2,\ 8^4$
9	$3,3$	9	$1,\ 2,\ 3,\ 4^2,\ 6^2,\ 9^2$
10	$2,5$	10	$1,\ 2,\ 4^2,\ 6^2,\ 10^4$
11	11	11	$1,\ 2,\ 3,\ 5^2,\ 7^2,\ 11^4$
12	$2,2,3$	12	$1,\ 2,\ 4^2,\ 8^4,\ 12^4$
13	13	13	$1,\ 2,\ 3,\ 5^2,\ 7^2,\ 11^4,\ 13^2$
14	$2,7$	14	$1,\ 2,\ 4^2,\ 6^2,\ 10^4,\ 14^4$
15	$3,5$	15	$1,\ 2,\ 3,\ 4^2,\ 6^2,\ 10^4,\ 9^2,\ 15^2$
16	$2,2,2,2$	16	$1,\ 2,\ 4^2,\ 8^4,\ 16^8$
17	17	17	$1,\ 2,\ 3,\ 5^2,\ 7^2,\ 11^4,\ 13^2,\ 17^4$
18	$2,3,3$	18	$1,\ 2,\ 4^2,\ 6^2,\ 8^4,\ 12^4,\ 18^4$
19	19	19	$1,\ 2,\ 3,\ 5^2,\ 7^2,\ 11^4,\ 13^2,\ 17^4,\ 19^2$
20	$2,2,5$	20	$1,\ 2,\ 4^2,\ 8^4,\ 12^4,\ 20^8$

For this lattice one finds

$$\det(G_{20}) = \prod_{n=1}^{20} g(n) = 2^{28}.$$

Hence L_{20} is an integral lattice of rank 20 and discriminant 2^{28}. In particular, it is not unimodular.

A computation of the Smith normal form of G_{20} gives

$$\operatorname{diag}(1,1,1,2,2,2,2,2,2,2,2,4,4,4,4,4,4,4,8,8),$$

so the discriminant group is

$$L_{20}^{\vee}/L_{20} \cong (\mathbb{Z}/2\mathbb{Z})^8 \oplus (\mathbb{Z}/4\mathbb{Z})^7 \oplus (\mathbb{Z}/8\mathbb{Z})^2.$$

Thus the discriminant is a finite 2-group.

12 The prime subkernel

Let

$$p_1 = 2 < p_2 = 3 < \cdots < p_k$$

be the first k primes, and consider the restriction of the kernel K to the prime numbers:

$$M_k := \big(K(p_i,p_j)\big)_{1\le i,j\le k}.$$

Since the prime-factor list of a prime consists of a single entry, we have

$$K(p_i,p_j) = \min(p_i,p_j).$$

Hence

$$M_k = \big(\min(p_i, p_j)\big)_{1\le i,j\le k}.$$

Set

$$\Delta_1 := p_1 = 2, \qquad \Delta_i := p_i - p_{i-1} \quad (i \ge 2).$$

Then

$$p_i = \Delta_1 + \cdots + \Delta_i.$$

Proposition 13. *Let*

$$L_k = (1_{j\le i})_{1\le i,j\le k}, \qquad D_k = \mathrm{diag}(\Delta_1, \dots, \Delta_k).$$

Then

$$M_k = L_k D_k L_k^T.$$

Proof. For every i, j,

$$\min(p_i, p_j) = p_{\min(i,j)} = \sum_{r=1}^{\min(i,j)} \Delta_r.$$

On the other hand,

$$(L_k D_k L_k^T)_{ij} = \sum_{r=1}^{k} 1_{\{r\le i\}}\, \Delta_r\, 1_{\{r\le j\}} = \sum_{r\le \min(i,j)} \Delta_r = \min(p_i, p_j).$$

Thus $M_k = L_k D_k L_k^T$. □

Corollary 2.

$$\det(M_k) = \prod_{i=1}^{k} \Delta_i = 2\prod_{i=2}^{k} (p_i - p_{i-1}).$$

Proof. Since L_k is unitriangular, $\det(L_k) = 1$. Therefore

$$\det(M_k) = \det(D_k) = \prod_{i=1}^{k} \Delta_i.$$

□

Remark 3. Thus the determinant of the prime Gram matrix is, up to the initial factor 2, exactly the product of the prime gaps. Equivalently,

$$\log\det(M_k) = \log 2 + \sum_{i=2}^{k} \log(p_i - p_{i-1}).$$

13 Inverse matrix and the grounded path Laplacian

Because L_k is invertible, one obtains

$$M_k^{-1} = L_k^{-T} D_k^{-1} L_k^{-1}.$$

Since L_k^{-1} is the first-difference matrix, this yields a tridiagonal inverse.

Proposition 14. *The inverse matrix is*

$$M_k^{-1} = \begin{pmatrix} \frac{1}{\Delta_1} + \frac{1}{\Delta_2} & -\frac{1}{\Delta_2} & 0 & \cdots & 0 \\ -\frac{1}{\Delta_2} & \frac{1}{\Delta_2} + \frac{1}{\Delta_3} & -\frac{1}{\Delta_3} & \ddots & \vdots \\ 0 & -\frac{1}{\Delta_3} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{1}{\Delta_{k-1}} + \frac{1}{\Delta_k} & -\frac{1}{\Delta_k} \\ 0 & \cdots & 0 & -\frac{1}{\Delta_k} & \frac{1}{\Delta_k} \end{pmatrix}.$$

Proof. Let

$$B = L_k^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

Then

$$M_k^{-1} = B^T D_k^{-1} B.$$

A direct multiplication gives the stated tridiagonal matrix. □

Remark 4. This is precisely the reduced Laplacian of the path graph

$$0 - 1 - 2 - \cdots - k$$

with edge conductances

$$c_i := \frac{1}{\Delta_i}$$

and edge resistances

$$r_i := \Delta_i.$$

The node 0 is grounded, and the reduced Laplacian acts on the potential vector $(u_1, \ldots, u_k)^T$.

14 Kirchhoff's law and the electrical interpretation

Let $u_0 := 0$ and let u_i be the electric potential at node i. The current through the edge $(i-1, i)$ is

$$I_i = c_i(u_i - u_{i-1}) = \frac{u_i - u_{i-1}}{\Delta_i}.$$

Proposition 15. *If $f = (f_1, \ldots, f_k)^T$ is the vector of injected currents at the nodes $1, \ldots, k$, then Kirchhoff's current law is exactly*

$$M_k^{-1} u = f.$$

Proof. At an interior node $1 < i < k$, the net outgoing current is

$$c_i(u_i - u_{i-1}) + c_{i+1}(u_i - u_{i+1}).$$

At the first node one has

$$c_1 u_1 + c_2(u_1 - u_2),$$

and at the last node

$$c_k(u_k - u_{k-1}).$$

These expressions are exactly the components of the tridiagonal matrix M_k^{-1} applied to u. □

Proposition 16. *The electrical energy is*

$$\mathcal{E}(u) = u^T M_k^{-1} u = \frac{u_1^2}{\Delta_1} + \sum_{i=2}^{k} \frac{(u_i - u_{i-1})^2}{\Delta_i}.$$

Proof. Since

$$M_k^{-1} = B^T D_k^{-1} B,$$

we obtain

$$u^T M_k^{-1} u = (Bu)^T D_k^{-1} (Bu).$$

Now

$$Bu = (u_1, u_2 - u_1, \ldots, u_k - u_{k-1})^T,$$

which gives the formula immediately. □

Remark 5. A large prime gap Δ_i makes the coefficient $1/\Delta_i$ small, so a potential jump across that edge is energetically cheap. Thus large prime gaps act as weak couplings or electrical bottlenecks in the chain.

15 Effective resistance

On a path, resistances in series add.

Proposition 17. *For $0 \leq a < b \leq k$, the effective resistance between the nodes a and b is*

$$R_{\text{eff}}(a, b) = \sum_{i=a+1}^{b} \Delta_i = p_b - p_a,$$

where $p_0 := 0$.

Proof. There is a unique path from a to b, and its edge resistances are

$$\Delta_{a+1}, \ldots, \Delta_b.$$

Resistances in series add, so

$$R_{\text{eff}}(a, b) = \Delta_{a+1} + \cdots + \Delta_b = p_b - p_a.$$

□

Corollary 3.

$$R_{\text{eff}}(0, j) = p_j.$$

In particular,

$$(M_k)_{jj} = p_j = R_{\text{eff}}(0, j).$$

Remark 6. Thus the primes themselves are exactly the cumulative effective resistances from the grounded boundary.

16 The Green matrix viewpoint

Proposition 18. *The matrix M_k is the Green matrix of the grounded path network with edge resistances given by the prime gaps.*

Proof. By the previous section, the reduced Laplacian of the path is exactly M_k^{-1}. Therefore

$$M_k = (L_k^{\mathrm{red}})^{-1},$$

which is the Green matrix of the network. □

Remark 7. This is the discrete network analogue of the classical fact that the kernel

$$(s,t) \mapsto \min(s,t)$$

is the Green kernel of the one-dimensional Dirichlet Laplacian and the covariance kernel of Brownian motion.

17 Spectral consequences

Let

$$\lambda_1(M_k) \geq \cdots \geq \lambda_k(M_k) > 0$$

be the eigenvalues of M_k, and

$$\mu_1 \leq \cdots \leq \mu_k$$

those of M_k^{-1}. Then

$$\lambda_i(M_k) = \frac{1}{\mu_{k+1-i}}.$$

Proposition 19. *The trace and determinant are*

$$\mathrm{tr}(M_k) = \sum_{i=1}^{k} p_i, \qquad \det(M_k) = 2\prod_{i=2}^{k}(p_i - p_{i-1}).$$

Proof. The trace is immediate from the diagonal entries

$$(M_k)_{ii} = p_i.$$

The determinant was proved above. □

Remark 8. Hence the prime Gram matrix packages two classical arithmetic statistics into linear algebra:

$$\text{trace} = \sum \text{primes}, \qquad \text{determinant} = \prod \text{prime gaps}.$$

Remark 9. Because M_k^{-1} is a weighted path Laplacian, large prime gaps correspond to small conductances and therefore to weak links in the network. One expects such weak links to influence the smallest nontrivial Laplace eigenvalues, and equivalently the largest eigenvalues of M_k.

18 Random walk interpretation

Associated with the conductances

$$c_i = \frac{1}{\Delta_i}$$

there is a reversible random walk on the path $\{0, 1, \dots, k\}$. At an interior node $1 \le i \le k-1$, the transition probabilities are

$$\mathbb{P}(i \to i-1) = \frac{c_i}{c_i + c_{i+1}}, \qquad \mathbb{P}(i \to i+1) = \frac{c_{i+1}}{c_i + c_{i+1}}.$$

Since

$$c_i = \frac{1}{\Delta_i},$$

a large gap Δ_{i+1} decreases the transition probability across the edge $(i, i+1)$.

Remark 10. In this probabilistic picture, large prime gaps are barriers for transport. They slow down passage, reduce conductance, and create bottlenecks for the walk.

19 Cramér's random model and a random conductance heuristic

A standard probabilistic model for the primes is Cramér's model, in which each integer $n \ge 3$ is independently selected with probability approximately $1/\log n$. This produces a random increasing sequence

$$P_1 < P_2 < \cdots$$

whose local spacing heuristically mimics the prime gaps.

Write

$$G_1 := P_1, \qquad G_i := P_i - P_{i-1} \quad (i \ge 2).$$

Then one may form the random matrix

$$\widetilde{M}_k = (\min(P_i, P_j))_{1 \le i,j \le k}.$$

Its inverse is the grounded Laplacian of a random path network with random resistances

$$r_i = G_i$$

and random conductances

$$c_i = \frac{1}{G_i}.$$

Remark 11. In Cramér's model, the local spacing near size x has mean order $\log x$. Thus the network is a slowly varying one-dimensional random medium whose conductances are typically of order $1/\log x$, but with large fluctuations.

20 Threshold models and percolation-inspired questions

Classical Bernoulli percolation on a one-dimensional path is too simple. However, the prime-gap network suggests more natural threshold models.

20.1 Gap-threshold model

Fix $T > 0$. Keep only those edges for which

$$\Delta_i \leq T.$$

Then the path breaks into clusters of consecutive small gaps, separated by large gaps.

Remark 12. This is a percolation-inspired decomposition of the primes into highly conducting regions separated by insulating defects.

20.2 Conductance-threshold model

Equivalently, keep only those edges whose conductance exceeds a threshold $\theta > 0$:

$$c_i \geq \theta \iff \Delta_i \leq \frac{1}{\theta}.$$

Remark 13. In this language, the same model is expressed directly in terms of the electrical couplings of the network.

21 Physically inspired questions

The previous sections suggest several concrete research questions.

Question 1: spectral bottlenecks

How strongly does a record prime gap influence the low spectrum of the weighted Laplacian M_k^{-1}, or equivalently the top spectrum of the Green matrix M_k?

Question 2: localization

Do eigenvectors of M_k^{-1} or M_k localize near unusually large prime gaps, in the same way that weak links localize modes in inhomogeneous one-dimensional media?

Question 3: free-energy analogue

Since

$$\log \det(M_k) = \log 2 + \sum_{i=2}^{k} \log \Delta_i,$$

is the quantity

$$\frac{1}{k} \log \det(M_k)$$

a natural "free energy" of the prime-gap medium?

Question 4: universality

Which spectral statistics of the true prime network agree with the analogous statistics in Cramér's random model, and which reflect finer arithmetic structure beyond the random heuristic?

Question 5: transport

How do hitting times, effective resistances, and spectral gaps compare between the true prime-gap network and the Cramér random conductance model?

22 Summary

The restriction of the kernel K to the primes yields the matrix

$$M_k = (\min(p_i, p_j))_{1 \le i,j \le k},$$

whose inverse is the grounded Laplacian of a path with edge resistances equal to the prime gaps. In this formulation:

- the prime gaps are the local resistances of the medium;
- the primes themselves are cumulative effective resistances;
- the determinant is the product of the prime gaps;
- the Green matrix encodes the global response of the network;
- Cramér's model leads naturally to a random conductance analogue.

This turns the sequence of prime gaps into a one-dimensional electrical medium, opening the door to questions from spectral theory, random media, transport, and statistical physics.

23 A polynomial lift of the meet kernel

We now introduce a polynomial-valued lift of the meet kernel. Let

$$f_1(x), f_2(x), f_3(x), \ldots$$

be the polynomial family attached to the natural numbers, and assume that

$$f_n(2) = n \qquad (n \ge 1).$$

We keep the same partial order $\preceq$ on $\mathbb{N}$ coming from the sorted prime-factor lists, and the same meet operation $m \wedge n$.

23.1 Polynomial Möbius coefficients

We define polynomial-valued coefficients $g(n; x) \in \mathbb{Z}[x]$ recursively by

$$g(1; x) := f_1(x),$$

and for $n \ge 2$,

$$g(n; x) := f_n(x) - \sum_{\substack{d \prec n \\ d \preceq n}} g(d; x).$$

Equivalently,

$$f_n(x) = \sum_{d \preceq n} g(d; x) \qquad (n \ge 1).$$

Thus $g(n;x)$ is the Möbius transform of the function

$$n \longmapsto f_n(x)$$

on the factorization poset.

Evaluating at $x=2$ gives

$$f_n(2) = \sum_{d \preceq n} g(d;2).$$

Since also

$$n = \sum_{d \preceq n} g(d),$$

uniqueness of Möbius inversion yields

$$\boxed{g(n;2) = g(n).}$$

Hence the polynomial coefficients $g(n;x)$ lift the arithmetic coefficients $g(n)$.

23.2 Definition of the lifted kernel

We define the polynomial kernel

$$K'(m,n;x) := \sum_{d \preceq m,\ d \preceq n} g(d;x).$$

Proposition 20. *For all $m,n \geq 1$,*

$$K'(m,n;x) = f_{m \wedge n}(x).$$

Proof. By definition of the meet, one has

$$d \preceq m \text{ and } d \preceq n \iff d \preceq (m \wedge n).$$

Therefore

$$K'(m,n;x) = \sum_{d \preceq m,\ d \preceq n} g(d;x) = \sum_{d \preceq m \wedge n} g(d;x).$$

Using the defining inversion identity

$$f_r(x) = \sum_{d \preceq r} g(d;x),$$

with $r = m \wedge n$, we obtain

$$K'(m,n;x) = f_{m \wedge n}(x).$$

□

Thus K' is exactly the polynomial realization of the meet operation under the map $n \mapsto f_n(x)$.

23.3 Recovery of the original kernel

Corollary 4. *For all $m, n \geq 1$,*

$$K'(m, n; 2) = K(m, n).$$

Proof. By the proposition,

$$K'(m, n; 2) = f_{m \wedge n}(2) = m \wedge n = K(m, n).$$

□

So K' is a genuine lift of the original meet kernel.

23.4 Diagonal values

Corollary 5. *For every $n \geq 1$,*

$$K'(n, n; x) = f_n(x).$$

Proof. Since $n \wedge n = n$, the previous proposition gives

$$K'(n, n; x) = f_n(x).$$

□

Hence the diagonal of the lifted Gram matrix is exactly the polynomial family itself.

23.5 Restriction to the prime layer

Let $p_1 < p_2 < \dots$ denote the primes. Since the meet of two primes is their minimum, one obtains:

Corollary 6. *For primes p_i, p_j,*

$$K'(p_i, p_j; x) = f_{\min(p_i, p_j)}(x).$$

Thus the prime-layer Gram matrix becomes

$$\big(f_{\min(p_i, p_j)}(x)\big)_{i,j},$$

which is the polynomial analogue of the classical matrix

$$(\min(p_i, p_j))_{i,j}.$$

23.6 Polynomial gap coefficients on the prime layer

The coefficients $g(n; x)$ restrict on the prime layer to a polynomial version of prime gaps.

Proposition 21. *Let p_k be the k-th prime. Then*

$$g(p_1; x) = f_{p_1}(x) - f_1(x),$$

and for every $k \geq 2$,

$$g(p_k; x) = f_{p_k}(x) - f_{p_{k-1}}(x).$$

Proof. If p_k is prime, then the predecessors of p_k in the factorization poset are exactly

$$1, p_1, \ldots, p_{k-1}.$$

Hence

$$g(p_k; x) = f_{p_k}(x) - \left(g(1; x) + \sum_{j=1}^{k-1} g(p_j; x) \right).$$

For $k = 1$, this gives

$$g(p_1; x) = f_{p_1}(x) - g(1; x) = f_{p_1}(x) - f_1(x).$$

Now assume $k \geq 2$ and inductively that

$$g(p_j; x) = f_{p_j}(x) - f_{p_{j-1}}(x) \qquad (2 \leq j \leq k-1).$$

Then

$$g(1; x) + \sum_{j=1}^{k-1} g(p_j; x) = f_1(x) + \big(f_{p_1}(x) - f_1(x)\big) + \sum_{j=2}^{k-1} \big(f_{p_j}(x) - f_{p_{j-1}}(x)\big).$$

This telescopes to

$$f_{p_{k-1}}(x).$$

Therefore

$$g(p_k; x) = f_{p_k}(x) - f_{p_{k-1}}(x).$$

□

So the polynomial Möbius coefficients extend the prime-gap function from integers to the polynomial family.

23.7 Lifted meet matrices

For $N \geq 1$, define

$$G'_N(x) := \big(K'(i, j; x)\big)_{1 \leq i,j \leq N} = \big(f_{i \wedge j}(x)\big)_{1 \leq i,j \leq N}.$$

If $x_1, \ldots, x_N$ is a linear extension of the factorization poset on $\{1, \ldots, N\}$, let

$$E_N = (1_{\{x_j \preceq x_i\}})_{1 \leq i,j \leq N}, \qquad D_N(x) = \operatorname{diag}(g(x_1; x), \ldots, g(x_N; x)).$$

Proposition 22. *The lifted Gram matrix factorizes as*

$$G'_N(x) = E_N \, D_N(x) \, E_N^T.$$

Consequently,

$$\det G'_N(x) = \prod_{n=1}^{N} g(n; x)$$

up to the chosen ordering of $\{1, \ldots, N\}$.

Proof. The proof is the same as for ordinary meet matrices. Indeed,

$$(E_N D_N(x) E_N^T)_{ij} = \sum_{d \preceq x_i,\ d \preceq x_j} g(d;x) = \sum_{d \preceq x_i \wedge x_j} g(d;x) = f_{x_i \wedge x_j}(x) = K'(x_i, x_j; x).$$

Hence

$$G'_N(x) = E_N D_N(x) E_N^T.$$

Since E_N is unitriangular in a linear extension, $\det(E_N) = 1$, and therefore

$$\det G'_N(x) = \det D_N(x) = \prod_{n=1}^{N} g(n;x).$$

□

23.8 Interpretation

The kernel $K'(m,n;x)$ is the polynomial lift of the arithmetic meet kernel. It has the following features:

- it is obtained by Möbius inversion on the same factorization poset;
- it satisfies
$$K'(m,n;x) = f_{m \wedge n}(x);$$
- it specializes at $x = 2$ to the original kernel:
$$K'(m,n;2) = K(m,n);$$
- on the prime layer, its diagonal Möbius coefficients are the polynomial prime gaps
$$g(p_k;x) = f_{p_k}(x) - f_{p_{k-1}}(x).$$

In this sense, K' transports the entire meet-geometry of the natural numbers into the polynomial family $\{f_n(x)\}_{n \geq 1}$.

24 Differentiation at $x = 2$ and an arithmetic derivation

The polynomial lift of the meet kernel admits a natural differentiation procedure. Evaluating derivatives at the distinguished point $x = 2$ produces a new arithmetic layer attached to the family $\{f_n(x)\}_{n \geq 1}$.

24.1 The arithmetic derivative induced by the polynomial lift

Assume that the polynomial family satisfies

$$f_n(2) = n \qquad (n \geq 1),$$

and that it is multiplicative:

$$f_{mn}(x) = f_m(x) f_n(x) \qquad (m, n \geq 1).$$

We define

$$\delta(n) := f'_n(2).$$

Proposition 23. *The function $\delta : \mathbb{N} \to \mathbb{Z}$ satisfies the Leibniz rule*

$$\delta(mn) = n\,\delta(m) + m\,\delta(n) \qquad (m, n \geq 1).$$

Thus δ is an arithmetic derivation on $\mathbb{N}$.

Proof. Differentiate the multiplicativity identity:

$$f'_{mn}(x) = f'_m(x) f_n(x) + f_m(x) f'_n(x).$$

Evaluating at $x = 2$ gives

$$\delta(mn) = f'_{mn}(2) = f'_m(2) f_n(2) + f_m(2) f'_n(2).$$

Since $f_n(2) = n$, we obtain

$$\delta(mn) = n\,\delta(m) + m\,\delta(n),$$

as claimed. □

Hence the polynomial family gives rise canonically to a derivation on the multiplicative semiring of natural numbers.

Remark 14. This is analogous to the classical arithmetic derivative, except that the values $\delta(p)$ on primes are not prescribed a priori. Instead they are determined by the polynomial family through

$$\delta(p) = f'_p(2).$$

Thus δ is a weighted arithmetic derivative whose prime values encode the local behavior of the polynomials $f_p(x)$ near $x = 2$.

24.2 Derived Möbius coefficients

Recall that the polynomial Möbius coefficients were defined by

$$f_n(x) = \sum_{d \preceq n} g(d; x).$$

Differentiating this identity with respect to x yields

$$f'_n(x) = \sum_{d \preceq n} \frac{\partial}{\partial x} g(d; x).$$

Evaluating at $x = 2$, we define

$$\gamma(n) := \left. \frac{\partial}{\partial x} g(n; x) \right|_{x=2}.$$

Proposition 24. *For every $n \geq 1$,*

$$\delta(n) = \sum_{d \preceq n} \gamma(d).$$

Equivalently, γ is the Möbius transform of δ on the factorization poset:

$$\gamma(n) = \delta(n) - \sum_{\substack{d \prec n \\ d \preceq n}} \gamma(d).$$

Proof. Differentiate

$$f_n(x) = \sum_{d \preceq n} g(d; x)$$

and evaluate at $x = 2$. This gives

$$f'_n(2) = \sum_{d \preceq n} g'_x(d; 2),$$

that is,

$$\delta(n) = \sum_{d \preceq n} \gamma(d).$$

The recursive formula is then just Möbius inversion on the poset. □

Thus $\gamma(n)$ plays for $\delta(n)$ the same role that $g(n)$ plays for the original arithmetic function n.

24.3 Differentiating the lifted kernel

Recall that the polynomial lift of the meet kernel is

$$K'(m, n; x) = f_{m \wedge n}(x).$$

We now define its derivative at $x = 2$ by

$$\partial K(m, n) := \frac{\partial}{\partial x} K'(m, n; x)\bigg|_{x=2}.$$

Proposition 25. *For all* $m, n \geq 1$,

$$\partial K(m, n) = \delta(m \wedge n).$$

Moreover,

$$\partial K(m, n) = \sum_{d \preceq m,\ d \preceq n} \gamma(d).$$

Proof. Since

$$K'(m, n; x) = f_{m \wedge n}(x),$$

differentiation gives

$$\partial K(m, n) = f'_{m \wedge n}(2) = \delta(m \wedge n).$$

On the other hand,

$$K'(m, n; x) = \sum_{d \preceq m,\ d \preceq n} g(d; x),$$

so differentiating and evaluating at $x = 2$ yields

$$\partial K(m, n) = \sum_{d \preceq m,\ d \preceq n} \gamma(d).$$

□

Hence the derived kernel is again a meet kernel, now associated with the function δ instead of the function n.

24.4 Behavior on the prime layer

The coefficients $g(n;x)$ restrict on the prime layer to polynomial prime gaps:

$$g(p_k;x) = f_{p_k}(x) - f_{p_{k-1}}(x) \qquad (k \geq 2).$$

Differentiating this identity gives:

Proposition 26. *For every prime p_k with $k \geq 2$,*

$$\gamma(p_k) = \delta(p_k) - \delta(p_{k-1}).$$

Also,

$$\gamma(p_1) = \delta(p_1) - \delta(1).$$

Proof. Differentiate

$$g(p_k;x) = f_{p_k}(x) - f_{p_{k-1}}(x)$$

and evaluate at $x = 2$. □

Thus the function γ is a derived gap function on the prime layer: it measures the successive differences of the arithmetic derivative δ along the sequence of primes.

24.5 Higher derivatives

The same construction may be iterated. For each integer $r \geq 0$, define

$$\delta_r(n) := f_n^{(r)}(2), \qquad \gamma_r(n) := g^{(r)}(n;2).$$

Then

$$\delta_r(n) = \sum_{d \preceq n} \gamma_r(d),$$

so that γ_r is the Möbius transform of δ_r on the factorization poset.

In particular:

- $\delta_0(n) = n$ and $\gamma_0(n) = g(n)$,
- $\delta_1(n) = \delta(n)$ and $\gamma_1(n) = \gamma(n)$.

This yields a hierarchy of arithmetic functions naturally attached to the polynomial lift.

24.6 Interpretation

Differentiation at $x = 2$ produces a second arithmetic layer from the polynomial family:

- the values
$$\delta(n) = f_n'(2)$$
form an arithmetic derivation;
- the coefficients
$$\gamma(n) = g_x'(n;2)$$
are its Möbius coefficients on the factorization poset;
- the derived kernel
$$\partial K(m,n) = \delta(m \wedge n)$$
is again a meet kernel.

Thus the polynomial lift does not merely recover the original kernel at $x = 2$; it also generates a natural differential refinement of the arithmetic structure.

25 Arithmetic properties inherited from the polynomial model

We now assume that the polynomial family

$$\{f_n(x)\}_{n\geq 1} \subset \mathbb{Z}[x]$$

satisfies the following properties:

1. $$f_{mn}(x) = f_m(x)f_n(x) \qquad (m,n \geq 1),$$

2. $$f_n(2) = n \qquad (n \geq 1),$$

3. $$\gcd(f_m(x), f_n(x)) = f_{\gcd(m,n)}(x),$$

4. $$f_n(x) \text{ is separable} \iff n = \operatorname{rad}(n),$$

5. $$f_n(x) \text{ is irreducible} \iff n \text{ is prime}.$$

Recall that the polynomial lift of the meet kernel is

$$K'(m,n;x) := f_{m\wedge n}(x).$$

25.1 Basic inherited properties

Proposition 27. *For all $m,n \geq 1$, one has*

$$K'(m,n;2) = K(m,n),$$

and

$$K'(n,n;x) = f_n(x).$$

Proof. Since

$$K'(m,n;x) = f_{m\wedge n}(x),$$

evaluation at $x = 2$ gives

$$K'(m,n;2) = f_{m\wedge n}(2) = m \wedge n = K(m,n).$$

Also,

$$K'(n,n;x) = f_{n\wedge n}(x) = f_n(x).$$

□

Thus the diagonal of the lifted kernel recovers the polynomial model itself, while evaluation at $x = 2$ recovers the original arithmetic meet kernel.

25.2 Multiplicativity on the diagonal

Proposition 28. *For all* $m, n \geq 1$*,*

$$K'(mn, mn; x) = K'(m, m; x)\, K'(n, n; x).$$

Proof. Using the diagonal identity and multiplicativity of $f_n(x)$,

$$K'(mn, mn; x) = f_{mn}(x) = f_m(x) f_n(x) = K'(m, m; x)\, K'(n, n; x).$$

□

Hence the diagonal of the lifted kernel is multiplicative.

25.3 GCD structure on the diagonal

Proposition 29. *For all* $m, n \geq 1$*,*

$$\gcd\bigl(K'(m, m; x), K'(n, n; x)\bigr) = f_{\gcd(m,n)}(x).$$

Proof. Since

$$K'(m, m; x) = f_m(x), \qquad K'(n, n; x) = f_n(x),$$

the claimed identity follows directly from

$$\gcd(f_m(x), f_n(x)) = f_{\gcd(m,n)}(x).$$

□

Thus the diagonal of the lifted kernel remembers the ordinary gcd-structure of the integers, now inside the polynomial ring $\mathbb{Z}[x]$.

25.4 Squarefreeness and separability

Proposition 30. *For every* $n \geq 1$*,*

$$K'(n, n; x) \text{ is separable} \iff n \text{ is squarefree.}$$

Proof. Again,

$$K'(n, n; x) = f_n(x),$$

so the statement is exactly the separability criterion

$$f_n(x) \text{ is separable} \iff n = \operatorname{rad}(n),$$

that is, iff n is squarefree. □

So squarefreeness of n is detected by the separability of the diagonal value of the lifted kernel.

25.5 Primality and irreducibility

Proposition 31. *For every $n \geq 1$,*

$$K'(n, n; x) \text{ is irreducible} \iff n \text{ is prime.}$$

Proof. Since

$$K'(n, n; x) = f_n(x),$$

the statement follows immediately from the irreducibility criterion

$$f_n(x) \text{ is irreducible} \iff n \text{ is prime.}$$

□

This gives a direct polynomial characterization of primality through the lifted kernel: the integer n is prime precisely when the diagonal entry $K'(n, n; x)$ is irreducible in $\mathbb{Z}[x]$.

25.6 The radical of the diagonal values

If one writes

$$\operatorname{rad}(f_n(x)) = \prod_{p|n} f_p(x),$$

then evaluating at $x = 2$ yields

$$\operatorname{rad}(f_n(x))\big|_{x=2} = \prod_{p|n} p = \operatorname{rad}(n).$$

Thus the polynomial radical lifts the ordinary squarefree kernel of n.

Equivalently, on the diagonal of the kernel one has

$$\operatorname{rad}(K'(n, n; x))\big|_{x=2} = \operatorname{rad}(n).$$

25.7 Interpretation

The lifted kernel

$$K'(m, n; x) = f_{m \wedge n}(x)$$

inherits a rich arithmetic structure from the polynomial model:

- evaluation at $x = 2$ recovers the original meet kernel;
- the diagonal is multiplicative;
- the diagonal encodes ordinary gcd-structure;
- squarefreeness is equivalent to separability of the diagonal value;
- primality is equivalent to irreducibility of the diagonal value.

In this sense, the polynomial lift does not merely refine the meet kernel: it transports fundamental arithmetic properties of the integers into intrinsic algebraic properties of the corresponding diagonal polynomials.

26 Characteristic polynomials: prime–min Gram matrices and meet Gram matrices

This section summarizes the characteristic-polynomial coefficients for two canonical positive definite matrices arising in our framework:

- the *prime–min Gram matrix*

$$M_k := (\min(p_i, p_j))_{1\le i,j\le k},$$

  where $p_1 < p_2 < \cdots$ are the primes;

- the *meet Gram matrix*

$$G_N := (K(i,j))_{1\le i,j\le N}, \qquad K(m,n) = \sum_{d\preceq m,\ d\preceq n} g(d),$$

  associated with the factorization poset $(\mathbb{N}, \preceq)$.

Throughout, for a $k \times k$ matrix A we write

$$\chi_A(\lambda) := \det(\lambda I - A) = \lambda^k - c_1(A)\lambda^{k-1} + c_2(A)\lambda^{k-2} - \cdots + (-1)^k c_k(A),$$

so that $c_r(A)$ is the r-th elementary symmetric polynomial in the eigenvalues of A.

26.1 A universal coefficient identity

We recall a standard identity (sometimes called the *principal minor expansion*):

Lemma 1 (Coefficients as sums of principal minors)**.** *Let A be a $k \times k$ matrix. Then for each $1 \le r \le k$,*

$$c_r(A) = \sum_{\substack{I\subseteq\{1,\dots,k\}\\ |I|=r}} \det(A[I]),$$

where $A[I]$ denotes the principal submatrix indexed by I.

Proof. Expanding $\det(\lambda I - A)$ by permutations and collecting the terms of degree $k-r$ shows that the coefficient of λ^{k-r} is $(-1)^r$ times the sum of all principal $r \times r$ minors. This yields the formula for $c_r(A)$. ☐

26.2 Prime–min Gram matrix: exact coefficient formula

Let $2 = p_1 < p_2 < \cdots < p_k$ be the first k primes, and set

$$M_k = (\min(p_i, p_j))_{1\le i,j\le k}.$$

Lemma 2 (Determinant of a min-matrix)**.** *Let $0 < a_1 < \cdots < a_r$ and set $B_r = (\min(a_i, a_j))_{1\le i,j\le r}$. Then*

$$\det(B_r) = a_1 \prod_{m=2}^{r} (a_m - a_{m-1}).$$

Proof. Define $\Delta_1 := a_1$ and $\Delta_m := a_m - a_{m-1}$ for $m \geq 2$, so that $a_j = \Delta_1 + \cdots + \Delta_j$. Let $L_r = (1_{j\leq i})_{1\leq i,j\leq r}$ and $D_r = \operatorname{diag}(\Delta_1, \ldots, \Delta_r)$. Then

$$(L_r D_r L_r^T)_{ij} = \sum_{t\leq \min(i,j)} \Delta_t = a_{\min(i,j)} = \min(a_i, a_j),$$

so $B_r = L_r D_r L_r^T$. Since L_r is unitriangular, $\det(L_r) = 1$, hence

$$\det(B_r) = \det(D_r) = \prod_{m=1}^{r} \Delta_m = a_1 \prod_{m=2}^{r} (a_m - a_{m-1}).$$

□

Theorem 3 (Prime–min characteristic polynomial coefficients)**.** *Let*

$$\chi_{M_k}(\lambda) = \lambda^k - c_1^{(k)}\lambda^{k-1} + c_2^{(k)}\lambda^{k-2} - \cdots + (-1)^k c_k^{(k)}.$$

Then for every $1 \leq r \leq k$*,*

$$c_r^{(k)} = \sum_{1\leq i_1<\cdots<i_r\leq k} p_{i_1} \prod_{m=2}^{r} (p_{i_m} - p_{i_{m-1}}).$$

In particular,

$$c_1^{(k)} = \sum_{i=1}^{k} p_i, \qquad c_k^{(k)} = \det(M_k) = 2\prod_{i=2}^{k}(p_i - p_{i-1}).$$

Proof. By the principal minor expansion,

$$c_r^{(k)} = \sum_{|I|=r} \det(M_k[I]).$$

For $I = \{i_1 < \cdots < i_r\}$, the principal submatrix is

$$M_k[I] = (\min(p_{i_a}, p_{i_b}))_{1\leq a,b\leq r}.$$

Applying the preceding lemma with $a_j = p_{i_j}$ yields

$$\det(M_k[I]) = p_{i_1} \prod_{m=2}^{r} (p_{i_m} - p_{i_{m-1}}).$$

Summing over all I gives the claimed formula. The special cases $r = 1$ and $r = k$ are immediate. □

26.3 Prime–min coefficients: divisibility and bounds

Proposition 32 (A 2-adic divisibility property)**.** *For every* k *and* $1 \leq r \leq k$*,*

$$2^{\,r-1} \mid c_r^{(k)}.$$

Proof. Fix $I = \{i_1 < \cdots < i_r\}$. The corresponding term

$$T_I := p_{i_1} \prod_{m=2}^{r} (p_{i_m} - p_{i_{m-1}})$$

is an integer. If $p_{i_1} = 2$, then p_{i_1} contributes one factor 2 and each difference $p_{i_m} - p_{i_{m-1}}$ with $m \geq 3$ is even (difference of two odd primes), yielding at least $1 + (r-2) = r-1$ powers of 2 in T_I. If p_{i_1} is odd, then all primes in the product are odd and each of the $r-1$ differences is even, again giving $2^{r-1} \mid T_I$. Thus every summand in $c_r^{(k)}$ is divisible by 2^{r-1}, hence so is their sum. □

Proposition 33 (Crude lower and upper bounds)**.** *For all* $1 \leq r \leq k$*,*

$$2^{r-1}\binom{k}{r} \;\leq\; c_r^{(k)} \;\leq\; \binom{k}{r} p_k^r.$$

Moreover, for each fixed r *the sequence* $c_r^{(k)}$ *is strictly increasing in* k*:*

$$c_r^{(k+1)} > c_r^{(k)} \qquad (1 \leq r \leq k).$$

Proof. Lower bound: in the explicit coefficient formula, each summand

$$T_I = p_{i_1} \prod_{m=2}^{r} (p_{i_m} - p_{i_{m-1}})$$

is positive. If $i_1 = 1$, then $p_{i_1} = 2$, the first difference is ≥ 1 and all remaining differences are ≥ 2, so $T_I \geq 2 \cdot 1 \cdot 2^{r-2} = 2^{r-1}$. If $i_1 > 1$, then all primes involved are odd and each difference is ≥ 2, hence $T_I \geq 3 \cdot 2^{r-1} \geq 2^{r-1}$. There are $\binom{k}{r}$ choices of I, giving $c_r^{(k)} \geq 2^{r-1}\binom{k}{r}$.

Upper bound: for every summand one has $p_{i_1} \leq p_k$ and $p_{i_m} - p_{i_{m-1}} \leq p_{i_m} \leq p_k$, so $T_I \leq p_k^r$. Summing over $\binom{k}{r}$ subsets yields $c_r^{(k)} \leq \binom{k}{r} p_k^r$.

Monotonicity: passing from k to $k+1$ adds new positive summands (those involving p_{k+1}), so $c_r^{(k+1)} > c_r^{(k)}$ for $1 \leq r \leq k$. □

26.4 Meet Gram matrix: coefficients and poset structure

Let

$$G_N = (K(i,j))_{1 \leq i,j \leq N}, \qquad K(m,n) = \sum_{d \preceq m,\ d \preceq n} g(d),$$

and write

$$\chi_{G_N}(\lambda) = \lambda^N - \alpha_1^{(N)} \lambda^{N-1} + \alpha_2^{(N)} \lambda^{N-2} - \cdots + (-1)^N \alpha_N^{(N)}.$$

Proposition 34 (Trace and determinant)**.** *One has*

$$\alpha_1^{(N)} = \operatorname{tr}(G_N) = \sum_{n=1}^{N} K(n,n) = \sum_{n=1}^{N} n = \frac{N(N+1)}{2},$$

and

$$\alpha_N^{(N)} = \det(G_N) = \prod_{n=1}^{N} g(n)$$

(up to the choice of a linear extension of the poset, which does not change the product).

Proof. The trace is immediate from $K(n,n) = n$. The determinant follows from the Lindström–Bhat factorization in a linear extension:

$$G_N = E_N D_N E_N^T, \qquad D_N = \operatorname{diag}(g(x_1), \dots, g(x_N)),$$

where E_N is unitriangular, hence $\det(E_N) = 1$, and therefore $\det(G_N) = \det(D_N) = \prod_{n \le N} g(n)$. □

Proposition 35 (Principal minors on lower ideals). *Let $X_N = \{x_1, \dots, x_N\}$ be a linear extension of $\preceq$ on $\{1, \dots, N\}$. If $I \subseteq X_N$ is* lower-closed *(an order ideal), meaning that $x \in I$ and $y \preceq x$ implies $y \in I$, then the principal submatrix $G_N[I]$ satisfies*

$$\det(G_N[I]) = \prod_{x \in I} g(x).$$

Proof. On a lower-closed set I, the restriction of K is again a meet kernel on the finite poset I. The same Lindström–Bhat factorization applies:

$$G_N[I] = E_I D_I E_I^T, \qquad D_I = \operatorname{diag}(g(x))_{x \in I},$$

with unitriangular E_I. Hence $\det(G_N[I]) = \det(D_I) = \prod_{x \in I} g(x)$. □

Remark 15. In contrast to the prime–min case, an arbitrary principal submatrix of G_N need not correspond to a meet-closed or lower-closed set; therefore there is no uniform closed formula for all coefficients $\alpha_r^{(N)}$. Nevertheless, the principal-minor expansion together with the preceding proposition shows that order ideals contribute explicitly computable product terms $\prod_{x \in I} g(x)$ to $\alpha_r^{(N)}$.

27 A 2-scaling law for the polynomial Möbius coefficients

In this section we prove a striking regularity visible in the table of the polynomial Möbius coefficients $g(n, x)$.

27.1 Setup

Let $\preceq$ be the factorization-poset order on $\mathbb{N}$ induced by componentwise comparison of sorted prime-factor lists (allowing shorter lengths). Assume that the polynomial family $\{f_n(x)\}_{n \ge 1} \subset \mathbb{Z}[x]$ is multiplicative and normalized by

$$f_{mn}(x) = f_m(x) f_n(x), \qquad f_1(x) = 1, \qquad f_2(x) = x.$$

Define the polynomial Möbius coefficients $g(n, x)$ by Möbius inversion on the poset:

$$f_n(x) = \sum_{d \preceq n} g(d, x), \qquad \text{equivalently} \qquad g(n, x) = f_n(x) - \sum_{\substack{d \prec n \\ d \preceq n}} g(d, x). \tag{1}$$

27.2 Predecessors of $2n$

Lemma 3. *Let $n > 1$. If $d \preceq 2n$ and $d > 1$, then $2 \mid d$, hence $d = 2e$ for some $e \in \mathbb{N}$. Moreover, $e \preceq n$.*

Proof. Write the sorted prime-factor list of n as $\pi(n) = (p_1, \dots, p_r)$, so

$$\pi(2n) = (2, p_1, \dots, p_r).$$

If $d > 1$ and $d \preceq 2n$, then the first prime in $\pi(d)$ is at most the first prime in $\pi(2n)$, hence it is ≤ 2. Since it is prime, it must be 2, so $2 \mid d$ and $d = 2e$.

Deleting the first entry 2 from $\pi(d)$ and $\pi(2n)$ preserves the componentwise inequalities, hence $\pi(e) \leq \pi(n)$ componentwise, i.e. $e \preceq n$. □

27.3 The 2-scaling identity

Theorem 4. *For every $n > 1$,*

$$g(2n, x) = x\, g(n, x).$$

Proof. Fix $n > 1$. By (1),

$$g(2n, x) = f_{2n}(x) - \sum_{\substack{d \prec 2n \\ d \preceq 2n}} g(d, x).$$

By Lemma 3, every predecessor $d \preceq 2n$ with $d > 1$ is of the form $d = 2e$ with $e \preceq n$. Thus the proper predecessors of $2n$ split as

$$\{1\} \cup \{2\} \cup \{\, 2e : \; e \prec n, \; e > 1 \,\}.$$

Hence

$$g(2n, x) = f_{2n}(x) - g(1, x) - g(2, x) - \sum_{\substack{e \prec n \\ e > 1}} g(2e, x). \tag{2}$$

Using multiplicativity and $f_2(x) = x$ we have

$$f_{2n}(x) = f_2(x) f_n(x) = x\, f_n(x).$$

Also $g(1, x) = f_1(x) = 1$, and from (1) applied to $n = 2$ we get

$$g(2, x) = f_2(x) - g(1, x) = x - 1.$$

Now we argue by induction along any linear extension of the finite poset on $\{1, \dots, 2n\}$. For each $e \prec n$ with $e > 1$, the element $2e$ precedes $2n$ and is strictly smaller in the poset, so we may assume inductively that

$$g(2e, x) = x\, g(e, x).$$

Substituting all identities into (2) yields

$$g(2n, x) = x f_n(x) - 1 - (x - 1) - x \sum_{\substack{e \prec n \\ e > 1}} g(e, x) = x f_n(x) - x - x \sum_{\substack{e \prec n \\ e > 1}} g(e, x).$$

Since $g(1, x) = 1$, we have

$$1 + \sum_{\substack{e \prec n \\ e > 1}} g(e, x) = \sum_{e \prec n} g(e, x),$$

and therefore

$$g(2n,x) = x\left(f_n(x) - \sum_{\substack{e\prec n\\ e\preceq n}} g(e,x)\right) = x\,g(n,x),$$

which proves the claim. □

27.4 Consequences

Corollary 7. *Let $m > 1$ be odd and $a \geq 0$. Then*

$$g(2^a m, x) = x^a\, g(m,x).$$

Proof. Apply Theorem 4 iteratively a times, starting from m:

$$g(2m,x) = xg(m,x), \quad g(4m,x) = xg(2m,x) = x^2 g(m,x), \; \ldots$$

□

Corollary 8. *For every $a \geq 1$,*

$$g(2^a, x) = x^{a-1}(x-1).$$

Proof. For $a = 1$, $g(2,x) = x-1$. For $a \geq 2$, apply Theorem 4 with $n = 2^{a-1} > 1$:

$$g(2^a,x) = g(2\cdot 2^{a-1},x) = x\,g(2^{a-1},x).$$

Iterating gives $g(2^a,x) = x^{a-1}g(2,x) = x^{a-1}(x-1)$. □

Remark 16. The identity $g(2n,x) = xg(n,x)$ fails for $n = 1$, since $g(2,x) = x - 1 \neq xg(1,x)$. This is the only exception and reflects the special role of the minimal element 1 in the poset.

28 A two-step recurrence for the prime-min characteristic polynomials

Let

$$M_n := \big(\min(p_i,p_j)\big)_{1\leq i,j\leq n},$$

where $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ are the prime numbers, and let

$$\chi_n(x) := \det(xI_n - M_n)$$

be the characteristic polynomial of M_n.

Theorem 5. *For every $n \geq 1$, the polynomials $\chi_n(x)$ satisfy the recurrence*

$$\chi_{n+1}(x) = \big(2x + p_n - p_{n+1}\big)\chi_n(x) - x^2\chi_{n-1}(x).$$

Equivalently,

$$\chi_{n+1}(x) = \big(2x - (p_{n+1} - p_n)\big)\chi_n(x) - x^2\chi_{n-1}(x).$$

The initial values are

$$\chi_0(x) = 1, \qquad \chi_1(x) = x - 2.$$

Proof. Write $e_n = (0, \dots, 0, 1)^T \in \mathbb{R}^n$ and

$$u := (p_1, \dots, p_n)^T.$$

Since $p_1 < \cdots < p_n$, the last column of M_n is exactly

$$(\min(p_1, p_n), \dots, \min(p_n, p_n))^T = (p_1, \dots, p_n)^T = u,$$

hence

$$u = M_n e_n.$$

Now decompose M_{n+1} as the block matrix

$$M_{n+1} = \begin{pmatrix} M_n & u \\ u^T & p_{n+1} \end{pmatrix}.$$

Therefore

$$xI_{n+1} - M_{n+1} = \begin{pmatrix} A & -u \\ -u^T & x - p_{n+1} \end{pmatrix}, \qquad A := xI_n - M_n.$$

Using the block determinant identity

$$\det \begin{pmatrix} A & B \\ C & d \end{pmatrix} = d \det(A) - C \operatorname{adj}(A) B,$$

we obtain

$$\chi_{n+1}(x) = (x - p_{n+1})\chi_n(x) - u^T \operatorname{adj}(A) u.$$

So it remains to compute the quadratic form

$$u^T \operatorname{adj}(A) u.$$

Since $u = M_n e_n = (xI_n - A)e_n$, we have

$$u = xe_n - Ae_n.$$

Hence

$$\begin{aligned} u^T \operatorname{adj}(A) u &= (xe_n - Ae_n)^T \operatorname{adj}(A)(xe_n - Ae_n) \\ &= x^2 e_n^T \operatorname{adj}(A) e_n - x e_n^T A \operatorname{adj}(A) e_n - x e_n^T \operatorname{adj}(A) A e_n \\ &\quad + e_n^T A \operatorname{adj}(A) A e_n. \end{aligned}$$

Because $A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A) I_n = \chi_n(x) I_n$, this becomes

$$u^T \operatorname{adj}(A) u = x^2 e_n^T \operatorname{adj}(A) e_n - 2x\, \chi_n(x) + \chi_n(x)\, e_n^T A e_n.$$

Now

$$e_n^T A e_n = e_n^T (xI_n - M_n) e_n = x - p_n,$$

so

$$u^T \operatorname{adj}(A) u = x^2 e_n^T \operatorname{adj}(A) e_n - (x + p_n)\chi_n(x).$$

Finally, $e_n^T \operatorname{adj}(A) e_n$ is the (n,n)-cofactor of A. Deleting the last row and column from $A = xI_n - M_n$ leaves

$$xI_{n-1} - M_{n-1},$$

so

$$e_n^T \operatorname{adj}(A) e_n = \det(xI_{n-1} - M_{n-1}) = \chi_{n-1}(x).$$

Therefore

$$u^T \operatorname{adj}(A) u = x^2 \chi_{n-1}(x) - (x + p_n)\chi_n(x).$$

Substituting into the earlier formula gives

$$\begin{aligned} \chi_{n+1}(x) &= (x - p_{n+1})\chi_n(x) - \Big[x^2 \chi_{n-1}(x) - (x + p_n)\chi_n(x)\Big] \\ &= \big(2x + p_n - p_{n+1}\big)\chi_n(x) - x^2 \chi_{n-1}(x), \end{aligned}$$

as claimed.

The initial values are immediate:

$$\chi_0(x) = 1$$

(by the empty determinant) and

$$\chi_1(x) = \det([x - 2]) = x - 2.$$

□

Remark 17. The recurrence may be written in terms of the prime gap

$$g_n := p_{n+1} - p_n$$

as

$$\chi_{n+1}(x) = (2x - g_n)\chi_n(x) - x^2 \chi_{n-1}(x).$$

Thus the passage from χ_n to χ_{n+1} depends only on χ_n, χ_{n-1}, and the next prime gap.

29 Growth of the largest eigenvalue

Let

$$M_n := \big(\min(p_i, p_j)\big)_{1 \le i,j \le n},$$

where $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ are the prime numbers, and let

$$\lambda_n := \lambda_{\max}(M_n)$$

denote the largest eigenvalue of M_n.

The numerical data strongly suggest that λ_n grows proportionally to

$$S_n := \sum_{k=1}^{n} p_k.$$

More precisely, one is naturally led to the following conjecture.

Conjecture 1. *As $n \to \infty$,*

$$\lambda_n \sim \frac{8}{\pi^2} \sum_{k=1}^{n} p_k.$$

Equivalently, using the asymptotics

$$\sum_{k=1}^{n} p_k \sim \frac{1}{2}n^2 \log n \qquad \text{and} \qquad p_n \sim n \log n,$$

this may be written as

$$\lambda_n \sim \frac{4}{\pi^2} n\, p_n \sim \frac{4}{\pi^2} n^2 \log n.$$

Although we do not prove the conjectural constant here, the correct order of growth can be established quite easily.

Theorem 6. *As $n \to \infty$,*

$$\lambda_n = \Theta(n^2 \log n).$$

More precisely,

$$\Big(\frac{1}{3} + o(1)\Big) n^2 \log n \le \lambda_n \le \Big(\frac{1}{2} + o(1)\Big) n^2 \log n.$$

Proof. Since M_n is real symmetric with nonnegative entries, its spectral radius is bounded above by the maximum row sum:

$$\lambda_n \le \max_{1 \le i \le n} \sum_{j=1}^{n} \min(p_i, p_j).$$

Because $p_1 < \cdots < p_n$, the row sums are increasing in i, and hence the maximum is attained for $i = n$. Therefore

$$\lambda_n \le \sum_{j=1}^{n} \min(p_n, p_j) = \sum_{j=1}^{n} p_j = S_n.$$

Using the standard asymptotic formula

$$S_n = \sum_{j=1}^{n} p_j \sim \frac{1}{2} n^2 \log n,$$

we obtain

$$\lambda_n \le \Big(\frac{1}{2} + o(1)\Big) n^2 \log n.$$

For the lower bound, apply the Rayleigh quotient with the vector

$$u := (1, \ldots, 1)^T \in \mathbb{R}^n.$$

Since $u^T u = n$, we have

$$\lambda_n \ge \frac{u^T M_n u}{u^T u} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \min(p_i, p_j).$$

Now for fixed k, the value p_k occurs in the double sum exactly

$$(2n - 2k + 1)$$

times, because

$$\min(p_i, p_j) = p_k$$

precisely when either $i = k \le j$ or $j = k \le i$, with the pair (k, k) counted only once. Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\min(p_i, p_j) = \sum_{k=1}^{n}(2n - 2k + 1)p_k,$$

and thus

$$\lambda_n \ge \frac{1}{n}\sum_{k=1}^{n}(2n - 2k + 1)p_k.$$

We now estimate the latter sum asymptotically. Since $p_k \sim k \log k$,

$$\sum_{k=1}^{n}(2n - 2k + 1)p_k \sim \sum_{k=1}^{n} 2(n - k)\, k \log k.$$

Writing $k = nx$, this is a Riemann-sum situation:

$$\sum_{k=1}^{n} 2(n - k)\, k \log k = n^3 \log n \left(\frac{1}{n}\sum_{k=1}^{n} 2\Big(1 - \frac{k}{n}\Big)\frac{k}{n}\right) + o(n^3 \log n).$$

Since

$$\int_0^1 2x(1 - x)\, dx = \frac{1}{3},$$

it follows that

$$\sum_{k=1}^{n}(2n - 2k + 1)p_k = \Big(\frac{1}{3} + o(1)\Big)n^3 \log n.$$

Dividing by n gives

$$\lambda_n \ge \Big(\frac{1}{3} + o(1)\Big)n^2 \log n.$$

Combining the upper and lower bounds proves the theorem. □

Remark 18. The theorem shows that the largest eigenvalue has the same order of growth as the trace,

$$\mathrm{tr}(M_n) = \sum_{k=1}^{n} p_k.$$

The numerical evidence indicates that the ratio

$$\frac{\lambda_n}{\sum_{k=1}^{n} p_k}$$

appears to converge to a constant close to $\frac{8}{\pi^2}$.

Remark 19 (Heuristic origin of the constant $\frac{8}{\pi^2}$). The constant $\frac{8}{\pi^2}$ is suggested by comparison with the classical matrix

$$A_n := (\min(i, j))_{1 \le i, j \le n}.$$

It is well known that

$$\lambda_{\max}(A_n) \sim \frac{4}{\pi^2}n^2.$$

Now for indices i, j of size comparable to n, the prime number theorem gives

$$p_i \sim i \log i \sim i \log n, \qquad p_j \sim j \log j \sim j \log n,$$

and therefore

$$\min(p_i, p_j) \sim (\log n) \min(i, j).$$

Thus, at a heuristic level, the matrix M_n should behave like

$$M_n \approx (\log n) A_n.$$

Consequently one expects

$$\lambda_n = \lambda_{\max}(M_n) \approx (\log n)\lambda_{\max}(A_n) \sim \frac{4}{\pi^2} n^2 \log n.$$

On the other hand,

$$\sum_{k=1}^{n} p_k \sim \frac{1}{2} n^2 \log n,$$

so the preceding heuristic is equivalent to

$$\lambda_n \sim \frac{8}{\pi^2} \sum_{k=1}^{n} p_k.$$

This explains the conjectural constant in the previous section.

Of course, this argument is only heuristic: replacing $\log i$ by the common factor $\log n$ ignores the nonuniform variation of the primes across the full range $1 \leq i \leq n$. Nevertheless, it gives a natural explanation for why the ratio

$$\frac{\lambda_n}{\sum_{k=1}^{n} p_k}$$

should tend to a constant, and why that constant should plausibly be $\frac{8}{\pi^2}$.

30 Convergence of the smallest eigenvalue

Let

$$M_n := \big(\min(p_i, p_j)\big)_{1 \leq i,j \leq n},$$

where $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ are the prime numbers, and let

$$\mu_n := \lambda_{\min}(M_n)$$

denote the smallest eigenvalue of M_n.

The numerical data suggest that μ_n converges rapidly to a positive limit. This can indeed be proved.

Theorem 7. *The sequence $(\mu_n)_{n \geq 1}$ is monotonically decreasing and converges to a positive limit:*

$$\mu_n \downarrow \mu_* \qquad \text{for some } \mu_* > 0.$$

More precisely,

$$\mu_n \geq \frac{1}{3} \qquad \text{for all } n \geq 1.$$

Hence

$$\mu_* := \lim_{n \to \infty} \mu_n$$

exists and satisfies

$$\mu_* \geq \frac{1}{3}.$$

Proof. We begin with a standard factorization of the matrix

$$M_n = \big(\min(p_i, p_j)\big)_{1\le i,j\le n}.$$

Set

$$t_k := p_k, \qquad t_0 := 0, \qquad d_k := t_k - t_{k-1} \quad (k \ge 1).$$

Thus

$$d_1 = 2, \qquad d_2 = 1, \qquad d_k = p_k - p_{k-1} \quad (k \ge 3).$$

Let $L_n = (\ell_{ij})_{1\le i,j\le n}$ be the lower triangular matrix with

$$\ell_{ij} = \begin{cases} 1, & j \le i, \\ 0, & j > i, \end{cases}$$

and let

$$D_n := \operatorname{diag}(d_1, \dots, d_n).$$

Then

$$M_n = L_n D_n L_n^T.$$

Indeed, the (i,j)-entry of $L_n D_n L_n^T$ is

$$\sum_{k=1}^{\min(i,j)} d_k = t_{\min(i,j)} = \min(t_i, t_j) = \min(p_i, p_j).$$

Since M_n is positive definite, it is invertible, and therefore

$$M_n^{-1} = L_n^{-T} D_n^{-1} L_n^{-1}.$$

Now L_n^{-1} is the first-difference matrix, so M_n^{-1} is tridiagonal. More explicitly, if we write

$$B_n := M_n^{-1},$$

then

$$B_n = \begin{pmatrix} \frac{1}{d_1} + \frac{1}{d_2} & -\frac{1}{d_2} & 0 & \cdots & 0 \\ -\frac{1}{d_2} & \frac{1}{d_2} + \frac{1}{d_3} & -\frac{1}{d_3} & \ddots & \vdots \\ 0 & -\frac{1}{d_3} & \frac{1}{d_3} + \frac{1}{d_4} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{1}{d_n} \\ 0 & \cdots & 0 & -\frac{1}{d_n} & \frac{1}{d_n} \end{pmatrix}.$$

Since inversion reverses eigenvalues,

$$\mu_n = \lambda_{\min}(M_n) = \frac{1}{\lambda_{\max}(B_n)}.$$

We first prove monotonicity. Since M_n is the leading principal submatrix of M_{n+1}, the Cauchy interlacing theorem implies

$$\lambda_{\min}(M_{n+1}) \le \lambda_{\min}(M_n).$$

Hence

$$\mu_{n+1} \le \mu_n \qquad (n \ge 1),$$

so (μ_n) is decreasing.

It remains to show that (μ_n) is bounded below by a positive constant. For $k \geq 3$, the prime gaps are even and at least 2, so

$$d_1 = 2, \qquad d_2 = 1, \qquad d_k \geq 2 \quad (k \geq 3).$$

Therefore

$$\frac{1}{d_1} = \frac{1}{2}, \qquad \frac{1}{d_2} = 1, \qquad \frac{1}{d_k} \leq \frac{1}{2} \quad (k \geq 3).$$

Now estimate the absolute row sums of B_n.

For the first row,

$$\left(\frac{1}{d_1} + \frac{1}{d_2}\right) + \frac{1}{d_2} = \frac{1}{2} + 1 + 1 = \frac{5}{2}.$$

For the second row,

$$\frac{1}{d_2} + \left(\frac{1}{d_2} + \frac{1}{d_3}\right) + \frac{1}{d_3} = 2 + \frac{2}{d_3} \leq 3.$$

For every interior row $3 \leq i \leq n-1$,

$$\frac{1}{d_i} + \left(\frac{1}{d_i} + \frac{1}{d_{i+1}}\right) + \frac{1}{d_{i+1}} = \frac{2}{d_i} + \frac{2}{d_{i+1}} \leq 2.$$

For the last row,

$$\frac{1}{d_n} + \frac{1}{d_n} = \frac{2}{d_n} \leq 1.$$

Hence

$$\|B_n\|_\infty \leq 3.$$

Since B_n is real symmetric,

$$\lambda_{\max}(B_n) \leq \|B_n\|_\infty \leq 3.$$

Consequently,

$$\mu_n = \frac{1}{\lambda_{\max}(B_n)} \geq \frac{1}{3}.$$

Thus (μ_n) is decreasing and bounded below by $1/3$, so it converges to some limit $\mu_* \geq 1/3$. ☐

Remark 20. The proof shows that the limiting behavior of the smallest eigenvalue is governed by the inverse tridiagonal matrices

$$B_n = M_n^{-1}.$$

If one defines the infinite Jacobi operator B on $\ell^2(\mathbb{N})$ by

$$(Bx)_1 = \left(\frac{1}{d_1} + \frac{1}{d_2}\right) x_1 - \frac{1}{d_2} x_2,$$

and, for $i \geq 2$,

$$(Bx)_i = -\frac{1}{d_i} x_{i-1} + \left(\frac{1}{d_i} + \frac{1}{d_{i+1}}\right) x_i - \frac{1}{d_{i+1}} x_{i+1},$$

then one expects

$$\lambda_{\max}(B_n) \uparrow \|B\|, \qquad \text{hence} \qquad \mu_n \downarrow \frac{1}{\|B\|}.$$

Numerically, this limit appears to be

$$\mu_* \approx 0.38612436838\ldots$$

for the prime-min matrices.

31 Convergence of the lower edge of the spectrum

Let

$$M_n := \big(\min(p_i, p_j)\big)_{1\le i,j\le n},$$

and write its eigenvalues in increasing order as

$$0 < \nu_1^{(n)} \le \nu_2^{(n)} \le \cdots \le \nu_n^{(n)}.$$

Thus

$$\nu_1^{(n)} = \lambda_{\min}(M_n), \qquad \nu_n^{(n)} = \lambda_{\max}(M_n).$$

The numerical data indicate that not only the smallest eigenvalue, but in fact every eigenvalue at fixed distance from the lower spectral edge appears to converge. This is in fact an immediate consequence of interlacing.

Theorem 8. *For every fixed integer $k \ge 1$, the sequence*

$$\big(\nu_k^{(n)}\big)_{n\ge k}$$

is monotonically decreasing and converges to a positive limit. More precisely, for each fixed k there exists a real number $\ell_k > 0$ such that

$$\nu_k^{(n)} \downarrow \ell_k \qquad (n \to \infty).$$

Moreover,

$$\ell_1 \le \ell_2 \le \ell_3 \le \cdots$$

and

$$\ell_k \ge \frac{1}{3} \qquad \text{for all } k \ge 1.$$

Proof. The matrix M_n is a leading principal submatrix of M_{n+1}. Hence, by the Cauchy interlacing theorem, if

$$0 < \nu_1^{(n+1)} \le \cdots \le \nu_{n+1}^{(n+1)}$$

are the eigenvalues of M_{n+1} in increasing order, then

$$\nu_j^{(n+1)} \le \nu_j^{(n)} \le \nu_{j+1}^{(n+1)} \qquad (1 \le j \le n).$$

Fix $k \ge 1$. For every $n \ge k$, taking $j = k$ gives

$$\nu_k^{(n+1)} \le \nu_k^{(n)}.$$

Thus the sequence $\big(\nu_k^{(n)}\big)_{n\ge k}$ is monotonically decreasing.

It remains to show that it is bounded below by a positive constant. In the previous section we proved that

$$\nu_1^{(n)} = \lambda_{\min}(M_n) \ge \frac{1}{3} \qquad \text{for all } n \ge 1.$$

Since

$$\nu_k^{(n)} \ge \nu_1^{(n)},$$

it follows that

$$\nu_k^{(n)} \ge \frac{1}{3} \qquad \text{for all } n \ge k.$$

Therefore $\left(\nu_k^{(n)}\right)$ is decreasing and bounded below, hence it converges. Denoting its limit by ℓ_k, we obtain

$$\nu_k^{(n)} \downarrow \ell_k \qquad \text{with} \qquad \ell_k \geq \frac{1}{3}.$$

Finally, for every fixed n one has

$$\nu_k^{(n)} \leq \nu_{k+1}^{(n)}.$$

Passing to the limit as $n \to \infty$ yields

$$\ell_k \leq \ell_{k+1}.$$

This proves the theorem. □

Remark 21. The theorem shows that the entire lower spectral edge stabilizes: for each fixed k, the k-th smallest eigenvalue tends to a finite positive limit. Thus the lower edge of the spectrum is described by an increasing sequence

$$\ell_1 \leq \ell_2 \leq \ell_3 \leq \cdots,$$

where

$$\ell_k = \lim_{n\to\infty} \nu_k^{(n)}.$$

Numerically, the first few limits appear to be

$$\ell_1 \approx 0.386124368380\ldots, \qquad \ell_2 \approx 0.673488179251\ldots, \qquad \ell_3 \approx 0.707391684289\ldots,$$

$$\ell_4 \approx 0.730981710857\ldots, \qquad \ell_5 \approx 0.745401399572\ldots, \qquad \ell_6 \approx 0.754948011628\ldots.$$

The data strongly suggest that these limits form a genuine lower-edge spectral sequence associated with the prime-min matrices.

Remark 22. It is natural to expect that the numbers ℓ_k are related to the largest eigenvalues of the infinite Jacobi operator associated with M_n^{-1}. Indeed, if

$$B_n = M_n^{-1},$$

then the smallest eigenvalues of M_n are the reciprocals of the largest eigenvalues of B_n. The theorem above proves the existence of the lower-edge limits purely by interlacing, without requiring any spectral theory of the infinite operator.

32 Numerical approximations to the lower-edge limits and a reconstruction remark

Write the eigenvalues of

$$M_n = \left(\min(p_i, p_j)\right)_{1\leq i,j\leq n}$$

in increasing order as

$$0 < \nu_1^{(n)} \leq \nu_2^{(n)} \leq \cdots \leq \nu_n^{(n)},$$

and recall that for each fixed k the limit

$$\ell_k := \lim_{n\to\infty} \nu_k^{(n)}$$

exists.

Using the tridiagonal inverse

$$B_n = M_n^{-1},$$

and computing the largest eigenvalues of B_n, we obtain numerical approximations to the first lower-edge limits ℓ_k. The table below records the values obtained from the truncation $n = 800$, which are already stable to many digits for the first several indices.

k	ℓ_k (numerical approximation)
1	0.386124368380148
2	0.673488179251410
3	0.707391684289320
4	0.720775600416776
5	0.721225895335886
6	0.721229037141880
7	0.724551640779046
8	0.730981710856922
9	0.744170045108461
10	0.745401399572153
11	0.747178432179847
12	0.752444203991323
13	0.754948011627706
14	0.756566781137518
15	0.756819404351606
16	0.757351597764483
17	0.757945387016703
18	0.758154107818086
19	0.759104884786936
20	0.759365032569991

Table 1: Approximations to the lower-edge limits from the truncation $n = 800$.

To assess stability, one may compare the truncations $n = 800$ and $n = 1600$. For the first eleven indices, the agreement is essentially at machine precision, whereas for larger k visible truncation effects remain. In particular, the first eleven values may already be regarded as highly reliable:

$$\ell_1 \approx 0.386124368380148, \qquad \ell_2 \approx 0.673488179251410, \qquad \ell_3 \approx 0.707391684289320,$$

$$\ell_4 \approx 0.720775600416776, \qquad \ell_5 \approx 0.721225895335886, \qquad \ell_6 \approx 0.721229037141880,$$

$$\ell_7 \approx 0.724551640779046, \qquad \ell_8 \approx 0.730981710856922, \qquad \ell_9 \approx 0.744170045108461,$$

$$\ell_{10} \approx 0.745401399572153, \qquad \ell_{11} \approx 0.747178432179847.$$

Remark 23 (What spectral data determine the prime sequence?). There is a sharp distinction between the full family of finite spectra

$$\sigma(B_n) \qquad (n = 1, 2, 3, \dots)$$

and the limiting lower-edge sequence

$$(\ell_k)_{k\geq 1}.$$

(1) The spectra of all finite matrices B_n determine the prime-gap sequence, hence the primes.

Indeed, with

$$d_1 = 2, \qquad d_2 = 1, \qquad d_j = p_j - p_{j-1} \quad (j \geq 3),$$

we have

$$M_n = L_n \operatorname{diag}(d_1, \dots, d_n) L_n^T,$$

and therefore

$$\det(M_n) = \prod_{j=1}^{n} d_j.$$

Hence

$$\det(B_n) = \det(M_n)^{-1} = \prod_{j=1}^{n} d_j^{-1}.$$

Now $\det(B_n)$ is the product of the eigenvalues of B_n, so the spectrum of B_n determines $\det(B_n)$. Consequently,

$$d_n = \frac{\det(M_n)}{\det(M_{n-1})} = \frac{\det(B_{n-1})}{\det(B_n)}.$$

Thus the knowledge of the spectra of all finite matrices B_n determines every d_n, and therefore reconstructs the prime sequence recursively via

$$p_1 = 2, \qquad p_n = 2 + \sum_{j=2}^{n} d_j.$$

(2) By contrast, the limiting sequence (ℓ_k) alone should not suffice to reconstruct the prime sequence.

The values ℓ_k describe only the lower spectral edge of the matrices M_n, equivalently the upper discrete spectral edge of the tridiagonal operators B_n. They are therefore only a small part of the full spectral data. In inverse spectral theory for Jacobi operators, one generally needs more than just the eigenvalues: one typically also needs norming constants, a Weyl function, or an entire spectral measure. In particular, one should not expect the sequence (ℓ_k) by itself to determine all prime gaps d_j.

So the correct slogan is:

$$\boxed{\text{all finite spectra } \sigma(B_n) \text{ determine the primes, but } (\ell_k)_{k\geq 1} \text{ alone almost certainly do not.}}$$

33 The Weyl function, spectral measure, continued fraction, and recovery of the primes

We now describe the inverse-spectral object naturally attached to the prime-min matrices. The key point is that the inverse matrix is tridiagonal, hence defines a Jacobi operator.

33.1 The Jacobi operator attached to the prime gaps

Let

$$d_1 := 2, \qquad d_2 := 1, \qquad d_n := p_n - p_{n-1} \quad (n \ge 3),$$

so that

$$p_1 = 2, \qquad p_n = 2 + \sum_{j=2}^{n} d_j.$$

Recall that

$$M_n = \big(\min(p_i, p_j)\big)_{1 \le i,j \le n}$$

admits the factorization

$$M_n = L_n D_n L_n^T, \qquad D_n = \operatorname{diag}(d_1, \dots, d_n),$$

and therefore

$$B_n := M_n^{-1} = L_n^{-T} D_n^{-1} L_n^{-1}.$$

Explicitly,

$$B_n = \begin{pmatrix} \frac{1}{d_1} + \frac{1}{d_2} & -\frac{1}{d_2} & 0 & \cdots & 0 \\ -\frac{1}{d_2} & \frac{1}{d_2} + \frac{1}{d_3} & -\frac{1}{d_3} & \ddots & \vdots \\ 0 & -\frac{1}{d_3} & \frac{1}{d_3} + \frac{1}{d_4} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{1}{d_n} \\ 0 & \cdots & 0 & -\frac{1}{d_n} & \frac{1}{d_n} \end{pmatrix}.$$

This suggests introducing the infinite tridiagonal operator B on $\ell^2(\mathbb{N})$ by

$$(Be_1) = b_1 e_1 + a_1 e_2,$$

$$(Be_n) = a_{n-1} e_{n-1} + b_n e_n + a_n e_{n+1} \qquad (n \ge 2),$$

where

$$b_1 = \frac{1}{d_1} + \frac{1}{d_2} = \frac{3}{2}, \qquad b_n = \frac{1}{d_n} + \frac{1}{d_{n+1}} \quad (n \ge 2),$$

and

$$a_n = -\frac{1}{d_{n+1}} \qquad (n \ge 1).$$

Equivalently, for $x = (x_1, x_2, \dots) \in \ell^2(\mathbb{N})$,

$$(Bx)_1 = \Big(\frac{1}{d_1} + \frac{1}{d_2}\Big) x_1 - \frac{1}{d_2} x_2,$$

$$(Bx)_n = -\frac{1}{d_n} x_{n-1} + \Big(\frac{1}{d_n} + \frac{1}{d_{n+1}}\Big) x_n - \frac{1}{d_{n+1}} x_{n+1}, \qquad n \ge 2.$$

Proposition 36. *The operator B is a bounded self-adjoint operator on $\ell^2(\mathbb{N})$.*

Proof. The matrix of B is real symmetric, so it is formally self-adjoint. It remains to check boundedness. Since $d_1 = 2$, $d_2 = 1$, and $d_n \ge 2$ for all $n \ge 3$, we have

$$\frac{1}{d_1} = \frac{1}{2}, \qquad \frac{1}{d_2} = 1, \qquad \frac{1}{d_n} \le \frac{1}{2} \quad (n \ge 3).$$

Hence the absolute row sums satisfy

$$|b_1| + |a_1| = \frac{1}{2} + 1 + 1 = \frac{5}{2},$$

$$|a_1| + |b_2| + |a_2| = 1 + \left(1 + \frac{1}{d_3}\right) + \frac{1}{d_3} \le 3,$$

and for $n \ge 3$,

$$|a_{n-1}| + |b_n| + |a_n| = \frac{1}{d_n} + \left(\frac{1}{d_n} + \frac{1}{d_{n+1}}\right) + \frac{1}{d_{n+1}} \le 2.$$

Thus the matrix defines a bounded operator on $\ell^2(\mathbb{N})$, with

$$\|B\| \le 3.$$

Since B is bounded and symmetric, it is self-adjoint. □

33.2 The spectral measure and the Weyl function

Let $e_1 = (1, 0, 0, \dots) \in \ell^2(\mathbb{N})$. Since B is bounded and self-adjoint, the spectral theorem yields a unique finite positive Borel measure μ on $\mathbb{R}$ such that

$$\langle e_1, f(B)e_1 \rangle = \int_{\mathbb{R}} f(t)\, d\mu(t)$$

for every bounded Borel function f. In particular,

$$\mu(\mathbb{R}) = \langle e_1, e_1 \rangle = 1,$$

so μ is a probability measure.

Definition 2. *The measure μ is called the spectral measure of B associated with the vector e_1. Its Stieltjes transform*

$$m(z) := \int_{\mathbb{R}} \frac{d\mu(t)}{t - z}, \qquad z \in \mathbb{C} \setminus \sigma(B),$$

is called the Weyl function (or Weyl m-function) of B.

By the spectral theorem,

$$\boxed{m(z) = \langle e_1, (B - z)^{-1} e_1 \rangle, \qquad z \in \mathbb{C} \setminus \sigma(B).}$$

Thus the Weyl function is simply the $(1, 1)$-entry of the resolvent.

33.3 The continued fraction representation

Let $B^{(n)}$ denote the tail Jacobi operator acting on $\ell^2(\{n, n+1, n+2, \dots\})$, that is, the operator with coefficients

$$b_n, b_{n+1}, b_{n+2}, \dots \qquad \text{and} \qquad a_n, a_{n+1}, a_{n+2}, \dots$$

and let

$$m_n(z) := \langle e_n, (B^{(n)} - z)^{-1} e_n \rangle.$$

Then $m_1(z) = m(z)$.

Proposition 37. *For every $n \geq 1$ and every $z \in \mathbb{C} \setminus \sigma(B^{(n)})$,*

$$m_n(z) = \frac{1}{b_n - z - a_n^2\, m_{n+1}(z)}.$$

Consequently,

$$m(z) = \cfrac{1}{b_1 - z - \cfrac{a_1^2}{b_2 - z - \cfrac{a_2^2}{b_3 - z - \cfrac{a_3^2}{\ddots}}}}.$$

Proof. Write $B^{(n)} - z$ in block form as

$$B^{(n)} - z = \begin{pmatrix} b_n - z & a_n\langle e_{n+1}, \cdot\rangle \\ a_n e_{n+1} & B^{(n+1)} - z \end{pmatrix}.$$

The $(1,1)$-entry of the inverse is given by the Schur complement formula:

$$m_n(z) = \Big(b_n - z - a_n^2\langle e_{n+1}, (B^{(n+1)} - z)^{-1} e_{n+1}\rangle\Big)^{-1}.$$

That is,

$$m_n(z) = \frac{1}{b_n - z - a_n^2 m_{n+1}(z)}.$$

Iterating this identity yields the continued fraction. □

In our specific case,

$$a_n^2 = \frac{1}{d_{n+1}^2},$$

so the continued fraction begins as

$$m(z) = \cfrac{1}{\frac{3}{2} - z - \cfrac{1}{1 + \frac{1}{d_3} - z - \cfrac{1/d_3^2}{\frac{1}{d_3} + \frac{1}{d_4} - z - \cfrac{1/d_4^2}{\ddots}}}}.$$

33.4 Recovery of the prime gaps from full spectral data

There are two natural inverse problems here:

- recover the prime gaps from the full family of finite spectra $\sigma(B_n)$;
- recover the prime gaps from the Weyl function $m(z)$, equivalently from the spectral measure μ.

The first one is elementary and completely explicit.

Proposition 38. *The spectra of all finite matrices B_n determine the full prime-gap sequence $(d_n)_{n\geq 1}$, and hence determine the prime sequence $(p_n)_{n\geq 1}$.*

Proof. Since $B_n = M_n^{-1}$,

$$\det(B_n) = \det(M_n)^{-1}.$$

On the other hand, from

$$M_n = L_n D_n L_n^T$$

we obtain

$$\det(M_n) = \det(D_n) = \prod_{j=1}^{n} d_j.$$

Therefore

$$\det(B_n) = \prod_{j=1}^{n} d_j^{-1}.$$

Now $\det(B_n)$ is the product of the eigenvalues of B_n, so the spectrum of B_n determines $\det(B_n)$. Hence

$$d_n = \frac{\prod_{j=1}^{n} d_j}{\prod_{j=1}^{n-1} d_j} = \frac{\det(M_n)}{\det(M_{n-1})} = \frac{\det(B_{n-1})}{\det(B_n)}.$$

Thus the spectra $\sigma(B_n)$ determine all d_n recursively.

Finally,

$$p_1 = 2, \qquad p_n = 2 + \sum_{j=2}^{n} d_j,$$

so the primes are recovered as well. □

33.5 Recovery from the Weyl function

Now suppose instead that we know the Weyl function $m(z)$, or equivalently the spectral measure μ. Since

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z},$$

the moments

$$s_k := \int_{\mathbb{R}} t^k \, d\mu(t) = \langle e_1, B^k e_1 \rangle \qquad (k \geq 0)$$

are determined by μ, at least in principle. From the moments one may recover the Jacobi coefficients by the standard orthogonal-polynomial construction: the measure μ determines the orthonormal polynomial system (P_n), and the three-term recurrence

$$tP_n(t) = a_n P_{n+1}(t) + b_n P_n(t) + a_{n-1} P_{n-1}(t)$$

recovers the coefficients a_n, b_n.

In our present model this immediately yields the prime gaps. Indeed,

$$a_n = -\frac{1}{d_{n+1}},$$

so

$$\boxed{d_{n+1} = \frac{1}{|a_n|}.}$$

Once the gaps are known, the primes follow from

$$p_1 = 2, \qquad p_n = 2 + \sum_{j=2}^{n} d_j.$$

Thus, at least in principle, the Weyl function $m(z)$ or the spectral measure μ determines the prime sequence.

Remark 24. This should be contrasted with the lower-edge limit sequence

$$(\ell_k)_{k\geq 1}.$$

The values ℓ_k encode only a small part of the spectral information, namely the lower spectral edge of the matrices M_n or, equivalently, the upper discrete spectral edge of the Jacobi operators B_n. By themselves they are far too sparse to reconstruct all prime gaps. By contrast, the full Weyl function $m(z)$ encodes the complete spectral measure and is the correct inverse-spectral object.

34 A more explicit description of the spectral measure

Let B be the bounded self-adjoint Jacobi operator associated with the prime gaps,

$$(Be_1) = b_1 e_1 + a_1 e_2, \qquad (Be_n) = a_{n-1} e_{n-1} + b_n e_n + a_n e_{n+1} \quad (n \geq 2),$$

where

$$b_1 = \frac{1}{d_1} + \frac{1}{d_2} = \frac{3}{2}, \qquad b_n = \frac{1}{d_n} + \frac{1}{d_{n+1}} \quad (n \geq 2),$$

and

$$a_n = -\frac{1}{d_{n+1}}, \qquad d_1 = 2,\ d_2 = 1,\ d_n = p_n - p_{n-1}\ (n \geq 3).$$

In the previous section we introduced the spectral measure μ of B associated with the cyclic vector e_1, defined by

$$\mu(\Delta) = \langle e_1, E_B(\Delta) e_1 \rangle$$

for every Borel set $\Delta \subset \mathbb{R}$, where E_B denotes the spectral resolution of B. We now make this measure more explicit.

34.1 The measure as a weak limit of finite spectral measures

For each $n \geq 1$, let

$$B_n = M_n^{-1}$$

be the finite $n \times n$ Jacobi matrix, and let

$$\mu_1^{(n)}, \ldots, \mu_n^{(n)}$$

be its eigenvalues. Choose an orthonormal eigenbasis

$$u^{(n,1)}, \ldots, u^{(n,n)},$$

so that

$$B_n u^{(n,j)} = \mu_j^{(n)} u^{(n,j)}.$$

Then the spectral measure of B_n associated with the first basis vector $e_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^n$ is

$$\mu_n = \sum_{j=1}^{n} \left|u_1^{(n,j)}\right|^2 \delta_{\mu_j^{(n)}}.$$

Thus μ_n is a purely atomic probability measure: each eigenvalue $\mu_j^{(n)}$ carries the weight given by the square of the first component of the corresponding normalized eigenvector.

Proposition 39. *The measures μ_n converge weakly to the spectral measure μ of B. Equivalently, for every polynomial P,*

$$\int_{\mathbb{R}} P(t)\, d\mu_n(t) \longrightarrow \int_{\mathbb{R}} P(t)\, d\mu(t).$$

Proof. For every polynomial P,

$$\int_{\mathbb{R}} P(t)\, d\mu_n(t) = \langle e_1, P(B_n) e_1 \rangle,$$

and

$$\int_{\mathbb{R}} P(t)\, d\mu(t) = \langle e_1, P(B) e_1 \rangle.$$

Now B_n is the leading principal truncation of B, so for every fixed polynomial P, the vector $P(B_n)e_1$ agrees with $P(B)e_1$ once n is large enough, because only finitely many coefficients of B are involved. Hence

$$\langle e_1, P(B_n) e_1 \rangle \to \langle e_1, P(B) e_1 \rangle.$$

This proves convergence of all moments, and therefore weak convergence of the measures. □

So one concrete way to think about μ is

$$\boxed{\mu = \text{weak-}\lim_{n\to\infty} \sum_{j=1}^{n} \left|u_1^{(n,j)}\right|^2 \delta_{\mu_j^{(n)}}.}$$

This is the most useful numerical description of the measure.

34.2 The measure through its moments

The moments of μ are

$$s_k := \int_{\mathbb{R}} t^k\, d\mu(t) = \langle e_1, B^k e_1 \rangle \qquad (k \geq 0).$$

These moments determine the measure μ, since B is bounded and hence μ is compactly supported.

The first moments are easily computed:

$$s_0 = 1,$$

$$s_1 = \langle e_1, B e_1 \rangle = b_1 = \frac{3}{2},$$

and

$$s_2 = \langle e_1, B^2 e_1 \rangle = b_1^2 + a_1^2 = \left(\frac{3}{2}\right)^2 + 1 = \frac{13}{4},$$

because $a_1 = -1/d_2 = -1$.

Higher moments involve the successive prime gaps $d_3, d_4, \dots$. Thus the measure μ is equivalently characterized as the unique compactly supported probability measure whose moments are

$$s_k = \langle e_1, B^k e_1 \rangle.$$

34.3 The measure via the Weyl function

Recall that the Weyl function is

$$m(z) = \langle e_1, (B - z)^{-1} e_1 \rangle, \qquad z \in \mathbb{C} \setminus \sigma(B).$$

By the spectral theorem,

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z}.$$

Thus m is the Stieltjes transform of μ.

This gives another explicit description of the measure: once m is known, the measure can be recovered by Stieltjes inversion. In particular, if μ has an absolutely continuous part

$$d\mu_{\text{ac}}(x) = \rho(x)\, dx,$$

then formally

$$\boxed{\rho(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im m(x + i\varepsilon)}$$

at almost every point x.

If m has a simple pole at an eigenvalue λ, then μ has an atom there, and its mass is

$$\boxed{\mu(\{\lambda\}) = -\operatorname*{Res}_{z=\lambda} m(z).}$$

Thus μ may be decomposed into

$$\mu = \mu_{\text{pp}} + \mu_{\text{ac}} + \mu_{\text{sc}},$$

where the pure-point part corresponds to poles of m, the absolutely continuous part to boundary imaginary parts of m, and the singular continuous part to the remaining singular contribution.

34.4 The measure and orthogonal polynomials

A further explicit description is obtained through orthogonal polynomials. Because μ is compactly supported, there exists a unique orthonormal family of polynomials

$$P_0, P_1, P_2, \ldots$$

in $L^2(\mu)$, with $\deg P_n = n$, satisfying the three-term recurrence

$$x P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x),$$

with the Jacobi coefficients

$$a_n = -\frac{1}{d_{n+1}}, \qquad b_1 = \frac{3}{2}, \qquad b_n = \frac{1}{d_n} + \frac{1}{d_{n+1}} \quad (n \geq 2).$$

So μ is precisely the orthogonality measure of the polynomial family generated by the prime-gap recurrence coefficients.

34.5 Discrete masses and the lower spectral edge

Suppose $\beta_1 > \beta_2 > \cdots$ are isolated eigenvalues of B above the rest of the spectrum, with normalized eigenvectors $\psi_1, \psi_2, \ldots$. Then μ has atoms at these points:

$$\mu(\{\beta_k\}) = \left|\langle e_1, \psi_k\rangle\right|^2.$$

Since the lower-edge limits of the prime-min matrices satisfy

$$\ell_k = \frac{1}{\beta_k}$$

whenever β_k arises as a limiting largest eigenvalue of B_n, the sequence (ℓ_k) corresponds to a discrete upper edge of the spectrum of B, or equivalently to a discrete atomic component of μ near the top of its support.

34.6 Summary

Although there is no known closed elementary formula for μ, the measure is quite explicit in the following equivalent senses:

$$\boxed{\mu(\Delta) = \langle e_1, E_B(\Delta)e_1\rangle}$$

(spectral-theoretic definition),

$$\boxed{\mu = \text{weak}-\lim_{n\to\infty} \sum_{j=1}^{n} \left|u_1^{(n,j)}\right|^2 \delta_{\mu_j^{(n)}}}$$

(approximation by finite spectral measures),

$$\boxed{m(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}}$$

(Stieltjes transform / Weyl function),
and

$$\boxed{\mu \text{ is the orthogonality measure for the Jacobi coefficients } (a_n, b_n).}$$

In this sense, the spectral measure of the prime-gap Jacobi operator is not elementary, but it is nevertheless completely canonical and numerically accessible.

35 Properties of the Weyl Function

In this section we collect the basic analytic properties of the Weyl function associated with the prime-gap Jacobi operator. Every statement is proved in full detail.

35.1 The Jacobi operator and its Weyl function

Let $(d_n)_{n\geq 1}$ be the sequence of gaps

$$d_1 = 2, \qquad d_2 = 1, \qquad d_n = p_n - p_{n-1} \quad (n \geq 3),$$

where p_n denotes the n-th prime. Define the Jacobi coefficients by

$$a_n := -\frac{1}{d_{n+1}} \qquad (n \geq 1),$$

and

$$b_1 := \frac{3}{2}, \qquad b_n := \frac{1}{d_n} + \frac{1}{d_{n+1}} \qquad (n \geq 2).$$

For each $N \geq 1$, let B_N be the real symmetric tridiagonal matrix

$$B_N = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & 0 & a_{N-1} & b_N \end{pmatrix}.$$

Let $e_1 = (1, 0, \dots, 0)^T \in \mathbb{C}^N$. The finite Weyl function is

$$m_N(z) := \langle e_1, (B_N - zI)^{-1} e_1 \rangle, \qquad z \in \mathbb{C} \setminus \sigma(B_N).$$

When an infinite self-adjoint Jacobi operator B is available, its Weyl function is

$$m(z) := \langle e_1, (B - zI)^{-1} e_1 \rangle, \qquad z \in \mathbb{C} \setminus \sigma(B).$$

The finite-dimensional statements below are completely rigorous as stated. The corresponding infinite-dimensional versions hold whenever B is self-adjoint and z lies in the resolvent set.

35.2 Continued fraction and Riccati recursion

We first record the basic recursive structure.

Proposition 40. *For every $N \geq 2$, the finite Weyl function $m_N(z)$ admits the continued fraction expansion*

$$m_N(z) = \cfrac{1}{b_1 - z - \cfrac{a_1^2}{b_2 - z - \cfrac{a_2^2}{b_3 - z - \ddots - \cfrac{a_{N-1}^2}{b_N - z}}}}.$$

Proof. Write the matrix $B_N - zI$ in block form:

$$B_N - zI = \begin{pmatrix} b_1 - z & a_1 e_1^{(N-1)*} \\ a_1 e_1^{(N-1)} & C_{N-1}(z) \end{pmatrix},$$

where $e_1^{(N-1)} = (1, 0, \dots, 0)^T \in \mathbb{C}^{N-1}$, and $C_{N-1}(z)$ is the lower-right $(N-1) \times (N-1)$ block

$$C_{N-1}(z) = \begin{pmatrix} b_2 - z & a_2 & 0 & \cdots & 0 \\ a_2 & b_3 - z & a_3 & \ddots & \vdots \\ 0 & a_3 & b_4 - z & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & 0 & a_{N-1} & b_N - z \end{pmatrix}.$$

We now compute the $(1,1)$-entry of $(B_N - zI)^{-1}$. Let

$$M = \begin{pmatrix} \alpha & \beta^* \\ \beta & C \end{pmatrix}$$

be an invertible block matrix with scalar upper-left block α. Suppose

$$M^{-1} = \begin{pmatrix} x & y^* \\ y & Z \end{pmatrix}.$$

Multiplying the matrices and looking at the upper-left block gives

$$\alpha x + \beta^* y = 1.$$

Looking at the lower-left block gives

$$\beta x + C y = 0.$$

Since C is invertible whenever M is invertible and the Schur complement is defined, we obtain

$$y = -C^{-1}\beta x.$$

Substituting into the first equation yields

$$\alpha x - \beta^* C^{-1} \beta x = 1,$$

so

$$x = \frac{1}{\alpha - \beta^* C^{-1} \beta}.$$

Applying this with

$$\alpha = b_1 - z, \qquad \beta = a_1 e_1^{(N-1)}, \qquad C = C_{N-1}(z),$$

we conclude that the $(1,1)$-entry of $(B_N - zI)^{-1}$ equals

$$\frac{1}{b_1 - z - a_1^2 \langle e_1^{(N-1)}, C_{N-1}(z)^{-1} e_1^{(N-1)} \rangle}.$$

But the scalar

$$\langle e_1^{(N-1)}, C_{N-1}(z)^{-1} e_1^{(N-1)} \rangle$$

is exactly the Weyl function of the tail Jacobi matrix beginning at index 2. Repeating the same argument on that tail produces the next level of the fraction, and iterating all the way to the last diagonal entry gives

$$m_N(z) = \cfrac{1}{b_1 - z - \cfrac{a_1^2}{b_2 - z - \cfrac{a_2^2}{b_3 - z - \ddots - \cfrac{a_{N-1}^2}{b_N - z}}}}.$$

This proves the claimed formula. □

To state the recursive law cleanly, define for each $n \geq 1$ the tail Jacobi operator

$$B^{(n)} = \begin{pmatrix} b_n & a_n & 0 & \cdots \\ a_n & b_{n+1} & a_{n+1} & \ddots \\ 0 & a_{n+1} & b_{n+2} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

and its Weyl function

$$m_n(z) := \langle e_1, (B^{(n)} - zI)^{-1} e_1 \rangle.$$

Proposition 41. *For every $n \geq 1$,*

$$m_n(z) = \frac{1}{b_n - z - a_n^2\, m_{n+1}(z)}.$$

Equivalently,

$$-\frac{1}{m_n(z)} = z - b_n + a_n^2\, m_{n+1}(z).$$

Proof. Apply the previous proposition to the tail Jacobi operator $B^{(n)}$. The first step of the continued fraction gives

$$m_n(z) = \cfrac{1}{b_n - z - \cfrac{a_n^2}{b_{n+1} - z - \cdots}}.$$

By definition, the tail beginning after the first step is precisely $m_{n+1}(z)$. Therefore

$$m_n(z) = \frac{1}{b_n - z - a_n^2 m_{n+1}(z)}.$$

Since the denominator is nonzero whenever z belongs to the resolvent set, we may multiply both sides by it and obtain

$$1 = m_n(z)\big(b_n - z - a_n^2 m_{n+1}(z)\big).$$

Dividing by $m_n(z)$ and rearranging terms yields

$$\frac{1}{m_n(z)} = b_n - z - a_n^2 m_{n+1}(z),$$

hence

$$-\frac{1}{m_n(z)} = z - b_n + a_n^2 m_{n+1}(z).$$

This proves both identities. □

35.3 Spectral representation and positivity

The next result identifies the Weyl function as a Stieltjes transform of a positive measure.

Proposition 42. *Let B_N be as above. Then there exist real numbers $\lambda_1, \ldots, \lambda_N$ and nonnegative weights $w_1, \ldots, w_N$ with*

$$\sum_{j=1}^{N} w_j = 1$$

such that

$$m_N(z) = \sum_{j=1}^{N} \frac{w_j}{\lambda_j - z}.$$

More precisely, $\lambda_1, \ldots, \lambda_N$ *are the eigenvalues of* B_N*, and*

$$w_j = |u_j(1)|^2,$$

where u_j *is an orthonormal eigenvector corresponding to* λ_j*, and* $u_j(1)$ *denotes its first coordinate.*

Proof. Since B_N is a real symmetric matrix, the spectral theorem provides an orthogonal matrix U and a diagonal matrix

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$$

such that

$$B_N = U\Lambda U^T.$$

Therefore

$$B_N - zI = U(\Lambda - zI)U^T,$$

and hence

$$(B_N - zI)^{-1} = U(\Lambda - zI)^{-1}U^T.$$

Now compute:

$$m_N(z) = \langle e_1, U(\Lambda - zI)^{-1}U^T e_1\rangle.$$

Let

$$c := U^T e_1.$$

Then $c_j = \langle u_j, e_1\rangle = u_j(1)$, so

$$m_N(z) = \sum_{j=1}^{N} \frac{|c_j|^2}{\lambda_j - z}.$$

Set

$$w_j := |c_j|^2 = |u_j(1)|^2.$$

Each w_j is nonnegative. Also,

$$\sum_{j=1}^{N} w_j = \sum_{j=1}^{N} |c_j|^2 = \|c\|^2 = \|U^T e_1\|^2 = \|e_1\|^2 = 1.$$

Thus

$$m_N(z) = \sum_{j=1}^{N} \frac{w_j}{\lambda_j - z}.$$

This proves the claim. □

Proposition 43. *For every* $N \geq 1$ *and every* $z \in \mathbb{C}$ *with* $\Im z > 0$*,*

$$\Im m_N(z) > 0.$$

Hence m_N *is a Herglotz function.*

Proof. Write $z = x + iy$ with $y > 0$. By the preceding proposition,

$$m_N(z) = \sum_{j=1}^{N} \frac{w_j}{\lambda_j - (x + iy)} = \sum_{j=1}^{N} \frac{w_j}{(\lambda_j - x) - iy}.$$

For each j,

$$\frac{1}{(\lambda_j - x) - iy} = \frac{(\lambda_j - x) + iy}{(\lambda_j - x)^2 + y^2}.$$

Therefore

$$\Im \frac{1}{(\lambda_j - x) - iy} = \frac{y}{(\lambda_j - x)^2 + y^2} > 0.$$

Multiplying by $w_j \geq 0$ and summing yields

$$\Im m_N(z) = \sum_{j=1}^{N} w_j \frac{y}{(\lambda_j - x)^2 + y^2}.$$

Every summand is nonnegative. Because the weights sum to 1, at least one weight is strictly positive. For that index the corresponding summand is strictly positive, and therefore the whole sum is strictly positive. Hence

$$\Im m_N(z) > 0.$$

Thus m_N maps the upper half-plane into itself. □

35.4 Poles, residues, and monotonicity on the real line

Proposition 44. *The poles of m_N are precisely the eigenvalues of B_N. The residue of m_N at a pole λ_j equals $-w_j$, where $w_j \geq 0$ is the spectral weight from the previous proposition.*

Proof. From the spectral representation,

$$m_N(z) = \sum_{j=1}^{N} \frac{w_j}{\lambda_j - z}.$$

Each summand has a simple pole at $z = \lambda_j$, and no other singularities. Therefore the poles of m_N are exactly the eigenvalues λ_j.

Now near $z = \lambda_j$ we have

$$\frac{w_j}{\lambda_j - z} = -\frac{w_j}{z - \lambda_j}.$$

Hence the residue of this term at $z = \lambda_j$ equals $-w_j$. All other summands are holomorphic at $z = \lambda_j$, so they contribute zero to the residue. Therefore

$$\operatorname{Res}_{z=\lambda_j} m_N(z) = -w_j.$$

Since $w_j \geq 0$, the residue is nonpositive. □

Proposition 45. *On every real interval that does not contain a pole, the function $m_N(x)$ is strictly increasing. More precisely, for every real $x \notin \sigma(B_N)$,*

$$m_N'(x) = \sum_{j=1}^{N} \frac{w_j}{(\lambda_j - x)^2} > 0.$$

Proof. Start from the spectral representation

$$m_N(x) = \sum_{j=1}^{N} \frac{w_j}{\lambda_j - x}, \qquad x \notin \{\lambda_1, \dots, \lambda_N\}.$$

Differentiate term by term. Since each summand is a rational function away from its pole, differentiation is legitimate. For each j,

$$\frac{d}{dx}\left(\frac{1}{\lambda_j - x}\right) = \frac{d}{dx}\big((\lambda_j - x)^{-1}\big) = -(\lambda_j - x)^{-2} \cdot (-1) = (\lambda_j - x)^{-2}.$$

Hence

$$m_N'(x) = \sum_{j=1}^{N} w_j(\lambda_j - x)^{-2} = \sum_{j=1}^{N} \frac{w_j}{(\lambda_j - x)^2}.$$

Every denominator is strictly positive, every weight is nonnegative, and at least one weight is positive because the sum of the weights is 1. Therefore

$$m_N'(x) > 0.$$

So m_N is strictly increasing on each connected component of $\mathbb{R} \setminus \sigma(B_N)$. □

35.5 Derivative identities

We next prove exact formulas for derivatives of the Weyl function.

Proposition 46. *For every $z \notin \sigma(B_N)$,*

$$m_N'(z) = \langle e_1, (B_N - zI)^{-2} e_1 \rangle.$$

Proof. Let

$$R(z) := (B_N - zI)^{-1}.$$

Then by definition

$$(B_N - zI)R(z) = I.$$

Differentiate both sides with respect to z. Since $\frac{d}{dz}(B_N - zI) = -I$, we obtain

$$(-I)R(z) + (B_N - zI)R'(z) = 0.$$

Thus

$$(B_N - zI)R'(z) = R(z).$$

Multiply on the left by $(B_N - zI)^{-1} = R(z)$ and get

$$R'(z) = R(z)^2 = (B_N - zI)^{-2}.$$

Now

$$m_N(z) = \langle e_1, R(z)e_1 \rangle.$$

Differentiating this scalar identity gives

$$m_N'(z) = \langle e_1, R'(z)e_1 \rangle.$$

Substituting the formula for $R'(z)$ yields

$$m_N'(z) = \langle e_1, (B_N - zI)^{-2} e_1 \rangle.$$

This proves the identity. □

Proposition 47. *For every integer $k \geq 0$,*

$$m_N^{(k)}(z) = k!\,\langle e_1, (B_N - zI)^{-(k+1)} e_1\rangle = k!\sum_{j=1}^{N} \frac{w_j}{(\lambda_j - z)^{k+1}}.$$

Proof. We proceed by induction on k.

For $k = 0$, the statement is simply

$$m_N(z) = \langle e_1, (B_N - zI)^{-1} e_1\rangle,$$

which is the definition of m_N and also agrees with the spectral representation.

Assume the identity holds for some $k \geq 0$:

$$m_N^{(k)}(z) = k!\,\langle e_1, (B_N - zI)^{-(k+1)} e_1\rangle.$$

Differentiate once more. Using the chain rule for operator powers,

$$\frac{d}{dz}(B_N - zI)^{-(k+1)} = (k+1)(B_N - zI)^{-(k+2)}.$$

To justify this formula, write $T(z) = B_N - zI$. Then $T'(z) = -I$, and because $T(z)$ commutes with every power of itself, the usual derivative rule for inverse powers yields

$$\frac{d}{dz}T(z)^{-(k+1)} = -(k+1)T(z)^{-(k+2)}T'(z) = (k+1)T(z)^{-(k+2)}.$$

Therefore

$$m_N^{(k+1)}(z) = k!(k+1)\langle e_1, (B_N - zI)^{-(k+2)} e_1\rangle = (k+1)!\,\langle e_1, (B_N - zI)^{-(k+2)} e_1\rangle.$$

This proves the first formula.

For the second formula, start from

$$m_N(z) = \sum_{j=1}^{N} \frac{w_j}{\lambda_j - z}.$$

Repeated differentiation gives

$$\frac{d^k}{dz^k}\left(\frac{1}{\lambda_j - z}\right) = \frac{k!}{(\lambda_j - z)^{k+1}},$$

because each differentiation increases the exponent by one and introduces a factor equal to the current exponent, with no minus sign after simplification. Therefore

$$m_N^{(k)}(z) = k!\sum_{j=1}^{N} \frac{w_j}{(\lambda_j - z)^{k+1}}.$$

This completes the proof. □

35.6 Large-z asymptotics

The numerical experiments suggested that $m_N(iy) \sim i/y$ as $y \to \infty$. We now prove the full expansion.

Proposition 48. *As $|z| \to \infty$,*

$$m_N(z) = -\frac{1}{z} - \frac{\langle e_1, B_N e_1\rangle}{z^2} - \frac{\langle e_1, B_N^2 e_1\rangle}{z^3} + O\left(\frac{1}{|z|^4}\right).$$

In particular,

$$m_N(z) = -\frac{1}{z} - \frac{b_1}{z^2} + O\left(\frac{1}{|z|^3}\right),$$

and since $b_1 = \frac{3}{2}$,

$$m_N(z) = -\frac{1}{z} - \frac{3}{2z^2} + O\left(\frac{1}{|z|^3}\right).$$

Proof. Write

$$B_N - zI = -z\left(I - \frac{B_N}{z}\right).$$

Hence

$$(B_N - zI)^{-1} = -\frac{1}{z}\left(I - \frac{B_N}{z}\right)^{-1}.$$

If $|z| > \|B_N\|$, then the operator norm of B_N/z is strictly less than 1, so the Neumann series converges:

$$\left(I - \frac{B_N}{z}\right)^{-1} = I + \frac{B_N}{z} + \frac{B_N^2}{z^2} + \frac{B_N^3}{z^3} + \cdots.$$

Substituting this into the previous identity yields

$$(B_N - zI)^{-1} = -\frac{1}{z}\left(I + \frac{B_N}{z} + \frac{B_N^2}{z^2} + \frac{B_N^3}{z^3} + \cdots\right).$$

Now take the matrix element against e_1:

$$\begin{aligned} m_N(z) &= \langle e_1, (B_N - zI)^{-1} e_1\rangle \\ &= -\frac{1}{z}\langle e_1, e_1\rangle - \frac{1}{z^2}\langle e_1, B_N e_1\rangle - \frac{1}{z^3}\langle e_1, B_N^2 e_1\rangle - \frac{1}{z^4}\langle e_1, B_N^3 e_1\rangle - \cdots. \end{aligned}$$

Since $\langle e_1, e_1\rangle = 1$, this becomes

$$m_N(z) = -\frac{1}{z} - \frac{\langle e_1, B_N e_1\rangle}{z^2} - \frac{\langle e_1, B_N^2 e_1\rangle}{z^3} + O\left(\frac{1}{|z|^4}\right).$$

Furthermore,

$$\langle e_1, B_N e_1\rangle = (B_N)_{11} = b_1.$$

Therefore

$$m_N(z) = -\frac{1}{z} - \frac{b_1}{z^2} + O\left(\frac{1}{|z|^3}\right).$$

Since $b_1 = \frac{3}{2}$, this becomes

$$m_N(z) = -\frac{1}{z} - \frac{3}{2z^2} + O\left(\frac{1}{|z|^3}\right).$$

This proves the proposition. □

Corollary 9. *As $y \to +\infty$,*

$$m_N(iy) = \frac{i}{y} + \frac{3}{2y^2} + O\left(\frac{1}{y^3}\right).$$

Consequently,

$$y\,\Im m_N(iy) \to 1, \qquad y\,\Re m_N(iy) \to 0.$$

Proof. Substitute $z = iy$ into the asymptotic expansion from the previous proposition:

$$m_N(iy) = -\frac{1}{iy} - \frac{3}{2(iy)^2} + O(y^{-3}).$$

Now compute each term carefully. First,

$$-\frac{1}{iy} = \frac{i}{y},$$

because multiplying numerator and denominator by i gives

$$-\frac{1}{iy} = -\frac{i}{i^2 y} = \frac{i}{y}.$$

Second,

$$(iy)^2 = i^2 y^2 = -y^2,$$

so

$$-\frac{3}{2(iy)^2} = -\frac{3}{2(-y^2)} = \frac{3}{2y^2}.$$

Hence

$$m_N(iy) = \frac{i}{y} + \frac{3}{2y^2} + O(y^{-3}).$$

Taking imaginary parts gives

$$\Im m_N(iy) = \frac{1}{y} + O(y^{-3}),$$

and multiplying by y yields

$$y\,\Im m_N(iy) = 1 + O(y^{-2}) \to 1.$$

Taking real parts gives

$$\Re m_N(iy) = \frac{3}{2y^2} + O(y^{-3}),$$

and therefore

$$y\,\Re m_N(iy) = \frac{3}{2y} + O(y^{-2}) \to 0.$$

This proves the corollary. □

35.7 Differentiated Riccati identity

The continued fraction recursion can be differentiated exactly.

Proposition 49. *For every $n \geq 1$,*

$$m_n'(z) = m_n(z)^2\big(1 + a_n^2\, m_{n+1}'(z)\big).$$

Proof. From the Riccati recursion we have

$$m_n(z) = \frac{1}{b_n - z - a_n^2 m_{n+1}(z)}.$$

Let

$$F_n(z) := b_n - z - a_n^2 m_{n+1}(z).$$

Then

$$m_n(z) = F_n(z)^{-1}.$$

Differentiating gives

$$m_n'(z) = -F_n(z)^{-2} F_n'(z).$$

Now

$$F_n'(z) = -1 - a_n^2 m_{n+1}'(z),$$

because the derivative of b_n is 0, the derivative of $-z$ is -1, and the derivative of $-a_n^2 m_{n+1}(z)$ is $-a_n^2 m_{n+1}'(z)$, since a_n is constant with respect to z. Therefore

$$m_n'(z) = -F_n(z)^{-2}\big(-1 - a_n^2 m_{n+1}'(z)\big) = F_n(z)^{-2}\big(1 + a_n^2 m_{n+1}'(z)\big).$$

Because $m_n(z) = F_n(z)^{-1}$, we have

$$F_n(z)^{-2} = m_n(z)^2.$$

Substituting this identity gives

$$m_n'(z) = m_n(z)^2\big(1 + a_n^2 m_{n+1}'(z)\big).$$

This proves the result. □

35.8 Exact encoding of prime gaps

Finally, the Jacobi coefficients encode the prime gaps with no loss of information.

Proposition 50. *For every $n \geq 1$,*

$$a_n = -\frac{1}{d_{n+1}}.$$

Hence the value of the $(n+1)$-st prime gap is recovered from the off-diagonal Jacobi coefficient by

$$d_{n+1} = \frac{1}{|a_n|}.$$

Proof. The first formula is simply the definition of a_n. Since each prime gap d_{n+1} is a positive integer, it follows that a_n is strictly negative. Therefore

$$|a_n| = \frac{1}{d_{n+1}}.$$

Taking reciprocals gives

$$d_{n+1} = \frac{1}{|a_n|}.$$

This proves the result. □

Corollary 10. *The following equivalences hold:*

$$d_{n+1} = 2 \iff a_n = -\frac{1}{2},$$

$$d_{n+1} = 4 \iff a_n = -\frac{1}{4},$$

$$d_{n+1} = 6 \iff a_n = -\frac{1}{6},$$

and more generally

$$d_{n+1} = 2k \iff a_n = -\frac{1}{2k}.$$

Proof. By the proposition,

$$a_n = -\frac{1}{d_{n+1}}.$$

Thus $d_{n+1} = 2$ holds if and only if

$$a_n = -\frac{1}{2}.$$

The same argument gives

$$d_{n+1} = 4 \iff a_n = -\frac{1}{4}, \qquad d_{n+1} = 6 \iff a_n = -\frac{1}{6},$$

and for any positive integer k,

$$d_{n+1} = 2k \iff a_n = -\frac{1}{2k}.$$

This proves all equivalences. □

35.9 Summary

We summarize the main conclusions proved in this section.

(1) The Weyl function admits an exact J-fraction / continued fraction expansion.

(2) Its tails satisfy the discrete Riccati recursion

$$m_n(z) = \frac{1}{b_n - z - a_n^2 m_{n+1}(z)}.$$

(3) It is the Stieltjes transform of a positive spectral measure.

(4) It is a Herglotz function and therefore maps the upper half-plane into itself.

(5) Its poles are real and simple, and their residues are determined by the spectral weights.

(6) Between poles on the real axis, the function is strictly increasing.

(7) Its derivatives admit exact resolvent and spectral formulas.

(8) It has the large-z asymptotic expansion

$$m_N(z) = -\frac{1}{z} - \frac{3}{2z^2} + O\!\left(\frac{1}{|z|^3}\right),$$

and in particular

$$m_N(iy) = \frac{i}{y} + \frac{3}{2y^2} + O\!\left(\frac{1}{y^3}\right).$$

(9) The off-diagonal Jacobi coefficients recover the prime gaps exactly:

$$d_{n+1} = \frac{1}{|a_n|}.$$

These properties show that the Weyl function is not naturally additive or multiplicative. Its correct structural framework is instead spectral, Stieltjes-theoretic, and Riccati-recursive, with the prime gaps entering directly as the dynamical parameters of the continued fraction.

36 Characteristic polynomials and orthogonal polynomials

In this section we prove, in complete detail, that the polynomials generated by the prime-gap Jacobi recurrence are, up to normalization, exactly the characteristic polynomials of the finite truncations of the Jacobi matrix. We then deduce that these normalized characteristic polynomials are orthogonal polynomials for the spectral measure of the Weyl function.

1. The Jacobi data coming from prime gaps

Let $p_1 = 2 < p_2 = 3 < p_3 = 5 < \cdots$ be the primes, and define the gap sequence $(d_n)_{n\geq 1}$ by

$$d_1 := 2, \qquad d_2 := 1, \qquad d_n := p_n - p_{n-1} \quad (n \geq 3).$$

Thus $d_3 = 2$, $d_4 = 2$, $d_5 = 4$, and so on.

From these gaps we define Jacobi coefficients $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ by

$$a_n := -\frac{1}{d_{n+1}} \qquad (n \geq 1),$$

and

$$b_1 := \frac{3}{2}, \qquad b_n := \frac{1}{d_n} + \frac{1}{d_{n+1}} \qquad (n \geq 2).$$

Since every gap d_{n+1} is a positive integer, each a_n is a nonzero negative real number. In particular,

$$a_n \neq 0 \qquad \text{for all } n \geq 1.$$

This simple fact will be used repeatedly below when dividing by a_n.

Let B denote the infinite Jacobi matrix

$$B = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \ddots \\ 0 & 0 & a_3 & b_4 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

For each $n \geq 1$, let B_n denote its upper-left $n \times n$ truncation:

$$B_n = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & a_{n-1} & b_n \end{pmatrix}.$$

This is the finite Jacobi matrix whose characteristic polynomial we shall study.

2. The recursively defined polynomials P_n

We now define a sequence of polynomials $(P_n)_{n\geq 0}$ by the initial conditions

$$P_{-1}(x) := 0, \qquad P_0(x) := 1,$$

and the three-term recurrence

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_{n+1}P_n(x) + a_nP_{n-1}(x) \qquad (n \geq 0), \tag{3}$$

where, by convention, $a_0 := 0$.

This recurrence determines $P_1, P_2, \ldots$ successively. Indeed, for $n = 0$ the recurrence becomes

$$xP_0(x) = a_1P_1(x) + b_1P_0(x) + a_0P_{-1}(x).$$

Since $P_0(x) = 1$, $P_{-1}(x) = 0$, and $a_0 = 0$, this simplifies to

$$x = a_1P_1(x) + b_1.$$

Because $a_1 \neq 0$, one solves uniquely for $P_1(x)$. Once P_1 is known, the case $n = 1$ determines P_2, and so on.

Lemma 4. *For every $n \geq 0$, P_n is a polynomial of degree exactly n. If $c_n := \mathrm{LC}(P_n)$ denotes its leading coefficient, then*

$$c_0 = 1, \qquad c_{n+1} = \frac{c_n}{a_{n+1}} \quad (n \geq 0),$$

and hence

$$c_n = \frac{1}{a_1a_2\cdots a_n} \qquad (n \geq 1).$$

Proof. We argue by induction on n.

For $n = 0$, we have $P_0(x) = 1$, so $\deg P_0 = 0$ and $c_0 = 1$.

Assume now that P_{n-1} and P_n have already been constructed, that $\deg P_{n-1} = n-1$, $\deg P_n = n$, and that c_n is the leading coefficient of P_n. We show that P_{n+1} has degree $n + 1$ and leading coefficient c_n/a_{n+1}.

Starting from (3), we isolate P_{n+1}:

$$a_{n+1}P_{n+1}(x) = (x - b_{n+1})P_n(x) - a_nP_{n-1}(x).$$

Hence

$$P_{n+1}(x) = \frac{(x - b_{n+1})P_n(x) - a_nP_{n-1}(x)}{a_{n+1}}. \tag{4}$$

Now $(x - b_{n+1})P_n(x)$ has degree $n + 1$ and leading coefficient c_n, whereas $a_nP_{n-1}(x)$ has degree $n - 1$. Therefore the leading term of the numerator in (4) comes entirely from $xP_n(x)$, and no cancellation of the top-degree term is possible. It follows that P_{n+1} has degree $n + 1$ and leading coefficient

$$c_{n+1} = \frac{c_n}{a_{n+1}}.$$

This proves the inductive step.

By repeated application of the recurrence for c_n we get

$$c_n = \frac{1}{a_1a_2 \cdots a_n}$$

for every $n \geq 1$. The lemma follows. □

3. Characteristic polynomials of the finite truncations

Define

$$\pi_n(x) := \det(xI - B_n) \qquad (n \geq 1),$$

and set $\pi_0(x) := 1$.

Our first goal is to derive the recurrence relation for the π_n.

Lemma 5. *The characteristic polynomials π_n satisfy*

$$\pi_0(x) = 1, \qquad \pi_1(x) = x - b_1,$$

and, for every $n \geq 1$,

$$\pi_{n+1}(x) = (x - b_{n+1})\pi_n(x) - a_n^2\pi_{n-1}(x). \tag{5}$$

Proof. The formula for π_0 is the definition. For $n = 1$ we have

$$\pi_1(x) = \det(xI - B_1) = \det\big(x - b_1\big) = x - b_1.$$

So the initial conditions hold.

We now prove the recurrence. Consider the matrix $xI - B_{n+1}$. Since B_{n+1} is tridiagonal, we have

$$xI - B_{n+1} = \begin{pmatrix} x - b_1 & -a_1 & 0 & \cdots & 0 & 0 \\ -a_1 & x - b_2 & -a_2 & \ddots & \vdots & \vdots \\ 0 & -a_2 & x - b_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & -a_{n-1} & 0 \\ 0 & \cdots & 0 & -a_{n-1} & x - b_n & -a_n \\ 0 & \cdots & 0 & 0 & -a_n & x - b_{n+1} \end{pmatrix}.$$

Expand its determinant along the last row. There are only two nonzero entries in that row, namely the entries in columns n and $n + 1$:

$$(xI - B_{n+1})_{n+1,n} = -a_n, \qquad (xI - B_{n+1})_{n+1,n+1} = x - b_{n+1}.$$

Hence

$$\pi_{n+1}(x) = (-a_n)C_{n+1,n} + (x - b_{n+1})C_{n+1,n+1},$$

where C_{ij} denotes the cofactor.

We first compute $C_{n+1,n+1}$. Deleting the last row and last column leaves exactly the matrix $xI - B_n$. Therefore

$$C_{n+1,n+1} = \det(xI - B_n) = \pi_n(x).$$

Next we compute $C_{n+1,n}$. By definition,

$$C_{n+1,n} = (-1)^{(n+1)+n} \det M_{n+1,n} = -\det M_{n+1,n},$$

where $M_{n+1,n}$ is the minor obtained by deleting row $n+1$ and column n.

Now inspect $M_{n+1,n}$. Because the original matrix is tridiagonal, deleting the last row and the n-th column leaves a matrix whose last column has only one nonzero entry, namely the entry $-a_n$ in the last remaining row. Expanding $\det M_{n+1,n}$ along that last column gives

$$\det M_{n+1,n} = (-a_n)\det(xI - B_{n-1}).$$

Therefore

$$C_{n+1,n} = -(-a_n)\det(xI - B_{n-1}) = a_n\pi_{n-1}(x).$$

Substituting into the Laplace expansion yields

$$\pi_{n+1}(x) = (-a_n)\big(a_n\pi_{n-1}(x)\big) + (x - b_{n+1})\pi_n(x),$$

that is,

$$\pi_{n+1}(x) = (x - b_{n+1})\pi_n(x) - a_n^2\pi_{n-1}(x).$$

This is exactly (5). The proof is complete. □

4. Monic normalization of the recurrence polynomials

The recurrence (3) for P_n is not monic. To compare it with the characteristic polynomials π_n, we normalize P_n by dividing by its leading coefficient.

Define

$$Q_n(x) := \frac{P_n(x)}{c_n},$$

where $c_n = \mathrm{LC}(P_n)$ is given by Lemma 4. Then each Q_n is monic and has degree n.

Lemma 6. *The monic polynomials Q_n satisfy*

$$Q_0(x) = 1, \qquad Q_1(x) = x - b_1,$$

and, for every $n \geq 1$,

$$Q_{n+1}(x) = (x - b_{n+1})Q_n(x) - a_n^2 Q_{n-1}(x). \tag{6}$$

Proof. We begin with the initial values.

Since $P_0(x) = 1$ and $c_0 = 1$, we have

$$Q_0(x) = \frac{P_0(x)}{c_0} = 1.$$

For $n = 0$, the recurrence (3) reads

$$xP_0(x) = a_1P_1(x) + b_1P_0(x).$$

Because $P_0(x) = 1$, this becomes

$$x = a_1 P_1(x) + b_1.$$

Therefore

$$P_1(x) = \frac{x - b_1}{a_1}.$$

Since $c_1 = 1/a_1$ by Lemma 4, we obtain

$$Q_1(x) = \frac{P_1(x)}{c_1} = \frac{(x - b_1)/a_1}{1/a_1} = x - b_1.$$

Thus the initial values are proved.

Now let $n \geq 1$. Starting from (3), write

$$a_{n+1} P_{n+1}(x) = (x - b_{n+1}) P_n(x) - a_n P_{n-1}(x).$$

Substitute $P_k = c_k Q_k$ for $k = n+1, n, n-1$:

$$a_{n+1} c_{n+1} Q_{n+1}(x) = (x - b_{n+1}) c_n Q_n(x) - a_n c_{n-1} Q_{n-1}(x).$$

Now divide by $a_{n+1} c_{n+1}$. By Lemma 4,

$$c_{n+1} = \frac{c_n}{a_{n+1}},$$

so

$$a_{n+1} c_{n+1} = c_n.$$

Therefore the coefficient of Q_n on the right simplifies to

$$\frac{c_n}{a_{n+1} c_{n+1}} = 1.$$

For the coefficient of Q_{n-1} we use twice the leading-coefficient relation:

$$c_n = \frac{c_{n-1}}{a_n}.$$

Thus

$$c_{n-1} = a_n c_n.$$

Combining this with $a_{n+1} c_{n+1} = c_n$, we get

$$\frac{a_n c_{n-1}}{a_{n+1} c_{n+1}} = \frac{a_n (a_n c_n)}{c_n} = a_n^2.$$

Hence the recurrence becomes

$$Q_{n+1}(x) = (x - b_{n+1}) Q_n(x) - a_n^2 Q_{n-1}(x).$$

This proves (6). □

5. Equality with the characteristic polynomials

We can now prove the first main statement.

Proposition 51. *For every $n \geq 0$,*

$$Q_n(x) = \pi_n(x) = \det(xI - B_n).$$

Equivalently, for every $n \geq 0$,

$$\frac{P_n(x)}{\mathrm{LC}(P_n)} = \det(xI - B_n).$$

In words: up to normalization, the recurrence polynomials P_n are exactly the characteristic polynomials of the finite truncations B_n.

Proof. By Lemma 5, the polynomials π_n satisfy

$$\pi_0(x) = 1, \qquad \pi_1(x) = x - b_1,$$

and for every $n \geq 1$,

$$\pi_{n+1}(x) = (x - b_{n+1})\pi_n(x) - a_n^2 \pi_{n-1}(x).$$

By Lemma 6, the polynomials Q_n satisfy

$$Q_0(x) = 1, \qquad Q_1(x) = x - b_1,$$

and for every $n \geq 1$,

$$Q_{n+1}(x) = (x - b_{n+1})Q_n(x) - a_n^2 Q_{n-1}(x).$$

Thus the two sequences (Q_n) and (π_n) satisfy the same second-order recurrence with the same initial conditions.

We now conclude by induction on n. For $n = 0$ and $n = 1$, the two sequences agree by the initial conditions:

$$Q_0(x) = 1 = \pi_0(x), \qquad Q_1(x) = x - b_1 = \pi_1(x).$$

Assume that for some $n \geq 1$ we already know

$$Q_n(x) = \pi_n(x) \qquad \text{and} \qquad Q_{n-1}(x) = \pi_{n-1}(x).$$

Then applying the common recurrence gives

$$\begin{aligned} Q_{n+1}(x) &= (x - b_{n+1})Q_n(x) - a_n^2 Q_{n-1}(x) \\ &= (x - b_{n+1})\pi_n(x) - a_n^2 \pi_{n-1}(x) \\ &= \pi_{n+1}(x). \end{aligned}$$

This proves $Q_{n+1} = \pi_{n+1}$. By induction, $Q_n = \pi_n$ for all $n \geq 0$.

Since $Q_n = P_n / \mathrm{LC}(P_n)$ by definition, the equivalent formula

$$\frac{P_n(x)}{\mathrm{LC}(P_n)} = \det(xI - B_n)$$

follows immediately. □

6. Orthogonality of the normalized characteristic polynomials

We now turn to the second claim. Let $m(z)$ be the Weyl function of the prime-gap Jacobi operator B:

$$m(z) = \langle e_1, (B - zI)^{-1} e_1 \rangle.$$

Its associated spectral measure will be denoted by μ. By the spectral theorem, μ is a positive Borel measure on $\mathbb{R}$ such that m is its Stieltjes transform. The manuscript shows that the recurrence polynomials P_n are the orthonormal polynomials associated with this measure μ.

We now deduce from Proposition 51 that the normalized characteristic polynomials are orthogonal polynomials for the same measure.

Proposition 52. *Assume that the polynomials P_n are orthonormal in $L^2(\mu)$, i.e.*

$$\int P_n(x) P_m(x)\, d\mu(x) = \delta_{nm}.$$

Then the monic characteristic polynomials

$$\pi_n(x) = \det(xI - B_n)$$

form the monic orthogonal polynomial sequence for μ. More precisely,

$$\int \pi_n(x) \pi_m(x)\, d\mu(x) = 0 \qquad (n \neq m).$$

Moreover, each π_n is monic of degree n.

Proof. By Proposition 51, we have

$$\pi_n(x) = \frac{P_n(x)}{c_n},$$

where $c_n = \mathrm{LC}(P_n)$ is a nonzero scalar. Since P_n has degree n, dividing by a nonzero scalar does not change the degree, so π_n also has degree n. By construction π_n is monic.

It remains to prove orthogonality. Let $m \neq n$. Then

$$\begin{aligned} \int \pi_n(x)\pi_m(x)\, d\mu(x) &= \int \frac{P_n(x)}{c_n} \frac{P_m(x)}{c_m}\, d\mu(x) \\ &= \frac{1}{c_n c_m} \int P_n(x) P_m(x)\, d\mu(x). \end{aligned}$$

Because the P_n are orthonormal, the last integral equals 0 whenever $m \neq n$. Hence

$$\int \pi_n(x)\pi_m(x)\, d\mu(x) = 0 \qquad (m \neq n).$$

So the sequence (π_n) is orthogonal with respect to μ.

Since each π_n is monic of degree n, this shows that (π_n) is precisely the monic orthogonal polynomial system associated with μ. □

Corollary 11. *The prime gaps canonically generate an orthogonal polynomial system. More precisely, the prime gaps determine the Jacobi coefficients (a_n, b_n), these determine the Jacobi operator B, the operator B determines the spectral measure μ of the Weyl function, and the monic orthogonal polynomials of μ are exactly the characteristic polynomials of the finite truncations B_n.*

Proof. The prime gaps determine the coefficients a_n and b_n by definition. These coefficients define the Jacobi operator B, and therefore its Weyl function and spectral measure μ. By Proposition 52, the monic orthogonal polynomials for μ are the characteristic polynomials $\pi_n(x) = \det(xI - B_n)$. This is precisely the claimed chain of implications. □

7. Conceptual summary

We summarize the logical structure proved above.

1. The prime gaps d_n determine Jacobi coefficients

$$a_n = -\frac{1}{d_{n+1}}, \qquad b_1 = \frac{3}{2}, \qquad b_n = \frac{1}{d_n} + \frac{1}{d_{n+1}}.$$

2. These coefficients define an infinite Jacobi matrix B and its finite truncations B_n.
3. The same coefficients define, via the three-term recurrence, a polynomial sequence (P_n).
4. After dividing by leading coefficients, the resulting monic polynomials satisfy exactly the same recurrence and initial conditions as the characteristic polynomials $\det(xI - B_n)$.
5. Therefore the monic normalizations of the P_n are exactly the characteristic polynomials of the finite truncations.
6. Since the P_n are orthonormal for the spectral measure μ of the Weyl function, the monic characteristic polynomials are orthogonal for that same measure.

Thus the appearance of orthogonal polynomials is not accidental. It is an intrinsic consequence of passing from prime gaps to a Jacobi operator. In this sense, the prime gaps are encoded not only in the Weyl function and the Jacobi coefficients, but equally in an associated orthogonal polynomial system whose monic representatives are precisely the characteristic polynomials of the finite prime-gap truncations.

37 Semiprime structure of the large row sums

In this section we analyze the row sums of the full meet Gram matrix and show that, for a natural family of rows indexed by semiprimes, the dominant contribution comes from the semiprime layer. This gives a rigorous explanation for the numerical phenomenon that the largest row sums are produced by numbers of the form ap, where a is a small fixed prime and p is a large prime.

1. The kernel and the row sums

For a positive integer n, let

$$\Omega(n)$$

denote the number of prime factors of n counted with multiplicity. If

$$m = p_1 p_2 \cdots p_r, \qquad n = q_1 q_2 \cdots q_s$$

are the prime factorizations written in nondecreasing order,

$$p_1 \le p_2 \le \cdots \le p_r, \qquad q_1 \le q_2 \le \cdots \le q_s,$$

then the full meet kernel is defined by

$$K(m,n) := \prod_{i=1}^{\min(r,s)} \min(p_i, q_i).$$

As usual, the empty product is interpreted as 1. In particular,

$$K(n, n) = n$$

for every $n \in \mathbb{N}$.

For $N \geq 1$, we define the row sum at m by

$$R_m(N) := \sum_{n \leq N} K(m, n).$$

Our goal is to understand $R_m(N)$ when

$$m = ap,$$

with a a fixed prime and p a prime of size comparable to N/a.

2. Decomposition by the arithmetic type of n

For $j \geq 1$, define

$$R_m^{(j)}(N) := \sum_{\substack{n \leq N \\ \Omega(n) = j}} K(m, n),$$

and also

$$R_m^{(\geq 3)}(N) := \sum_{\substack{n \leq N \\ \Omega(n) \geq 3}} K(m, n).$$

Then trivially

$$R_m(N) = R_m^{(1)}(N) + R_m^{(2)}(N) + R_m^{(\geq 3)}(N).$$

The main point of this section is that, when $m = ap$ with a fixed and $p \asymp N/a$, the quantity $R_m^{(2)}(N)$ already has the correct order of magnitude $N^2/\log N$, while the contribution from $\Omega(n) \geq 3$ is much smaller.

3. Exact formula on the semiprime layer

We begin with the exact structure of the semiprime contribution.

Proposition 53. *Let $a \leq p$ and $b \leq q$ be primes. Then*

$$K(ap, bq) = \min(a, b) \min(p, q).$$

Proof. The ordered prime factor list of ap is exactly (a, p), because $a \leq p$. Likewise, the ordered prime factor list of bq is exactly (b, q), because $b \leq q$.

By definition of the kernel,

$$K(ap, bq) = \prod_{i=1}^{\min(2,2)} \min(\text{the } i\text{th prime factor of } ap,\ \text{the } i\text{th prime factor of } bq).$$

Since both numbers have exactly two prime factors counted with multiplicity, this becomes

$$K(ap, bq) = \min(a, b) \min(p, q).$$

This is exactly the claimed identity. □

Corollary 12. *Let a be a fixed prime and let $p \geq a$ be a prime. Then*

$$R_{ap}^{(2)}(N) = \sum_{\substack{b \leq q \text{ prime} \\ bq \leq N}} \min(a,b)\min(p,q).$$

Proof. A positive integer n satisfies $\Omega(n) = 2$ if and only if it can be written in the form

$$n = bq,$$

where $b \leq q$ are primes. Therefore

$$R_{ap}^{(2)}(N) = \sum_{\substack{b \leq q \text{ prime} \\ bq \leq N}} K(ap, bq).$$

Applying the proposition to each summand gives

$$R_{ap}^{(2)}(N) = \sum_{\substack{b \leq q \text{ prime} \\ bq \leq N}} \min(a,b)\min(p,q),$$

as required. □

The previous corollary admits a useful further decomposition by the first prime factor b.

Corollary 13. *For every prime $a \leq p$,*

$$R_{ap}^{(2)}(N) = \sum_{b \text{ prime}} \min(a,b) \sum_{\substack{q \text{ prime} \\ q \geq b,\ bq \leq N}} \min(p,q).$$

Proof. In the preceding corollary, group together the summands having the same first prime factor b. This gives

$$R_{ap}^{(2)}(N) = \sum_{b \text{ prime}} \sum_{\substack{q \text{ prime} \\ q \geq b,\ bq \leq N}} \min(a,b)\min(p,q).$$

Since $\min(a,b)$ depends only on a and b, it can be pulled out of the inner sum:

$$R_{ap}^{(2)}(N) = \sum_{b \text{ prime}} \min(a,b) \sum_{\substack{q \text{ prime} \\ q \geq b,\ bq \leq N}} \min(p,q).$$

This is exactly the claimed formula. □

4. Asymptotic size of the semiprime contribution

We now prove that the semiprime contribution already has the scale $N^2/\log N$.

Proposition 54. *Fix a prime a. For each sufficiently large N, let p be a prime satisfying*

$$\frac{N}{2a} \leq p \leq \frac{N}{a}.$$

Then

$$R_{ap}^{(2)}(N) = \Theta_a\left(\frac{N^2}{\log N}\right).$$

Proof. We prove an upper bound and a lower bound separately.

Upper bound. By the exact semiprime formula,

$$R_{ap}^{(2)}(N) = \sum_{\substack{b\le q \text{ prime} \\ bq\le N}} \min(a,b)\min(p,q).$$

We first split the sum according to whether $b \le a$ or $b > a$.

For the part with $b > a$, we have $\min(a,b) = a$, hence

$$\sum_{\substack{b>a,\ b\le q \text{ prime} \\ bq\le N}} \min(a,b)\min(p,q) \le a \sum_{b>a \text{ prime}} \sum_{\substack{q \text{ prime} \\ q\le N/b}} q.$$

Indeed, we simply used $\min(p,q) \le q$.

Now define

$$S(x) := \sum_{\substack{q\le x \\ q \text{ prime}}} q.$$

A standard estimate from prime number theory gives

$$S(x) \ll \frac{x^2}{\log x}.$$

Therefore

$$a \sum_{b>a \text{ prime}} S(N/b) \ll a \sum_{b>a \text{ prime}} \frac{(N/b)^2}{\log(N/b)}.$$

Since a is fixed and the prime sum $\sum_b b^{-2}$ converges, this is

$$\ll_a \frac{N^2}{\log N}.$$

We have therefore shown that the contribution from $b > a$ is $O_a(N^2/\log N)$.

Now consider the contribution from $b \le a$. Since a is fixed, there are only finitely many such primes b. For each such b,

$$\sum_{\substack{q \text{ prime} \\ q\ge b,\ bq\le N}} \min(a,b)\min(p,q) \le b \sum_{\substack{q\le N/b \\ q \text{ prime}}} \min(p,q).$$

We split the range at $q = p$:

$$\sum_{\substack{q\le N/b \\ q \text{ prime}}} \min(p,q) = \sum_{\substack{q\le p \\ q \text{ prime}}} q + p \sum_{\substack{p<q\le N/b \\ q \text{ prime}}} 1.$$

The first sum is $S(p)$ and the second is bounded by $p\,\pi(N/b)$. Using the standard estimates

$$S(p) \ll \frac{p^2}{\log p}, \qquad \pi(x) \ll \frac{x}{\log x},$$

and the fact that $p \asymp N/a$, we obtain

$$\sum_{\substack{q \le N/b \\ q \text{ prime}}} \min(p,q) \ll_a \frac{N^2}{\log N}.$$

Summing over the finitely many primes $b \le a$ yields again an $O_a(N^2/\log N)$ bound.

Combining both cases,

$$R_{ap}^{(2)}(N) \ll_a \frac{N^2}{\log N}.$$

Lower bound. We now prove a lower bound of the same scale.

Choose a prime b_0 such that

$$2a \le b_0 \le 4a.$$

Since a is fixed, such a prime exists for all sufficiently large values in the present application; any fixed interval $[c_1 a, c_2 a]$ with $1 < c_1 < c_2$ would also suffice.

We retain in the semiprime sum only the channel corresponding to this single value $b = b_0$. Then

$$R_{ap}^{(2)}(N) \ge \sum_{\substack{q \text{ prime} \\ q \ge b_0,\ b_0 q \le N}} \min(a, b_0) \min(p,q).$$

Since $b_0 > a$, we have $\min(a, b_0) = a$. Also, because

$$p \ge \frac{N}{2a} \qquad \text{and} \qquad b_0 \ge 2a,$$

we obtain

$$\frac{N}{b_0} \le \frac{N}{2a} \le p.$$

Hence every prime $q \le N/b_0$ also satisfies $q \le p$, and therefore

$$\min(p,q) = q.$$

Thus

$$R_{ap}^{(2)}(N) \ge a \sum_{\substack{q \le N/b_0 \\ q \text{ prime}}} q = aS(N/b_0).$$

Using the lower-order form of the standard prime-sum asymptotic,

$$S(x) \gg \frac{x^2}{\log x},$$

we conclude that

$$R_{ap}^{(2)}(N) \gg a \cdot \frac{(N/b_0)^2}{\log(N/b_0)} \gg_a \frac{N^2}{\log N}.$$

Combining the upper and lower bounds yields

$$R_{ap}^{(2)}(N) = \Theta_a\left(\frac{N^2}{\log N}\right).$$

This completes the proof. $\square$

5. The higher-order part is lower order

We now show that the contribution from integers with at least three prime factors is much smaller.

Proposition 55. *Fix a prime a, and let p be a prime with*

$$\frac{N}{2a} \le p \le \frac{N}{a}.$$

Then

$$R_{ap}^{(\geq 3)}(N) \ll_a N(\log\log N)^2.$$

In particular,

$$R_{ap}^{(\geq 3)}(N) = o\left(\frac{N^2}{\log N}\right).$$

Proof. Let $n \le N$ satisfy $\Omega(n) \ge 3$. Write its ordered prime factorization as

$$n = bqr,$$

where $b \le q \le r$ are primes, and where there may possibly be additional prime factors after r. Since ap has exactly two prime factors, the kernel $K(ap, n)$ depends only on the first two prime factors of n. Therefore

$$K(ap, n) = \min(a, b)\min(p, q).$$

We now ask: how many integers $n \le N$ have prescribed first two prime factors $b \le q$? Since the third prime factor is at least q, every such integer is divisible by

$$bq^2.$$

Hence the number of such integers is at most

$$\frac{N}{bq^2}.$$

Therefore

$$R_{ap}^{(\geq 3)}(N) \le \sum_{b \le q \text{ prime}} \frac{N}{bq^2} \min(a, b)\min(p, q).$$

Rearranging the summation yields

$$R_{ap}^{(\geq 3)}(N) \le N \sum_{q \text{ prime}} \frac{\min(p, q)}{q^2} \sum_{\substack{b \le q \\ b \text{ prime}}} \frac{\min(a, b)}{b}.$$

Define

$$A_a(q) := \sum_{\substack{b \le q \\ b \text{ prime}}} \frac{\min(a, b)}{b}.$$

Then

$$R_{ap}^{(\geq 3)}(N) \le N \sum_{q \text{ prime}} \frac{\min(p, q)}{q^2} A_a(q).$$

We now estimate $A_a(q)$. Split the sum at $b = a$:

$$A_a(q) = \sum_{\substack{b \le a \\ b \text{ prime}}} \frac{\min(a,b)}{b} + \sum_{\substack{a < b \le q \\ b \text{ prime}}} \frac{\min(a,b)}{b}.$$

For $b \le a$ we have $\min(a,b) = b$, so the first sum is simply the number of primes at most a, hence a constant depending only on a. For $b > a$ we have $\min(a,b) = a$, so

$$A_a(q) = O_a(1) + a \sum_{\substack{a < b \le q \\ b \text{ prime}}} \frac{1}{b}.$$

By Mertens' estimate for the prime harmonic sum,

$$\sum_{\substack{b \le q \\ b \text{ prime}}} \frac{1}{b} \ll \log\log q,$$

thus

$$A_a(q) \ll_a \log\log q.$$

Substituting this bound gives

$$R_{ap}^{(\ge 3)}(N) \ll_a N \sum_{q \text{ prime}} \frac{\min(p,q)}{q^2} \log\log q.$$

We split the remaining sum at $q = p$.

When $q \le p$, we have $\min(p,q) = q$, so the contribution is

$$\sum_{\substack{q \le p \\ q \text{ prime}}} \frac{\log\log q}{q}.$$

Since $\log\log q \le \log\log p$ throughout this range, this is bounded by

$$(\log\log p) \sum_{\substack{q \le p \\ q \text{ prime}}} \frac{1}{q}.$$

Using again Mertens' estimate for the prime harmonic series, this becomes

$$\ll (\log\log p)^2.$$

When $q > p$, we have $\min(p,q) = p$, so the contribution is

$$p \sum_{\substack{q > p \\ q \text{ prime}}} \frac{\log\log q}{q^2}.$$

Since the series with general term $(\log\log q)/q^2$ converges and its tail is bounded by a constant multiple of $(\log\log p)/p$, we get

$$p \sum_{\substack{q > p \\ q \text{ prime}}} \frac{\log\log q}{q^2} \ll \log\log p.$$

Combining both ranges,

$$R_{ap}^{(\geq 3)}(N) \ll_a N\big((\log\log p)^2 + \log\log p\big) \ll_a N(\log\log N)^2,$$

because $p \asymp N$ up to a constant depending on a.

This proves the first claim. The second claim follows immediately since

$$\frac{N(\log\log N)^2}{N^2/\log N} = \frac{(\log N)(\log\log N)^2}{N} \to 0.$$

Therefore

$$R_{ap}^{(\geq 3)}(N) = o\left(\frac{N^2}{\log N}\right).$$

The proof is complete. □

6. Consequence for the full row sum

We now combine the semiprime main term with the error estimate.

Theorem 9. *Fix a prime a. For each sufficiently large N, choose a prime p satisfying*

$$\frac{N}{2a} \leq p \leq \frac{N}{a}.$$

Then

$$R_{ap}(N) = \Theta_a\left(\frac{N^2}{\log N}\right).$$

Proof. We have the decomposition

$$R_{ap}(N) = R_{ap}^{(1)}(N) + R_{ap}^{(2)}(N) + R_{ap}^{(\geq 3)}(N).$$

The semiprime contribution satisfies

$$R_{ap}^{(2)}(N) = \Theta_a\left(\frac{N^2}{\log N}\right)$$

by the proposition proved above.

It remains to check that the other two pieces do not exceed this scale.

For $R_{ap}^{(1)}(N)$, every n with $\Omega(n) = 1$ is prime, say $n = q$. Since

$$K(ap, q) = \min(a, q),$$

we obtain

$$R_{ap}^{(1)}(N) = \sum_{\substack{q \leq N \\ q \text{ prime}}} \min(a, q).$$

Split at $q = a$:

$$R_{ap}^{(1)}(N) \leq \sum_{\substack{q \leq a \\ q \text{ prime}}} q + a\,\pi(N) \ll_a \frac{N}{\log N}.$$

In particular,

$$R_{ap}^{(1)}(N) = o\left(\frac{N^2}{\log N}\right).$$

For the higher-order part, we already proved that

$$R_{ap}^{(\geq 3)}(N) \ll_a N(\log\log N)^2 = o\left(\frac{N^2}{\log N}\right).$$

Therefore the asymptotic behavior of the full row sum is governed entirely by the semiprime term:

$$R_{ap}(N) = R_{ap}^{(2)}(N) + o\left(\frac{N^2}{\log N}\right).$$

Since the leading term is itself of size $\Theta_a(N^2/\log N)$, it follows that

$$R_{ap}(N) = \Theta_a\left(\frac{N^2}{\log N}\right).$$

This proves the theorem. □

7. Summary

The argument above shows that the dominant large row sums in the full meet Gram matrix are not driven by primes alone. Instead, they come from the interaction between semiprimes of the form ap and the semiprime layer $n = bq$. The key identity

$$K(ap, bq) = \min(a, b)\min(p, q)$$

reduces the semiprime contribution to an explicit two-parameter min-kernel. This semiprime layer already has the correct scale $N^2/\log N$, while the contribution from $\Omega(n) \geq 3$ is only of size $N(\log\log N)^2$ and is therefore negligible on the main scale.

In particular, the large row sums are genuinely produced by the presence of composite numbers. This is precisely the extra arithmetic structure that disappears when one restricts the theory to the prime layer alone.

38 The Young-lattice Möbius lemma and the entry-wise limit of the inverse Gram matrices

In this section we isolate a structural fact that explains the empirical stabilization of the inverse Gram matrices G_N^{-1}. The key point is that the factorization poset on $\mathbb{N}$ is naturally isomorphic to a lower ideal in the Young lattice, and that the Möbius function of the Young lattice is supported only on rook strips. This implies that, for every fixed pair of indices (i, j), the entry $(G_N^{-1})_{ij}$ is eventually constant.

38.1 The factorization poset

We work with the partial order $\prec$ on $\mathbb{N}$ defined as follows. Write

$$n = p_1 p_2 \cdots p_r, m = q_1 q_2 \cdots q_s,$$

where $p_1 \leq p_2 \leq \cdots \leq p_r$ and $q_1 \leq q_2 \leq \cdots \leq q_s$ are the prime factorizations with multiplicity written in nondecreasing order. We say that

$$n \prec m$$

if and only if $r \le s$ and

$$p_k \le q_k \qquad (1 \le k \le r).$$

Thus $n \prec m$ means that the ordered prime-factor list of n embeds coordinatewise into that of m.

For each $N \ge 1$, let

$$P_N := \{1, 2, \dots, N\}$$

with the induced order. Let E_N be the incidence matrix of P_N in the natural order $1, 2, \dots, N$:

$$(E_N)_{xy} = \begin{cases} 1, & y \prec x, \\ 0, & \text{otherwise.} \end{cases}$$

Since P_N is lower triangular in the natural order and $(E_N)_{xx} = 1$, the matrix E_N is unit lower triangular and therefore invertible over $\mathbb{Z}$. Its inverse is the Möbius matrix of the finite poset P_N.

The full meet Gram matrix G_N admits the factorization

$$G_N = E_N D_N E_N^T,$$

where $D_N = \operatorname{diag}(g(1), \dots, g(N))$ and g is the Möbius transform of the meet kernel weight function. Hence

$$G_N^{-1} = E_N^{-T} D_N^{-1} E_N^{-1}.$$

Our goal is to show that for fixed i, j, the quantity $(G_N^{-1})_{ij}$ stabilizes as $N \to \infty$.

38.2 From factorization types to partitions

Let $\pi_1 < \pi_2 < \pi_3 < \cdots$ denote the increasing sequence of primes. If

$$n = \pi_{a_1} \pi_{a_2} \cdots \pi_{a_r} \qquad (a_1 \le a_2 \le \cdots \le a_r),$$

then define

$$\lambda(n) := (a_r, a_{r-1}, \dots, a_1).$$

This is a partition, because reversing a nondecreasing sequence produces a nonincreasing one.

Lemma 7. *The map $n \mapsto \lambda(n)$ is an order embedding of $(\mathbb{N}, \prec)$ into the Young lattice Y of all partitions ordered by inclusion of Ferrers diagrams. More precisely, for all $d, n \in \mathbb{N}$,*

$$d \prec n \iff \lambda(d) \subseteq \lambda(n).$$

Proof. Write

$$d = \pi_{a_1} \cdots \pi_{a_r}, \qquad n = \pi_{b_1} \cdots \pi_{b_s},$$

with

$$a_1 \le \cdots \le a_r, \qquad b_1 \le \cdots \le b_s.$$

By definition,

$$d \prec n$$

means exactly that $r \le s$ and $a_k \le b_k$ for $1 \le k \le r$. Now

$$\lambda(d) = (a_r, \dots, a_1), \qquad \lambda(n) = (b_s, \dots, b_1).$$

In the Young lattice, inclusion of Ferrers diagrams is coordinatewise comparison of partition parts, after padding the shorter partition by trailing zeros. Thus

$$\lambda(d) \subseteq \lambda(n)$$

means precisely that $r \le s$ and

$$a_{r+1-k} \le b_{s+1-k} \qquad (1 \le k \le r).$$

Reindexing this inequality gives exactly $a_k \le b_k$ for $1 \le k \le r$. Hence the two conditions are equivalent. □

38.3 A lower-ideal property

Lemma 8. *For every $N \ge 1$, the finite set $P_N = \{1, \ldots, N\}$ is a lower ideal in $(\mathbb{N}, \prec)$. That is, if $d \prec n$ and $n \le N$, then necessarily $d \le N$.*

Proof. Write

$$d = \pi_{a_1} \cdots \pi_{a_r}, \qquad n = \pi_{b_1} \cdots \pi_{b_s},$$

with nondecreasing prime-index sequences. Since $d \prec n$, we have $r \le s$ and $a_k \le b_k$ for all $1 \le k \le r$. Therefore

$$\pi_{a_k} \le \pi_{b_k} \qquad (1 \le k \le r).$$

Hence

$$d = \prod_{k=1}^{r} \pi_{a_k} \le \prod_{k=1}^{r} \pi_{b_k} \le \prod_{k=1}^{s} \pi_{b_k} = n.$$

So $d \le n \le N$, and therefore $d \in P_N$. □

This lower-ideal property has an important consequence: if $x, y \le N$, then the interval $[x, y]$ computed inside the finite poset P_N is the same as the interval $[x, y]$ in the infinite poset $(\mathbb{N}, \prec)$. Thus the Möbius value computed in P_N agrees with the Möbius value in the ambient infinite poset.

38.4 A separate Young-lattice lemma

We now state the needed Young-lattice fact as a separate auxiliary lemma.

Definition 3. *Let $\alpha \subseteq \beta$ be partitions, viewed as Ferrers diagrams. The skew diagram β/α is called a* rook strip *if it contains at most one box in each row and at most one box in each column.*

Lemma 9 (Möbius function of the Young lattice)**.** *Let $\alpha \subseteq \beta$ be partitions in the Young lattice Y. Then*

$$\mu_Y(\alpha, \beta) = 0$$

unless the skew diagram β/α is a rook strip. In the rook-strip case,

$$\mu_Y(\alpha, \beta) = (-1)^{|\beta|-|\alpha|}.$$

Proof. We give a detailed proof by induction on

$$r := |\beta| - |\alpha|.$$

Recall that the Möbius function of any locally finite poset is defined recursively by

$$\mu_Y(\alpha, \alpha) = 1, \qquad \sum_{\alpha \subseteq \gamma \subseteq \beta} \mu_Y(\alpha, \gamma) = 0 \quad (\alpha \subsetneq \beta).$$

Equivalently,

$$\mu_Y(\alpha, \beta) = -\sum_{\alpha \subseteq \gamma \subsetneq \beta} \mu_Y(\alpha, \gamma).$$

If $r = 0$, then $\alpha = \beta$ and the formula gives

$$\mu_Y(\alpha, \alpha) = 1 = (-1)^0,$$

which is correct.

Assume now $r \geq 1$ and that the statement is already proved for all pairs with strictly smaller difference in size.

Case 1: β/α is a rook strip.

Then every intermediate partition γ with

$$\alpha \subseteq \gamma \subseteq \beta$$

is obtained by choosing a subset of the boxes of β/α. Because β/α contains at most one box in each row and each column, any subset of those boxes again forms a valid Ferrers extension of α. Hence the interval $[\alpha, \beta]$ is isomorphic to a Boolean lattice on r elements.

Indeed, list the r boxes of β/α as

$$B_1, \ldots, B_r.$$

To any intermediate partition γ we associate the subset

$$S(\gamma) := \{k : B_k \subseteq \gamma/\alpha\}.$$

This yields an order-preserving bijection between $[\alpha, \beta]$ and the Boolean lattice of all subsets of $\{1, \ldots, r\}$. The Möbius function of a Boolean lattice is classical:

$$\mu(S, T) = (-1)^{|T| - |S|}.$$

Thus

$$\mu_Y(\alpha, \beta) = (-1)^r = (-1)^{|\beta| - |\alpha|}.$$

Case 2: β/α is not a rook strip.

Then either two boxes of β/α lie in the same row, or two boxes lie in the same column. We treat the row case; the column case is symmetric. So assume there are two distinct added boxes in a common row. Let u be the *rightmost* row in which this happens, and within that row let x be the leftmost among the new boxes that still has another new box to its right. Let y be the unique new box immediately to the right of x in the same row.

Consider the involution on intermediate partitions

$$\gamma \longmapsto \gamma^*$$

defined as follows: if $x \in \gamma/\alpha$, remove x; if $x \notin \gamma/\alpha$, add x. Because of the special choice of x and y, toggling x preserves the Ferrers property for intermediate partitions between α and β. Indeed:

- if x is present and we remove it, the result is still a Ferrers shape since removing a rightmost available box from a row preserves the partition condition;
- if x is absent and we add it, then the box to its left is already present (either in α or in γ), and because y exists to the right inside β/α, the choice of the rightmost bad row ensures that adding x does not violate the nonincreasing row-length condition.

Thus $\gamma \mapsto \gamma^*$ is a well-defined sign-reversing involution on the proper interval

$$\{\gamma : \alpha \subseteq \gamma \subsetneq \beta\}$$

when weighted by the Boolean sign $(-1)^{|\gamma|-|\alpha|}$. By the induction hypothesis, only rook-strip subintervals contribute to the Möbius recursion, and these contributions cancel in pairs under the involution. Hence

$$\sum_{\alpha \subseteq \gamma \subsetneq \beta} \mu_Y(\alpha, \gamma) = 0,$$

and therefore

$$\mu_Y(\alpha, \beta) = -0 = 0.$$

This completes the induction. □

Corollary 14 (Finite support above a fixed partition)**.** *Fix a partition α. Then there are only finitely many partitions $\beta \supseteq \alpha$ such that*

$$\mu_Y(\alpha, \beta) \neq 0.$$

Proof. By the lemma, nonvanishing can occur only when β/α is a rook strip. Such a strip can add at most one box to each existing row of α, and can create at most one new row of length 1. Therefore there are only finitely many possible shapes β. □

38.5 Finite support of the Möbius columns in the factorization poset

Let $P = (\mathbb{N}, \prec)$ denote the infinite factorization poset, and let μ_P be its Möbius function. Via the embedding $n \mapsto \lambda(n)$ into the Young lattice, the previous corollary implies the following.

Lemma 10. *For each fixed $x \in \mathbb{N}$, the set*

$$S_x := \{y \in \mathbb{N} : \mu_P(x, y) \neq 0\}$$

is finite.

Proof. By the first lemma, the interval structure above x in P is isomorphic to the interval structure above $\lambda(x)$ in the Young lattice. Therefore

$$\mu_P(x, y) = \mu_Y(\lambda(x), \lambda(y)).$$

If $\mu_P(x, y) \neq 0$, then by the Young-lattice lemma the skew diagram $\lambda(y)/\lambda(x)$ must be a rook strip. By the previous corollary there are only finitely many such $\lambda(y)$, hence only finitely many such integers y. □

38.6 Entry-wise stabilization of G_N^{-1}

We now prove the main statement of this section.

Theorem 10 (Entry-wise eventually constant limit of the inverse Gram matrices)**.** *For every fixed pair $i, j \in \mathbb{N}$, the limit*

$$\lim_{N\to\infty} (G_N^{-1})_{ij}$$

exists. More precisely, there is an integer $N_0(i,j)$ such that

$$(G_N^{-1})_{ij} = (G_{N_0(i,j)}^{-1})_{ij} \qquad \text{for all } N \geq N_0(i,j).$$

Proof. Fix $i, j \in \mathbb{N}$. Since

$$G_N^{-1} = E_N^{-T} D_N^{-1} E_N^{-1},$$

we may write the (i,j)-entry as

$$(G_N^{-1})_{ij} = \sum_{k=1}^{N} (E_N^{-1})_{ki} \, \frac{1}{g(k)} \, (E_N^{-1})_{kj}.$$

Because P_N is a lower ideal of the infinite factorization poset, the Möbius value computed in P_N agrees with the global Möbius value whenever the arguments lie in P_N. Hence

$$(E_N^{-1})_{ki} = \mu_P(i,k), \qquad (E_N^{-1})_{kj} = \mu_P(j,k).$$

Therefore

$$(G_N^{-1})_{ij} = \sum_{k=1}^{N} \mu_P(i,k) \, \frac{1}{g(k)} \, \mu_P(j,k).$$

Now define

$$T_{ij} := S_i \cap S_j = \{k \in \mathbb{N} : \mu_P(i,k) \neq 0 \text{ and } \mu_P(j,k) \neq 0\}.$$

By the previous lemma, both S_i and S_j are finite, hence T_{ij} is finite. For $k \notin T_{ij}$, the summand vanishes. So in fact

$$(G_N^{-1})_{ij} = \sum_{\substack{k \leq N \\ k \in T_{ij}}} \mu_P(i,k) \, \frac{1}{g(k)} \, \mu_P(j,k).$$

Let

$$N_0(i,j) := \max T_{ij}$$

if $T_{ij} \neq \varnothing$, and let $N_0(i,j) := \max\{i,j\}$ if $T_{ij} = \varnothing$. Then for every $N \geq N_0(i,j)$, the above sum already contains all nonzero contributions, and so

$$(G_N^{-1})_{ij} = \sum_{k \in T_{ij}} \mu_P(i,k) \, \frac{1}{g(k)} \, \mu_P(j,k),$$

which is independent of N. Thus $(G_N^{-1})_{ij}$ is eventually constant, and therefore the entry-wise limit exists. □

Corollary 15. *There exists an infinite matrix*

$$G_\infty^{-1} = (a_{ij})_{i,j\geq 1}$$

given by

$$a_{ij} := \lim_{N\to\infty} (G_N^{-1})_{ij},$$

and it admits the explicit finite-sum representation

$$a_{ij} = \sum_{k\in T_{ij}} \mu_P(i,k)\,\frac{1}{g(k)}\,\mu_P(j,k).$$

Proof. This is immediate from the theorem and the definition of T_{ij}. □

38.7 Interpretation

The theorem shows that the inverse Gram matrices do not merely converge in a vague heuristic sense. Instead, each fixed matrix entry stabilizes after finitely many truncation steps. Thus the infinite object G_∞^{-1} is naturally a *local Möbius-weighted precision operator* on the factorization poset. The locality comes from the finite support of the Möbius function above each fixed element, and the arithmetic weights come from the diagonal coefficients $1/g(k)$.

This gives a rigorous explanation of the empirical phenomenon that the upper-left corner of G_N^{-1} becomes stable very quickly as N grows.

39 The infinite inverse Gram operator

We work with the factorization poset $(\mathbb{N}, \preceq)$, where for

$$m = p_1 \cdots p_r, \qquad n = q_1 \cdots q_s$$

with ordered prime factors $p_1 \leq \cdots \leq p_r$ and $q_1 \leq \cdots \leq q_s$, one defines

$$m \preceq n$$

if and only if $r \leq s$ and $p_i \leq q_i$ for all $1 \leq i \leq r$. For each $N \geq 1$, let

$$P_N := \{1, 2, \ldots, N\},$$

and let G_N denote the finite meet Gram matrix on P_N. We assume the already established factorization

$$G_N = E_N D_N E_N^T,$$

where E_N is the incidence matrix of the finite poset P_N in the natural order $1, 2, \ldots, N$, and

$$D_N = \operatorname{diag}(g(1), g(2), \ldots, g(N))$$

with $g(n) > 0$ for all n. Hence

$$G_N^{-1} = E_N^{-T} D_N^{-1} E_N^{-1}.$$

In the previous section we proved that, for every fixed $i, j \in \mathbb{N}$, the entry $(G_N^{-1})_{ij}$ is eventually constant as $N \to \infty$. Therefore the entrywise limit

$$a_{ij} := \lim_{N\to\infty} (G_N^{-1})_{ij}$$

exists. In this section we show that the resulting infinite matrix defines a natural positive symmetric operator on $\ell^2(\mathbb{N})$.

39.1 The limiting matrix

Definition 4. *Define the infinite matrix*

$$A = (a_{ij})_{i,j\geq 1}, \qquad a_{ij} := \lim_{N\to\infty} (G_N^{-1})_{ij}.$$

We denote this matrix by

$$A = G_\infty^{-1}.$$

We now establish its basic structure.

Lemma 11 (finite support of the rows and columns)**.** *For every fixed $i \in \mathbb{N}$, the set*

$$\{j \in \mathbb{N} : a_{ij} \neq 0\}$$

is finite. By symmetry, the same holds for every fixed column.

Proof. From the finite factorization

$$G_N^{-1} = E_N^{-T} D_N^{-1} E_N^{-1}$$

and the entrywise stabilization proved earlier, we know that for fixed i, j one has the formula

$$a_{ij} = \sum_{k\geq 1} \mu(i,k) \, \frac{1}{g(k)} \, \mu(j,k),$$

where $\mu(\cdot,\cdot)$ is the Möbius function of the infinite factorization poset. Moreover, this sum is in fact finite, because the Möbius function in our poset is transported from the Möbius function of the Young lattice, and the Young-lattice lemma shows that for fixed i the set

$$S_i := \{k \in \mathbb{N} : \mu(i,k) \neq 0\}$$

is finite.

Fix i. If $a_{ij} \neq 0$, then there exists at least one $k \in \mathbb{N}$ such that

$$\mu(i,k) \neq 0 \qquad \text{and} \qquad \mu(j,k) \neq 0.$$

Thus $k \in S_i$, so k belongs to a finite set. For a fixed such k, the condition $\mu(j,k) \neq 0$ implies in particular $j \preceq k$. But for fixed k, the set

$$\{j \in \mathbb{N} : j \preceq k\}$$

is finite, because every lower interval in the factorization poset is finite. Therefore

$$\{j \in \mathbb{N} : a_{ij} \neq 0\} \subseteq \bigcup_{k\in S_i} \{j \in \mathbb{N} : j \preceq k\},$$

and the right-hand side is a finite union of finite sets. Hence the row support is finite.

Since each finite matrix G_N^{-1} is symmetric, its entrywise limit A is symmetric as well. Therefore row-finiteness implies column-finiteness. □

Corollary 16. *Every row and every column of A has finite support.*

39.2 Definition of the operator on c_{00}

Definition 5. *Let*

$$c_{00}(\mathbb{N}) := \{x = (x_n)_{n\geq 1} : x_n = 0 \text{ for all but finitely many } n\}.$$

For $x \in c_{00}(\mathbb{N})$, define

$$(Ax)_i := \sum_{j=1}^{\infty} a_{ij} x_j.$$

Proposition 56 (well-definedness on $c_{00}(\mathbb{N})$)**.** *The map A is a well-defined linear operator*

$$A : c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N}).$$

Proof. Fix $x \in c_{00}(\mathbb{N})$. Since x has finite support, say contained in $\{1, \dots, M\}$, the formula for $(Ax)_i$ becomes

$$(Ax)_i = \sum_{j=1}^{M} a_{ij} x_j,$$

which is a finite sum. Hence $(Ax)_i$ is well-defined for every i.

It remains to show that Ax is again finitely supported. Let

$$J := \operatorname{supp}(x) \subset \mathbb{N}.$$

Then J is finite. If $(Ax)_i \neq 0$, then there exists some $j \in J$ with $a_{ij} \neq 0$. Thus

$$\operatorname{supp}(Ax) \subseteq \bigcup_{j\in J} \{i \in \mathbb{N} : a_{ij} \neq 0\}.$$

By Lemma 11, each set on the right-hand side is finite, and since J is finite, the union is finite. Therefore $Ax \in c_{00}(\mathbb{N})$.

Linearity is immediate from the linearity of matrix multiplication. □

Corollary 17. *The operator A is densely defined on $\ell^2(\mathbb{N})$, because $c_{00}(\mathbb{N})$ is dense in $\ell^2(\mathbb{N})$.*

39.3 Symmetry and positivity

Proposition 57 (symmetry)**.** *The operator A is symmetric on $c_{00}(\mathbb{N})$; that is,*

$$\langle Ax, y\rangle = \langle x, Ay\rangle \qquad (x, y \in c_{00}(\mathbb{N})).$$

Proof. For every N, the finite matrix G_N^{-1} is symmetric. Passing to the entrywise limit gives

$$a_{ij} = a_{ji} \qquad (i, j \geq 1).$$

Now let $x, y \in c_{00}(\mathbb{N})$. Since both vectors are finitely supported and each row/column of A is finitely supported, all sums below are finite, so rearrangement is legitimate:

$$\langle Ax, y\rangle = \sum_{i\geq 1} (Ax)_i \overline{y_i} = \sum_{i\geq 1} \sum_{j\geq 1} a_{ij} x_j \overline{y_i} = \sum_{j\geq 1} x_j \sum_{i\geq 1} a_{ji} \overline{y_i} = \langle x, Ay\rangle.$$

Hence A is symmetric. □

Proposition 58 (positivity)**.** *The operator A is positive on $c_{00}(\mathbb{N})$; namely,*

$$\langle Ax, x\rangle \geq 0 \qquad (x \in c_{00}(\mathbb{N})).$$

Proof. Fix $x \in c_{00}(\mathbb{N})$, and let $J := \operatorname{supp}(x)$. Since J is finite and each entry a_{ij} stabilizes eventually, there exists N_0 such that for all $i, j \in J$ and all $N \geq N_0$ one has

$$(G_N^{-1})_{ij} = a_{ij}.$$

Therefore, for all $N \geq N_0$,

$$\langle Ax, x\rangle = \sum_{i,j\in J} a_{ij} x_j \overline{x_i} = \sum_{i,j\in J} (G_N^{-1})_{ij} x_j \overline{x_i} = \langle G_N^{-1} x, x\rangle.$$

Since G_N^{-1} is positive definite as the inverse of a positive definite Gram matrix, we obtain

$$\langle Ax, x\rangle = \langle G_N^{-1} x, x\rangle \geq 0.$$

Hence A is positive on $c_{00}(\mathbb{N})$. □

Remark 25. The preceding proof also shows that the quadratic form of A on finitely supported vectors is the eventual stabilization of the finite quadratic forms of G_N^{-1}.

39.4 Closability and selfadjoint extensions

Proposition 59 (closability)**.** *The operator A is closable on $\ell^2(\mathbb{N})$.*

Proof. This is a standard fact from operator theory: every densely defined symmetric operator on a Hilbert space is closable. By Corollary 3.4 and Proposition 57, A is densely defined and symmetric on $\ell^2(\mathbb{N})$. Hence A is closable. □

Theorem 11 (existence as an infinite operator)**.** *The entrywise limit matrix*

$$A = G_\infty^{-1}$$

defines a densely defined linear operator on $\ell^2(\mathbb{N})$ with domain $c_{00}(\mathbb{N})$. This operator is symmetric, positive, and closable. Moreover, it admits a positive selfadjoint Friedrichs extension.

Proof. By Proposition 56, A defines a linear operator on $c_{00}(\mathbb{N})$, and $c_{00}(\mathbb{N})$ is dense in $\ell^2(\mathbb{N})$. By Proposition 57, the operator is symmetric, and by Proposition 58, it is positive. By Proposition 59, it is closable.

It remains to justify the existence of a positive selfadjoint extension. Since A is densely defined, symmetric, and semibounded below by 0, the Friedrichs extension theorem applies. Hence A admits a positive selfadjoint extension, namely its Friedrichs extension. □

39.5 Interpretation

The finite inverse Gram matrices satisfy

$$G_N^{-1} = E_N^{-T} D_N^{-1} E_N^{-1},$$

where E_N^{-1} is the Möbius matrix of the finite factorization poset and D_N^{-1} carries the reciprocal arithmetic weights $1/g(n)$ on the diagonal. The infinite operator $A = G_\infty^{-1}$ therefore has the natural interpretation of a *Möbius-weighted precision operator* attached to the infinite factorization poset.

This should be compared with the prime-subkernel situation, where the inverse finite Gram matrix becomes a weighted grounded path Laplacian. In the present setting, the underlying combinatorial object is no longer a chain but the full factorization poset on $\mathbb{N}$. Thus the limiting operator is best viewed as an *arithmetic poset Laplacian* or, equivalently, as the precision operator associated with the meet-kernel Green matrix.

Remark 26 (what is proved and what remains open). Theorem 11 does *not* assert that the minimal operator A on $c_{00}(\mathbb{N})$ is essentially selfadjoint. What is proved is:

1. A exists as a densely defined positive symmetric operator;
2. A is closable;
3. A has a canonical positive selfadjoint Friedrichs extension.

The question whether the minimal operator is already essentially selfadjoint remains open at this stage.

Theorem 12 (Existence of the infinite Gram kernel as a Green operator)**.** *Let*

$$K(m,n) := m \wedge n \qquad (m,n \in \mathbb{N}),$$

be the meet-kernel on the factorization poset, and let

$$G_N := \big(K(i,j)\big)_{1\le i,j\le N}$$

be the corresponding finite Gram matrices. Assume that the inverse matrices

$$G_N^{-1} = \big(a_{ij}^{(N)}\big)_{1\le i,j\le N}$$

stabilize entrywise, i.e. for every fixed $i,j \in \mathbb{N}$ the limit

$$a_{ij} := \lim_{N\to\infty} a_{ij}^{(N)}$$

exists, and let

$$A = (a_{ij})_{i,j\ge 1}.$$

Assume moreover that every row and every column of A has finite support, so that

$$A : c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N})$$

is a well-defined linear operator.

Define, for $x = (x_j)_{j\ge1} \in c_{00}(\mathbb{N})$,

$$(G_\infty x)_i := \sum_{j\ge1} K(i,j)x_j, \qquad i \ge 1.$$

Then the following hold:

1. *$G_\infty : c_{00}(\mathbb{N}) \to \mathbb{C}^{\mathbb{N}}$ is a well-defined linear operator.*
2. *G_∞ is positive in the kernel sense, i.e.*

$$\sum_{i,j\ge1} \overline{x_i}\, K(i,j)\, x_j \ge 0 \qquad \textit{for all } x \in c_{00}(\mathbb{N}).$$

3. G_∞ is a two-sided algebraic inverse of A on $c_{00}(\mathbb{N})$, i.e.

$$AG_\infty x = x \qquad \text{and} \qquad G_\infty A x = x \qquad \text{for all } x \in c_{00}(\mathbb{N}).$$

In particular, the infinite Gram matrix $G_\infty = (K(i,j))_{i,j\geq 1}$ exists canonically as a Green operator on finitely supported vectors, and A is its algebraic inverse on $c_{00}(\mathbb{N})$.

Proof. We proceed in several steps.

Step 1: Well-definedness of G_∞. Let $x \in c_{00}(\mathbb{N})$. Then x has finite support, say

$$J := \operatorname{supp}(x) \subset \mathbb{N}, \qquad |J| < \infty.$$

Hence for every fixed $i \geq 1$,

$$(G_\infty x)_i = \sum_{j\geq 1} K(i,j)x_j = \sum_{j\in J} K(i,j)x_j,$$

which is a finite sum. Therefore $G_\infty x$ is well defined as an element of $\mathbb{C}^{\mathbb{N}}$. Linearity is immediate from the definition.

Step 2: Positivity of G_∞. Let $x \in c_{00}(\mathbb{N})$, and again let $J = \operatorname{supp}(x)$. Choose $N \in \mathbb{N}$ so large that $J \subset \{1, \ldots, N\}$. Then

$$\sum_{i,j\geq 1} \overline{x_i}\, K(i,j)\, x_j = \sum_{i,j\in J} \overline{x_i}\, K(i,j)\, x_j.$$

Since $K(i,j) = (G_N)_{ij}$ for all $1 \leq i,j \leq N$, the right-hand side equals

$$x^* G_N x,$$

where we view x as a vector in $\mathbb{C}^N$ by padding with zeros outside J. Because G_N is a finite Gram matrix, it is positive semidefinite. Therefore

$$x^* G_N x \geq 0.$$

This proves

$$\sum_{i,j\geq 1} \overline{x_i}\, K(i,j)\, x_j \geq 0 \qquad (x \in c_{00}(\mathbb{N})).$$

Step 3: Proof of $AG_\infty x = x$. Fix $x \in c_{00}(\mathbb{N})$, and write $J := \operatorname{supp}(x)$. Let $i \geq 1$ be arbitrary. Since the i-th row of A has finite support, the set

$$S_i := \{j \in \mathbb{N} : a_{ij} \neq 0\}$$

is finite. Thus

$$(AG_\infty x)_i = \sum_{j\geq 1} a_{ij}(G_\infty x)_j = \sum_{j\in S_i} a_{ij}(G_\infty x)_j = \sum_{j\in S_i} a_{ij} \sum_{k\in J} K(j,k)x_k.$$

Since both S_i and J are finite, we may interchange the sums:

$$(AG_\infty x)_i = \sum_{k\in J} \left(\sum_{j\in S_i} a_{ij} K(j,k) \right) x_k.$$

Now choose N so large that

$$N \geq \max\bigl(\{i\} \cup J \cup S_i\bigr),$$

and, in addition, large enough so that

$$a_{ij} = a_{ij}^{(N)} \qquad \text{for all } j \in S_i.$$

This is possible because the entries $a_{ij}^{(N)}$ stabilize for every fixed pair (i, j).

For $j \in S_i$ and $k \in J$, we then have $j, k \leq N$, hence

$$K(j,k) = (G_N)_{jk}, \qquad a_{ij} = a_{ij}^{(N)} = (G_N^{-1})_{ij}.$$

Therefore

$$\sum_{j \in S_i} a_{ij} K(j,k) = \sum_{j \in S_i} (G_N^{-1})_{ij} (G_N)_{jk}.$$

Since $(G_N^{-1})_{ij} = 0$ for $j \notin S_i$, this equals

$$\sum_{j=1}^{N} (G_N^{-1})_{ij} (G_N)_{jk} = \delta_{ik}.$$

Substituting this into the previous expression gives

$$(AG_\infty x)_i = \sum_{k \in J} \delta_{ik} x_k = x_i.$$

Since i was arbitrary, we conclude that

$$AG_\infty x = x \qquad \text{for all } x \in c_{00}(\mathbb{N}).$$

Step 4: Proof of $G_\infty Ax = x$. Let $x \in c_{00}(\mathbb{N})$, and set

$$y := Ax.$$

By assumption, $A : c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N})$, so $y \in c_{00}(\mathbb{N})$. Let

$$I := \operatorname{supp}(y), \qquad |I| < \infty.$$

Fix $i \geq 1$. Then

$$(G_\infty Ax)_i = (G_\infty y)_i = \sum_{j \in I} K(i,j) y_j.$$

Since $y = Ax$, we have

$$y_j = \sum_{k \geq 1} a_{jk} x_k.$$

Let $J := \operatorname{supp}(x)$. Because x has finite support, this reduces to

$$y_j = \sum_{k \in J} a_{jk} x_k.$$

Hence

$$(G_\infty Ax)_i = \sum_{j \in I} K(i,j) \sum_{k \in J} a_{jk} x_k = \sum_{k \in J} \left(\sum_{j \in I} K(i,j) a_{jk} \right) x_k.$$

Choose N so large that

$$N \geq \max\big(\{i\} \cup I \cup J\big),$$

and large enough so that

$$a_{jk} = a_{jk}^{(N)} \qquad \text{for all } j \in I,\ k \in J.$$

Again this is possible by entrywise stabilization.

For $j \in I$, $k \in J$, we then have

$$K(i,j) = (G_N)_{ij}, \qquad a_{jk} = a_{jk}^{(N)} = (G_N^{-1})_{jk}.$$

Therefore

$$\sum_{j \in I} K(i,j) a_{jk} = \sum_{j \in I} (G_N)_{ij} (G_N^{-1})_{jk}.$$

Since $a_{jk} = 0$ for $j \notin I$, equivalently $y_j = 0$ outside I, we may extend the sum to all $1 \leq j \leq N$:

$$\sum_{j \in I} (G_N)_{ij} (G_N^{-1})_{jk} = \sum_{j=1}^{N} (G_N)_{ij} (G_N^{-1})_{jk} = \delta_{ik}.$$

Thus

$$(G_\infty A x)_i = \sum_{k \in J} \delta_{ik} x_k = x_i.$$

Since i was arbitrary, it follows that

$$G_\infty A x = x \qquad \text{for all } x \in c_{00}(\mathbb{N}).$$

Combining Steps 3 and 4, we conclude that G_∞ is a two-sided algebraic inverse of A on $c_{00}(\mathbb{N})$. This proves the theorem. □

Remark 27. The theorem does not assert that G_∞ defines an operator on $\ell^2(\mathbb{N})$. What it shows is that the infinite meet Gram matrix exists canonically as a Green object on the core $c_{00}(\mathbb{N})$, where it is the algebraic inverse of the finitely supported Möbius-weighted precision operator A.

Proposition 60 (Feature embedding, infinite Gram kernel, and precision operator)**.** *Let* $(\mathbb{N}, \preceq)$ *be the factorization poset, let*

$$K(m,n) = \sum_{d \preceq m,\ d \preceq n} g(d) = m \wedge n, \qquad m, n \in \mathbb{N},$$

and assume that $g(d) \in \mathbb{Z}_{>0}$ *for all* $d \in \mathbb{N}$.

For every $d \in \mathbb{N}$ *and* $1 \leq j \leq g(d)$, *let* $e_{d,j}$ *denote the standard basis vectors of*

$$\ell^2(\mathcal{D}), \qquad \mathcal{D} := \{(d,j) : d \in \mathbb{N},\ 1 \leq j \leq g(d)\}.$$

Define the feature map

$$\Phi : \mathbb{N} \to \ell^2(\mathcal{D}), \qquad \Phi(n) := \sum_{d \preceq n} \sum_{j=1}^{g(d)} e_{d,j}.$$

Then the following hold.

1. *For every* $n \in \mathbb{N}$, *the vector* $\Phi(n)$ *is well defined and finitely supported.*

2. *The kernel K is the Gram kernel of the family $(\Phi(n))_{n\geq 1}$, i.e.*

$$\langle \Phi(m), \Phi(n)\rangle = K(m,n) \qquad (m,n \in \mathbb{N}).$$

3. *The infinite Gram matrix*

$$G_\infty := \big(K(i,j)\big)_{i,j\geq 1}$$

therefore exists canonically as a positive kernel on $\mathbb{N}$.

4. *For $x \in c_{00}(\mathbb{N})$, the formula*

$$(G_\infty x)_i := \sum_{j\geq 1} K(i,j)x_j$$

defines a linear map

$$G_\infty : c_{00}(\mathbb{N}) \to \mathbb{C}^{\mathbb{N}}.$$

5. *If $A = (a_{ij})_{i,j\geq 1}$ is the limiting inverse matrix constructed from the stabilized entries*

$$a_{ij} = \lim_{N\to\infty} (G_N^{-1})_{ij},$$

and if $A : c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N})$ is the corresponding finitely supported operator, then A is the algebraic precision operator associated with the embedding Φ, whereas G_∞ is the corresponding Green operator on $c_{00}(\mathbb{N})$.

Proof. For fixed $n \in \mathbb{N}$, the set

$$\{d \in \mathbb{N} : d \preceq n\}$$

is finite, since every lower interval in the factorization poset is finite. Hence the defining sum

$$\Phi(n) = \sum_{d\preceq n} \sum_{j=1}^{g(d)} e_{d,j}$$

contains only finitely many terms, so $\Phi(n) \in \ell^2(\mathcal{D})$ and has finite support.

Now let $m, n \in \mathbb{N}$. Using orthonormality of the basis vectors $e_{d,j}$, we obtain

$$\langle \Phi(m), \Phi(n)\rangle = \sum_{d} \sum_{j=1}^{g(d)} \mathbf{1}_{\{d\preceq m\}} \mathbf{1}_{\{d\preceq n\}} = \sum_{d\preceq m,\ d\preceq n} g(d) = K(m,n).$$

This proves that K is the Gram kernel of the family $(\Phi(n))_{n\geq 1}$. In particular,

$$G_\infty = (K(i,j))_{i,j\geq 1}$$

is a well-defined positive kernel.

Next let $x \in c_{00}(\mathbb{N})$, and write $J = \operatorname{supp}(x)$. Then for every $i \geq 1$,

$$(G_\infty x)_i = \sum_{j\geq 1} K(i,j)x_j = \sum_{j\in J} K(i,j)x_j,$$

which is a finite sum. Hence $G_\infty x$ is well defined as an element of $\mathbb{C}^{\mathbb{N}}$, and linearity is immediate.

Finally, the limiting matrix

$$A = (a_{ij})_{i,j\geq 1}, \qquad a_{ij} = \lim_{N\to\infty} (G_N^{-1})_{ij},$$

is constructed in the manuscript as the stabilized inverse Gram matrix, and its rows and columns have finite support, so it defines an operator

$$A : c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N}).$$

Thus A is the inverse object attached to the Gram kernel generated by Φ: G_∞ is the global Gram/Green object of the embedding, and A is its local precision operator on the core $c_{00}(\mathbb{N})$. $\square$

Proposition 61. *Let*

$$p_1 = 2 < p_2 = 3 < \cdots$$

be the primes, and set

$$\Delta_k := p_k - p_{k-1} \qquad (k \geq 1,\ p_0 := 1).$$

If

$$i_1 < i_2 < \cdots < i_r,$$

then

$$g(p_{i_1} p_{i_2} \cdots p_{i_r}) = \prod_{m=1}^{r} g(p_{i_m}) = \prod_{m=1}^{r} \Delta_{i_m}.$$

Proof. We prove the stronger cumulative identity

$$H_r(i_1, \ldots, i_r) := \sum_{\substack{1 \leq a_1 \leq \cdots \leq a_r \\ a_m \leq i_m}} g(p_{a_1} \cdots p_{a_r}) = p_{i_1} \cdots p_{i_{r-1}}(p_{i_r} - 1) \tag{$*$}$$

for every strictly increasing tuple $i_1 < \cdots < i_r$.

For $r = 1$, the predecessors of p_i are exactly

$$1, p_1, \ldots, p_{i-1},$$

hence Möbius inversion gives

$$p_i = g(1) + \sum_{a \leq i} g(p_a) = 1 + \sum_{a \leq i} g(p_a),$$

so

$$H_1(i) = \sum_{a \leq i} g(p_a) = p_i - 1.$$

Thus $(*)$ holds for $r = 1$.

Now let $r \geq 2$, and set

$$n_r := p_{i_1} \cdots p_{i_r}, \qquad n_{r-1} := p_{i_1} \cdots p_{i_{r-1}}.$$

By definition of the factorization-poset order, the predecessors $d \preceq n_r$ having exactly r prime factors are precisely the numbers

$$d = p_{a_1} \cdots p_{a_r} \quad \text{with} \quad 1 \leq a_1 \leq \cdots \leq a_r, \quad a_m \leq i_m.$$

Therefore their total g-contribution is $H_r(i_1, \ldots, i_r)$.

The predecessors of n_r having fewer than r prime factors are exactly the predecessors of n_{r-1}. Hence Möbius inversion yields

$$\sum_{\substack{d \preceq n_r \\ \Omega(d) \le r-1}} g(d) = n_{r-1}, \qquad \sum_{d \preceq n_r} g(d) = n_r.$$

Subtracting, we obtain

$$H_r(i_1, \ldots, i_r) = n_r - n_{r-1} = p_{i_1} \cdots p_{i_{r-1}}(p_{i_r} - 1),$$

which proves $(*)$.

Now define

$$h_r(i_1, \ldots, i_r) := g(p_{i_1} \cdots p_{i_r}) \qquad (i_1 < \cdots < i_r).$$

Since H_r is the cumulative sum of the h_r-terms over the simplex

$$1 \le a_1 \le \cdots \le a_r, \qquad a_m \le i_m,$$

successive backward differences isolate the top term:

$$h_r(i_1, \ldots, i_r) = (\nabla_1 \cdots \nabla_r H_r)(i_1, \ldots, i_r),$$

where

$$(\nabla_m F)(i_1, \ldots, i_r) = F(i_1, \ldots, i_m, \ldots, i_r) - F(i_1, \ldots, i_m - 1, \ldots, i_r).$$

Applying this to $(*)$ gives

$$h_r(i_1, \ldots, i_r) = \prod_{m=1}^{r} (p_{i_m} - p_{i_m - 1}) = \prod_{m=1}^{r} \Delta_{i_m}.$$

Finally, since on the prime layer

$$g(p_{i_m}) = p_{i_m} - p_{i_m-1} = \Delta_{i_m},$$

we conclude that

$$g(p_{i_1} \cdots p_{i_r}) = \prod_{m=1}^{r} g(p_{i_m}).$$

□

Proposition 62 (Prime powers)**.** *Let*

$$p_1 = 2 < p_2 = 3 < p_3 = 5 < \cdots$$

be the primes, and set

$$\Delta_i := p_i - p_{i-1} \qquad (i \ge 1,\ p_0 := 1).$$

Then

$$g(2^k) = 2^{k-1} \qquad (k \ge 1),$$

and for every $i \ge 2$ and every $k \ge 2$,

$$g(p_i^k) = \Delta_i\,(p_i - 1)\,p_i^{k-2}.$$

Equivalently,

$$g(p_i) = \Delta_i, \qquad g(p_i^k) = g(p_i)\,(p_i - 1)\,p_i^{k-2} \qquad (i \ge 2,\ k \ge 2).$$

Proof. We first treat the prime 2, and then the odd primes.

Step 1: The case $p = 2$. By the doubling identity proved earlier,

$$g(2n) = 2g(n) \qquad (n > 1).$$

Applying this recursively gives

$$g(2^k) = 2g(2^{k-1}) = 2^{k-1}g(2).$$

Since

$$g(2) = 2 - g(1) = 1,$$

it follows that

$$g(2^k) = 2^{k-1} \qquad (k \geq 1).$$

Step 2: A cumulative identity. Fix $r \geq 1$, and let

$$i_1 \leq i_2 \leq \cdots \leq i_r$$

be weakly increasing indices. Define

$$S(i_1, \ldots, i_r) := \sum_{\substack{1 \leq a_1 \leq \cdots \leq a_r \\ a_m \leq i_m}} g(p_{a_1} \cdots p_{a_r}).$$

We claim that

$$S(i_1, \ldots, i_r) = p_{i_1} \cdots p_{i_{r-1}}(p_{i_r} - 1). \qquad (*)$$

Indeed, let

$$n := p_{i_1} \cdots p_{i_r}.$$

By definition of the factorization-poset order, the predecessors $d \preceq n$ having exactly r prime factors are precisely the numbers

$$d = p_{a_1} \cdots p_{a_r} \quad \text{with} \quad 1 \leq a_1 \leq \cdots \leq a_r, \quad a_m \leq i_m.$$

Hence their total g-contribution is exactly $S(i_1, \ldots, i_r)$.

On the other hand, the predecessors of n having fewer than r prime factors are precisely the predecessors of

$$p_{i_1} \cdots p_{i_{r-1}}.$$

Therefore Möbius inversion gives

$$\sum_{\substack{d \preceq n \\ \Omega(d) \leq r-1}} g(d) = p_{i_1} \cdots p_{i_{r-1}}, \qquad \sum_{d \preceq n} g(d) = n = p_{i_1} \cdots p_{i_r}.$$

Subtracting these identities yields

$$S(i_1, \ldots, i_r) = p_{i_1} \cdots p_{i_r} - p_{i_1} \cdots p_{i_{r-1}} = p_{i_1} \cdots p_{i_{r-1}}(p_{i_r} - 1),$$

which proves $(*)$.

Step 3: Isolation of the prime-power term. Now fix $i \geq 2$ and $k \geq 2$. Applying $(*)$ to the k-tuple $(i, \ldots, i)$ gives

$$S(\underbrace{i, \ldots, i}_{k}) = p_i^{k-1}(p_i - 1).$$

Likewise,

$$S(i-1,\underbrace{i,\dots,i}_{k-1}) = p_{i-1}\,p_i^{k-2}(p_i-1).$$

We now observe that the difference

$$S(\underbrace{i,\dots,i}_{k}) - S(i-1,\underbrace{i,\dots,i}_{k-1})$$

isolates exactly the single term $g(p_i^k)$. Indeed, in the simplex

$$1 \le a_1 \le \cdots \le a_k, \qquad a_m \le i,$$

the condition $a_1 = i$ forces

$$a_1 = \cdots = a_k = i.$$

Hence the only term present in $S(i,\dots,i)$ but absent from $S(i-1,i,\dots,i)$ is $g(p_i^k)$. Therefore

$$g(p_i^k) = S(\underbrace{i,\dots,i}_{k}) - S(i-1,\underbrace{i,\dots,i}_{k-1}).$$

Substituting the two formulas above gives

$$g(p_i^k) = p_i^{k-1}(p_i-1) - p_{i-1}p_i^{k-2}(p_i-1).$$

Factoring out $p_i^{k-2}(p_i-1)$, we obtain

$$g(p_i^k) = (p_i - p_{i-1})(p_i-1)p_i^{k-2} = \Delta_i\,(p_i-1)\,p_i^{k-2}.$$

Since on the prime layer $g(p_i) = p_i - p_{i-1} = \Delta_i$, this is equivalently

$$g(p_i^k) = g(p_i)\,(p_i-1)\,p_i^{k-2}.$$

This completes the proof. □

Remark 28. For odd primes p_i, the prime-power values grow geometrically:

$$g(p_i^k) = g(p_i^2)\,p_i^{k-2} \qquad (k \ge 2),$$

with

$$g(p_i^2) = (p_i - p_{i-1})(p_i - 1).$$

Thus the entire p_i-power tower is determined by the single prime-gap value $g(p_i) = p_i - p_{i-1}$.

40 Formula for g on natural numbers

Proposition 63 (General formula for prime blocks)**.** *Let*

$$n = p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_s}^{\alpha_s}, \qquad i_1 < i_2 < \cdots < i_s, \quad \alpha_j \ge 1,$$

and set

$$\Delta_i := p_i - p_{i-1} \qquad (i \ge 1,\ p_0 := 1).$$

Then

$$g(n) = \left(\prod_{j=1}^{s-1} \Delta_{i_j}\, p_{i_j}^{\alpha_j - 1}\right) \cdot \begin{cases} \Delta_{i_s}, & \alpha_s = 1, \\ \Delta_{i_s}(p_{i_s}-1)p_{i_s}^{\alpha_s-2}, & \alpha_s \ge 2. \end{cases}$$

Proof. Write

$$r := \alpha_1 + \cdots + \alpha_s,$$

and consider the weakly increasing index tuple

$$\mathbf{i} := (\underbrace{i_1, \ldots, i_1}_{\alpha_1}, \underbrace{i_2, \ldots, i_2}_{\alpha_2}, \ldots, \underbrace{i_s, \ldots, i_s}_{\alpha_s}).$$

For a weakly increasing r-tuple

$$j_1 \le j_2 \le \cdots \le j_r$$

define

$$h(j_1, \ldots, j_r) := g(p_{j_1} \cdots p_{j_r}),$$

and its cumulative sum

$$S(j_1, \ldots, j_r) := \sum_{\substack{1 \le a_1 \le \cdots \le a_r \\ a_m \le j_m}} h(a_1, \ldots, a_r).$$

Exactly as in the squarefree and prime-power cases, Möbius inversion on the factorization poset gives

$$S(j_1, \ldots, j_r) = p_{j_1} \cdots p_{j_{r-1}}(p_{j_r} - 1). \tag{1}$$

Now let

$$\beta_t := \alpha_1 + \cdots + \alpha_t, \qquad \beta_0 := 0,$$

so that the first coordinate of the t-th prime block is at position

$$m_t := \beta_{t-1} + 1.$$

We claim that

$$h(\mathbf{i}) = (\nabla_{m_1} \nabla_{m_2} \cdots \nabla_{m_s} S)(\mathbf{i}), \tag{2}$$

where ∇_m denotes backward difference in the m-th coordinate:

$$(\nabla_m F)(j_1, \ldots, j_r) = F(j_1, \ldots, j_m, \ldots, j_r) - F(j_1, \ldots, j_m - 1, \ldots, j_r).$$

Indeed, taking the backward difference at the first coordinate of a block forces that block to attain its maximal value. Since the tuple is weakly increasing, once the first coordinate in the block equals i_t, all subsequent coordinates in that block are also forced to equal i_t, because they are both $\ge i_t$ and $\le i_t$. Applying this successively for all s blocks isolates exactly the top term

$$h(\mathbf{i}) = g\big(p_{i_1}^{\alpha_1} \cdots p_{i_s}^{\alpha_s}\big),$$

which proves (2).

It remains to evaluate these block differences on the explicit formula (1).

For each block $t < s$, the variable i_t occurs in (1) through the factor

$$p_{i_t}^{\alpha_t}.$$

Therefore the corresponding backward difference contributes

$$p_{i_t}^{\alpha_t} - p_{i_t - 1} p_{i_t}^{\alpha_t - 1} = (p_{i_t} - p_{i_t - 1}) p_{i_t}^{\alpha_t - 1} = \Delta_{i_t} p_{i_t}^{\alpha_t - 1}.$$

For the last block $t = s$, there are two cases.

If $\alpha_s = 1$, then the last block contributes only the final factor

$$p_{i_s} - 1,$$

so the last backward difference gives

$$(p_{i_s} - 1) - (p_{i_s - 1} - 1) = p_{i_s} - p_{i_s - 1} = \Delta_{i_s}.$$

If $\alpha_s \geq 2$, then the last block contributes

$$p_{i_s}^{\alpha_s - 1}(p_{i_s} - 1),$$

and the last backward difference gives

$$p_{i_s}^{\alpha_s - 1}(p_{i_s} - 1) - p_{i_s - 1} p_{i_s}^{\alpha_s - 2}(p_{i_s} - 1) = (p_{i_s} - p_{i_s - 1})(p_{i_s} - 1) p_{i_s}^{\alpha_s - 2} = \Delta_{i_s}(p_{i_s} - 1) p_{i_s}^{\alpha_s - 2}.$$

Multiplying the block contributions proves the claimed formula. □

Corollary 18. *For*

$$n = p_{i_1}^{\alpha_1} \cdots p_{i_s}^{\alpha_s}, \qquad i_1 < \cdots < i_s,$$

the value $g(n)$ depends only on the prime gaps $\Delta_{i_j} = p_{i_j} - p_{i_j - 1}$, the exponents α_j, and the prime values p_{i_j}, via the explicit block formula of Proposition 63.

41 A Reduction of the Equation $g(n) = z$ to Prime-Gap Patterns

In this section we show that the equation

$$g(n) = z$$

for a fixed integer $z \geq 1$ can be reduced, up to finitely many exceptional cases, to a finite collection of prime-gap pattern problems on odd squarefree integers.

Recall first that

$$g(2n) = 2g(n) \qquad (n > 1),$$

and that for odd primes one has

$$g(p_i) = p_i - p_{i-1} =: \Delta_i, \qquad i \geq 2.$$

Moreover, for

$$n = p_{i_1}^{\alpha_1} \cdots p_{i_s}^{\alpha_s}, \qquad i_1 < \cdots < i_s, \quad \alpha_j \geq 1,$$

Proposition 63 gives the explicit formula

$$g(n) = \left(\prod_{j=1}^{s-1} \Delta_{i_j} p_{i_j}^{\alpha_j - 1} \right) \cdot \begin{cases} \Delta_{i_s}, & \alpha_s = 1, \\ \Delta_{i_s}(p_{i_s} - 1) p_{i_s}^{\alpha_s - 2}, & \alpha_s \geq 2. \end{cases}$$

The next result shows that, for fixed z, any infinite family of solutions to $g(n) = z$ must eventually come from odd squarefree integers, and hence from factorizations of z into prime gaps.

Theorem 13. *Fix $z \in \mathbb{N}$. Then the following hold.*

1. *There are only finitely many integers $n \in \mathbb{N}$ with $g(n) = z$ for which some odd prime occurs in n with exponent at least 2.*

2. *Consequently, if the set*
$$\{n \in \mathbb{N} : g(n) = z\}$$
is infinite, then there exists an integer a with $0 \le a \le v_2(z)$ such that there are infinitely many odd squarefree integers m satisfying
$$n = 2^a m, \qquad g(m) = \frac{z}{2^a}.$$

3. *Equivalently, if $\{n : g(n) = z\}$ is infinite, then for some $a \in \{0, \dots, v_2(z)\}$ there exist infinitely many odd squarefree integers*
$$m = p_{i_1} \cdots p_{i_r} \qquad (2 \le i_1 < \cdots < i_r)$$
such that
$$\prod_{j=1}^{r} \Delta_{i_j} = \frac{z}{2^a}.$$

Proof. Let $n \in \mathbb{N}$ satisfy $g(n) = z$, and write
$$n = 2^a m, \qquad m \text{ odd}.$$
If $m = 1$, then $n = 2^a$, so there is at most one such n for each a. Hence the only interesting case is $m > 1$.

By repeated application of the doubling identity $g(2n) = 2g(n)$, we obtain
$$g(n) = g(2^a m) = 2^a g(m),$$
and therefore
$$2^a \mid z.$$
In particular,
$$0 \le a \le v_2(z),$$
so only finitely many values of a are possible.

Now factor the odd part m as
$$m = p_{i_1}^{\alpha_1} \cdots p_{i_s}^{\alpha_s}, \qquad 2 \le i_1 < \cdots < i_s, \quad \alpha_j \ge 1.$$
By Proposition 63,
$$g(m) = \left(\prod_{j=1}^{s-1} \Delta_{i_j} p_{i_j}^{\alpha_j - 1} \right) \cdot \begin{cases} \Delta_{i_s}, & \alpha_s = 1, \\ \Delta_{i_s}(p_{i_s} - 1) p_{i_s}^{\alpha_s - 2}, & \alpha_s \ge 2. \end{cases} \tag{$*$}$$

We first prove (1). Suppose that some odd prime occurs in m with exponent at least 2. Then one of the following happens.

Case 1: $\alpha_j \ge 2$ for some $j < s$.

Then the factor $p_{i_j}^{\alpha_j-1}$ appears explicitly in $(*)$, hence

$$p_{i_j} \mid g(m).$$

Since $g(n) = 2^a g(m) = z$, it follows that

$$p_{i_j} \mid z.$$

Thus p_{i_j} must be one of the finitely many odd prime divisors of z. Moreover,

$$p_{i_j}^{\alpha_j-1} \le g(m) \le z,$$

so α_j is bounded as well.

Case 2: $\alpha_s \ge 2$.

Then $(*)$ contains the factor

$$\Delta_{i_s}(p_{i_s} - 1)p_{i_s}^{\alpha_s-2}.$$

In particular,

$$p_{i_s}^{\alpha_s-2}(p_{i_s} - 1) \le g(m) \le z.$$

Since $p_{i_s} - 1 \ge p_{i_s}/2$ for odd primes $p_{i_s} \ge 3$, we obtain

$$p_{i_s}^{\alpha_s-1} \ll z,$$

hence both p_{i_s} and α_s are bounded in terms of z.

Thus in either case every odd prime appearing with exponent at least 2 is drawn from a finite set of primes and can occur only with bounded exponent. Therefore only finitely many such odd integers m can occur, and since only finitely many values of a are possible, there are only finitely many integers n with $g(n) = z$ for which some odd prime appears with exponent at least 2. This proves (1).

For (2), assume now that the set

$$\{n \in \mathbb{N} : g(n) = z\}$$

is infinite. By (1), all but finitely many such n have odd part m squarefree. Since only finitely many values of $a \in \{0, \dots, v_2(z)\}$ are possible, the pigeonhole principle implies that for at least one fixed a there are infinitely many solutions of the form

$$n = 2^a m, \qquad m \text{ odd and squarefree.}$$

For these,

$$z = g(n) = 2^a g(m),$$

hence

$$g(m) = \frac{z}{2^a}.$$

This proves (2).

Finally, if m is odd squarefree, then

$$m = p_{i_1} \cdots p_{i_r} \qquad (2 \le i_1 < \cdots < i_r),$$

and the squarefree product formula yields

$$g(m) = \prod_{j=1}^{r} g(p_{i_j}) = \prod_{j=1}^{r} \Delta_{i_j}.$$

Substituting this into (2) proves (3). $\square$

Remark 29. Theorem 13 shows that the infinitude problem for a fixed value z is, up to finitely many exceptional nonsquarefree cases, a finite prime-gap pattern problem. More precisely, for each fixed z, the question whether

$$g(n) = z \quad \text{for infinitely many } n$$

reduces to asking whether one of finitely many factorizations

$$\frac{z}{2^a} = d_1 \cdots d_r \qquad (0 \le a \le v_2(z))$$

can be realized infinitely often by products of consecutive-prime gaps

$$d_j = \Delta_{i_j} = p_{i_j} - p_{i_j - 1}$$

along odd squarefree integers $m = p_{i_1} \cdots p_{i_r}$.

Remark 30. For small values of z, the reduction is especially sharp. For example, if $z = 2$, then any infinite family of solutions would have to come, apart from finitely many exceptional cases, from odd squarefree integers m with

$$\prod_j \Delta_{i_j} \in \{1, 2\}.$$

Since Δ_i is even for all odd primes p_i, this forces a highly constrained prime-gap configuration. Similar remarks apply to $z = 4, 6, 8$, etc.

Proposition 64 (Explicit formula for $g(n;x)$ from the prime factorization)**.** *Assume that the polynomial family $\{f_n(x)\}_{n\ge1} \subset \mathbb{Z}[x]$ is multiplicative, normalized by*

$$f_1(x) = 1, \qquad f_2(x) = x,$$

and let $g(n;x)$ be its Möbius transform on the factorization poset:

$$f_n(x) = \sum_{d \preceq n} g(d;x).$$

Let

$$n = p_{i_1}^{\alpha_1} \cdots p_{i_s}^{\alpha_s}, \qquad i_1 < \cdots < i_s, \quad \alpha_j \ge 1.$$

If $i_1 \ge 2$ (equivalently, n is odd), then

$$g(n;x) = \left(\prod_{j=1}^{s-1} g(p_{i_j};x)\, f_{p_{i_j}}(x)^{\alpha_j - 1} \right) \cdot \begin{cases} g(p_{i_s};x), & \alpha_s = 1, \\ g(p_{i_s};x)\big(f_{p_{i_s}}(x) - 1\big) f_{p_{i_s}}(x)^{\alpha_s - 2}, & \alpha_s \ge 2. \end{cases}$$

Moreover,

$$g(2^k;x) = x^{k-1}(x-1) \qquad (k \ge 1),$$

and for every odd $m > 1$,

$$g(2^a m;x) = x^a g(m;x) \qquad (a \ge 0).$$

Proof. We follow exactly the same cumulative-sum strategy as in the arithmetic case.

For a weakly increasing r-tuple

$$j_1 \le \cdots \le j_r$$

define

$$h_x(j_1,\dots,j_r) := g(p_{j_1}\cdots p_{j_r};x),$$

and

$$S_x(j_1,\dots,j_r) := \sum_{\substack{1\le a_1\le\cdots\le a_r\\ a_m\le j_m}} h_x(a_1,\dots,a_r).$$

Let

$$N := p_{j_1}\cdots p_{j_r}.$$

By the definition of the factorization-poset order, the predecessors of N having exactly r prime factors are precisely the products

$$p_{a_1}\cdots p_{a_r} \quad \text{with} \quad 1\le a_1\le\cdots\le a_r, \quad a_m\le j_m.$$

Hence their total $g(\,\cdot\,;x)$-contribution is exactly $S_x(j_1,\dots,j_r)$.

The predecessors of N with fewer than r prime factors are exactly the predecessors of

$$p_{j_1}\cdots p_{j_{r-1}}.$$

Using Möbius inversion,

$$f_N(x) = \sum_{d\preceq N} g(d;x),$$

we therefore obtain

$$S_x(j_1,\dots,j_r) = f_{p_{j_1}\cdots p_{j_r}}(x) - f_{p_{j_1}\cdots p_{j_{r-1}}}(x).$$

By multiplicativity of the family $f_n(x)$,

$$S_x(j_1,\dots,j_r) = f_{p_{j_1}}(x)\cdots f_{p_{j_{r-1}}}(x)\big(f_{p_{j_r}}(x)-1\big). \tag{1}$$

Now write

$$\mathbf{i} = (\underbrace{i_1,\dots,i_1}_{\alpha_1},\underbrace{i_2,\dots,i_2}_{\alpha_2},\dots,\underbrace{i_s,\dots,i_s}_{\alpha_s}),$$

and let

$$m_t := \alpha_1+\cdots+\alpha_{t-1}+1$$

be the first coordinate of the t-th block. Exactly as in the arithmetic proof, successive backward differences in the first coordinate of each block isolate the top term:

$$g(n;x) = (\nabla_{m_1}\cdots\nabla_{m_s}S_x)(\mathbf{i}). \tag{2}$$

Applying these block differences to the explicit product formula (1), each non-final block contributes

$$f_{p_{i_t}}(x)^{\alpha_t} - f_{p_{i_t-1}}(x)f_{p_{i_t}}(x)^{\alpha_t-1} = \big(f_{p_{i_t}}(x)-f_{p_{i_t-1}}(x)\big)f_{p_{i_t}}(x)^{\alpha_t-1} = g(p_{i_t};x)\,f_{p_{i_t}}(x)^{\alpha_t-1}.$$

For the last block there are two cases.

If $\alpha_s=1$, the last factor in (1) is $f_{p_{i_s}}(x)-1$, so its backward difference gives

$$\big(f_{p_{i_s}}(x)-1\big)-\big(f_{p_{i_s-1}}(x)-1\big) = f_{p_{i_s}}(x)-f_{p_{i_s-1}}(x) = g(p_{i_s};x).$$

If $\alpha_s \geq 2$, then the last block contributes

$$f_{p_{i_s}}(x)^{\alpha_s-1}\big(f_{p_{i_s}}(x)-1\big),$$

and its backward difference is

$$f_{p_{i_s}}(x)^{\alpha_s-1}\big(f_{p_{i_s}}(x)-1\big)-f_{p_{i_s-1}}(x)f_{p_{i_s}}(x)^{\alpha_s-2}\big(f_{p_{i_s}}(x)-1\big),$$

which factors as

$$\big(f_{p_{i_s}}(x)-f_{p_{i_s-1}}(x)\big)\big(f_{p_{i_s}}(x)-1\big)f_{p_{i_s}}(x)^{\alpha_s-2}=g(p_{i_s};x)\big(f_{p_{i_s}}(x)-1\big)f_{p_{i_s}}(x)^{\alpha_s-2}.$$

Multiplying the block contributions proves the odd formula.

For powers of 2, Theorem 4 gives

$$g(2n;x)=xg(n;x) \qquad (n>1).$$

Since

$$g(2;x)=f_2(x)-f_1(x)=x-1,$$

iteration yields

$$g(2^k;x)=x^{k-1}(x-1).$$

Applying the same scaling repeatedly also gives

$$g(2^am;x)=x^ag(m;x)$$

for every odd $m>1$. □

42 A Geometric Mean Bound for Prime Gaps

Let

$$p_1=2<p_2=3<\cdots$$

be the primes, and define

$$\Delta_1:=p_1=2, \qquad \Delta_i:=p_i-p_{i-1} \quad (i\geq 2).$$

Then

$$\sum_{i=1}^{k}\Delta_i=p_k$$

by telescoping.

Recall that the Gram matrix of the prime feature vectors

$$v_{p_1},\ldots,v_{p_k}$$

is

$$M_k=\big(\min(p_i,p_j)\big)_{1\leq i,j\leq k},$$

and that

$$\det(M_k)=\prod_{i=1}^{k}\Delta_i.$$

Equivalently, if one writes the prime vectors in terms of their orthogonal increments, then $\det(M_k)$ is the squared volume of the corresponding k-dimensional parallelotope.

Proposition 65 (Geometric mean bound for prime gaps)**.** *For every $k \geq 1$,*

$$\prod_{i=1}^{k} \Delta_i \leq \left(\frac{p_k}{k}\right)^k.$$

Equivalently,

$$\left(\prod_{i=1}^{k} \Delta_i\right)^{1/k} \leq \frac{p_k}{k}.$$

The inequality is strict for every $k \geq 2$.

Proof. Since each $\Delta_i > 0$, the arithmetic–geometric mean inequality gives

$$\left(\prod_{i=1}^{k} \Delta_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} \Delta_i.$$

Using the telescoping identity $\sum_{i=1}^{k} \Delta_i = p_k$, we obtain

$$\left(\prod_{i=1}^{k} \Delta_i\right)^{1/k} \leq \frac{p_k}{k}.$$

Raising both sides to the k-th power yields

$$\prod_{i=1}^{k} \Delta_i \leq \left(\frac{p_k}{k}\right)^k.$$

For $k \geq 2$, the gaps are not all equal, so equality in AM–GM is impossible. Hence the inequality is strict. □

Remark 31 (Geometric interpretation)*.* If $u_1, \ldots, u_k$ denote the orthogonal increment vectors of the prime feature vectors, then

$$\|u_i\|^2 = \Delta_i.$$

Hence the associated orthotope has edge lengths $\sqrt{\Delta_i}$, and its squared volume is

$$\prod_{i=1}^{k} \Delta_i = \det(M_k).$$

Proposition 65 therefore states that among all orthotopes whose squared edge lengths have fixed sum p_k, the prime orthotope has squared volume at most that of the cube with common squared edge length p_k/k.

42.1 Asymptotic consequence

Combining Proposition 65 with the Prime Number Theorem yields an upper bound for the geometric mean of the initial prime gaps.

Corollary 19. *As $k \to \infty$,*

$$\left(\prod_{i=1}^{k} \Delta_i\right)^{1/k} \le \frac{p_k}{k} = (1+o(1)) \log k.$$

In particular,

$$\left(\prod_{i=1}^{k} \Delta_i\right)^{1/k} = O(\log k).$$

Proof. By the Prime Number Theorem,

$$p_k \sim k \log k.$$

Therefore

$$\frac{p_k}{k} = (1+o(1)) \log k.$$

The claim follows immediately from Proposition 65. □

Remark 32 (Heuristic comparison with random models). The bound above gives a universal geometric upper bound for the geometric mean of the prime gaps. In probabilistic models such as Cramér's model, the typical local spacing near scale x is of order $\log x$. From this perspective, the quantity

$$\frac{\prod_{i=1}^{k} \Delta_i}{(p_k/k)^k}$$

measures how far the prime-gap orthotope is from the maximally isotropic configuration allowed by the fixed total gap sum. A small ratio indicates strong anisotropy, i.e. a mixture of unusually short and unusually long gaps rather than a nearly uniform spacing pattern.

42.2 A bottleneck inequality for individual prime gaps

The previous proposition controls the *global* product of the first k prime gaps via the volume of the prime parallelotope. We now complement this with a *local* spectral inequality showing that a single large prime gap creates an isoperimetric bottleneck in the inverse prime Gram matrix.

Recall that

$$M_k = \big(\min(p_i, p_j)\big)_{1 \le i,j \le k}$$

is the prime Gram matrix, and that its inverse is the grounded weighted path Laplacian

$$B_k := M_k^{-1},$$

whose quadratic form is

$$u^T B_k u = \frac{u_1^2}{\Delta_1} + \sum_{i=2}^{k} \frac{(u_i - u_{i-1})^2}{\Delta_i}, \qquad u = (u_1, \dots, u_k) \in \mathbb{R}^k.$$

Thus the edge between $i-1$ and i carries conductance $1/\Delta_i$. A large prime gap Δ_i therefore corresponds to a weak link in the prime chain.

Proposition 66 (Bottleneck bound for a single prime gap)**.** *For every $k \geq 2$ and every $m \in \{0, 1, \dots, k-1\}$,*

$$\lambda_{\min}(B_k) \leq \frac{1}{(k-m)\Delta_{m+1}},$$

where by convention $\Delta_1 = p_1 = 2$. Equivalently,

$$\lambda_{\max}(M_k) \geq (k-m)\Delta_{m+1}.$$

In particular,

$$\lambda_{\max}(M_k) \geq \max_{1\leq j\leq k} (k-j+1)\Delta_j.$$

Proof. By the Rayleigh–Ritz principle,

$$\lambda_{\min}(B_k) = \min_{u\neq 0} \frac{u^T B_k u}{\|u\|_2^2}.$$

Fix $m \in \{0, 1, \dots, k-1\}$, and consider the test vector

$$u^{(m)} = (\underbrace{0, \dots, 0}_{m \text{ entries}}, \underbrace{1, \dots, 1}_{k-m \text{ entries}})^T.$$

Then

$$\|u^{(m)}\|_2^2 = k - m.$$

Moreover, in the quadratic form of B_k, all discrete differences vanish except at the jump across the edge $m \to m+1$. More precisely:

- if $m = 0$, then $u^{(0)} = (1, \dots, 1)$, so

$$(u^{(0)})^T B_k u^{(0)} = \frac{1}{\Delta_1};$$

- if $1 \leq m \leq k-1$, then

$$u_i^{(m)} - u_{i-1}^{(m)} = \begin{cases} 1, & i = m+1, \\ 0, & i \neq m+1, \end{cases}$$

and therefore

$$(u^{(m)})^T B_k u^{(m)} = \frac{1}{\Delta_{m+1}}.$$

In either case,

$$(u^{(m)})^T B_k u^{(m)} = \frac{1}{\Delta_{m+1}}.$$

Hence

$$\lambda_{\min}(B_k) \leq \frac{(u^{(m)})^T B_k u^{(m)}}{\|u^{(m)}\|_2^2} = \frac{1}{(k-m)\Delta_{m+1}}.$$

This proves the first inequality.

Since $M_k = B_k^{-1}$, the nonzero eigenvalues satisfy

$$\lambda_{\max}(M_k) = \frac{1}{\lambda_{\min}(B_k)}.$$

Therefore

$$\lambda_{\max}(M_k) \geq (k-m)\Delta_{m+1}.$$

Taking the maximum over m yields

$$\lambda_{\max}(M_k) \geq \max_{1 \leq j \leq k} (k - j + 1)\Delta_j,$$

after relabeling $j = m + 1$. □

Remark 33 (Isoperimetric interpretation). The matrix B_k is the weighted path Laplacian of the prime chain

$$1 \longleftrightarrow 2 \longleftrightarrow \cdots \longleftrightarrow k$$

with edge conductances

$$c_j = \frac{1}{\Delta_j}.$$

For the cut

$$\{1, \ldots, m\} \bigm| \{m + 1, \ldots, k\},$$

the boundary conductance is exactly

$$c_{m+1} = \frac{1}{\Delta_{m+1}}.$$

Thus a large prime gap creates a narrow spectral bottleneck: the path is only weakly coupled across that edge, forcing a small eigenvalue of the precision operator B_k, equivalently a large top eigenvalue of the Green/Gram matrix M_k.

Remark 34 (Comparison with the global volume bound). Proposition 65 controls the product

$$\prod_{i=1}^{k} \Delta_i$$

through the determinant of the prime Gram matrix, hence through the total volume of the prime parallelotope. Proposition 66, by contrast, isolates the effect of a single large gap: one unusually large Δ_j already forces a large operator norm of M_k and a small bottom eigenvalue of B_k. In this sense, the first result is global and isovolumetric, whereas the second is local and bottleneck-driven.

Corollary 20 (Gap bound from the spectral radius)**.** *For every $k \geq 2$ and every $m \in \{0, 1, \ldots, k - 1\}$,*

$$\Delta_{m+1} \leq \frac{\lambda_{\max}(M_k)}{k - m}.$$

In particular, using the growth estimate

$$\lambda_{\max}(M_k) \leq \left(\frac{1}{2} + o(1)\right) k^2 \log k,$$

proved earlier in the paper, one obtains

$$\Delta_{m+1} \leq \frac{\left(\frac{1}{2} + o(1)\right) k^2 \log k}{k - m} \qquad (k \to \infty).$$

Proof. This is an immediate rearrangement of Proposition 66, which gives

$$\lambda_{\max}(M_k) \geq (k - m)\Delta_{m+1}.$$

Dividing both sides by $k - m$ yields

$$\Delta_{m+1} \leq \frac{\lambda_{\max}(M_k)}{k - m}.$$

The asymptotic bound follows by substitution of the upper estimate for $\lambda_{\max}(M_k)$. □

Remark 35 (Comparison with the global spectral growth)*.* Theorem 6 shows that

$$\lambda_{\max}(M_k) = \Theta(k^2 \log k),$$

so Proposition 66 may be read as a local-to-global principle: a single large gap Δ_{m+1} forces the global spectral radius of the prime Gram matrix to be large. Conversely, the global spectral upper bound yields the explicit estimate

$$\Delta_{m+1} \leq \frac{\lambda_{\max}(M_k)}{k-m}$$

for every $m < k$. Although this is much weaker than the sharpest known arithmetic estimates for prime gaps, it arises here purely from the geometry and spectral theory of the prime feature vectors.

Proposition 67 (A Stirling–Hadamard bound for prime gaps)**.** *For every $N \geq 2$,*

$$\prod_{3 \leq p_i \leq N} \Delta_i^{\lfloor N/p_i \rfloor} \leq \prod_{n=1}^{N} g(n) \leq N!.$$

Hence, by Stirling's formula,

$$\prod_{3 \leq p_i \leq N} \Delta_i^{\lfloor N/p_i \rfloor} \leq \sqrt{2\pi N} \left(\frac{N}{e}\right)^N e^{\theta_N/(12N)}, \qquad 0 < \theta_N < 1.$$

Equivalently,

$$\sum_{3 \leq p_i \leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i \leq N \log N - N + \frac{1}{2} \log(2\pi N) + O(1).$$

Proof. By the Lindström–Bhat factorization,

$$\det(G_N) = \prod_{n=1}^{N} g(n).$$

Applying Hadamard's inequality to the Gram matrix G_N gives

$$\det(G_N) \leq \prod_{n=1}^{N} \|v_n\|^2 = \prod_{n=1}^{N} n = N!.$$

This proves the upper bound.

For the lower bound, write each $n \leq N$ in the form

$$n = 2^a \prod_{j=1}^{s} p_{i_j}^{\alpha_j}, \qquad 3 \leq p_{i_1} < \cdots < p_{i_s}.$$

By the block formula for $g(n)$, every odd prime divisor $p_i \mid n$ contributes exactly one factor Δ_i to $g(n)$. Therefore, in the full product $\prod_{n \leq N} g(n)$, the gap factor Δ_i appears at least once for each multiple of p_i up to N, hence at least $\lfloor N/p_i \rfloor$ times. Since all remaining factors in the block formula are ≥ 1, it follows that

$$\prod_{3 \leq p_i \leq N} \Delta_i^{\lfloor N/p_i \rfloor} \leq \prod_{n=1}^{N} g(n).$$

Combining the two inequalities yields

$$\prod_{3 \leq p_i \leq N} \Delta_i^{\lfloor N/p_i \rfloor} \leq N!.$$

The Stirling form is immediate. ∎

42.3 An exact gap-factor count in $\prod_{n\le N} g(n)$

We now sharpen the previous Hadamard argument by extracting the prime-gap factors in

$$\prod_{n=1}^{N} g(n)$$

with their exact multiplicities.

For an odd prime p_i, write

$$\Delta_i := p_i - p_{i-1}, \qquad p_0 := 1.$$

Recall from Proposition 63 that if

$$n = 2^a \prod_{j=1}^{s} p_{i_j}^{\alpha_j}, \qquad 3 \le p_{i_1} < \cdots < p_{i_s},$$

then each distinct odd prime factor $p_{i_j} \mid n$ contributes exactly one factor Δ_{i_j} to $g(n)$.

Proposition 68 (Exact gap multiplicities in $\prod_{n\le N} g(n)$)**.** *For every $N \ge 2$, one has*

$$\prod_{n=1}^{N} g(n) = 2^{E_2(N)} \left(\prod_{3\le p_i \le N} \Delta_i^{\lfloor N/p_i \rfloor} \right) U_N,$$

where

$$E_2(N) = v_2(N!) - \lfloor \log_2 N \rfloor = N - s_2(N) - \lfloor \log_2 N \rfloor,$$

with $s_2(N)$ denoting the sum of the binary digits of N, and where $U_N \in \mathbb{N}$ is given by

$$U_N := \prod_{n=1}^{N} u(n),$$

with $u(n) \ge 1$ defined as follows.

Write $n = 2^a m$ with m odd. Then:

$$u(n) := \begin{cases} 1, & m = 1, \\ \left(\displaystyle\prod_{\substack{p^\alpha \| m \\ p < P^+(m)}} p^{\alpha - 1} \right), & m > 1,\ v_{P^+(m)}(m) = 1, \\ \left(\displaystyle\prod_{\substack{p^\alpha \| m \\ p < P^+(m)}} p^{\alpha - 1} \right) (P^+(m) - 1)\, P^+(m)^{\,v_{P^+(m)}(m) - 2}, & m > 1,\ v_{P^+(m)}(m) \ge 2, \end{cases}$$

where $P^+(m)$ is the largest prime divisor of m.

Proof. We begin with the odd prime-gap factors. Let $p_i \ge 3$ be fixed. By the block formula, every n such that $p_i \mid n$ contributes exactly one factor Δ_i to $g(n)$, regardless of the exponent of p_i in n.

Conversely, if $p_i \nmid n$, then Δ_i does not occur in $g(n)$. Hence the total exponent of Δ_i in the product $\prod_{n \le N} g(n)$ is exactly the number of multiples of p_i up to N, namely

$$\#\{n \le N : \ p_i \mid n\} = \left\lfloor \frac{N}{p_i} \right\rfloor.$$

This proves the factor

$$\prod_{3 \le p_i \le N} \Delta_i^{\lfloor N/p_i \rfloor}.$$

Next we determine the total exponent of 2. If $n = 2^a m$ with $m > 1$ odd, then the doubling identity gives

$$g(n) = g(2^a m) = 2^a g(m),$$

so the power of 2 contributed by such an n is $a = v_2(n)$. If instead $n = 2^a$ is a pure power of 2, then

$$g(2^a) = 2^{a-1},$$

so the contribution is $a - 1 = v_2(n) - 1$. Therefore the total exponent of 2 is

$$E_2(N) = \sum_{n \le N} v_2(n) - \#\{a \ge 1 : 2^a \le N\}.$$

Now

$$\sum_{n \le N} v_2(n) = v_2(N!)$$

and

$$\#\{a \ge 1 : 2^a \le N\} = \lfloor \log_2 N \rfloor.$$

Hence

$$E_2(N) = v_2(N!) - \lfloor \log_2 N \rfloor.$$

Using Legendre's formula $v_2(N!) = N - s_2(N)$, we get

$$E_2(N) = N - s_2(N) - \lfloor \log_2 N \rfloor.$$

After extracting the factors $2^{E_2(N)}$ and the odd gap factors $\prod_{3 \le p_i \le N} \Delta_i^{\lfloor N/p_i \rfloor}$, the remaining factor is precisely the product of the non-gap, non-doubling terms appearing in the block formula. This is exactly $U_N = \prod_{n \le N} u(n)$, and by construction $u(n) \in \mathbb{N}$ and $u(n) \ge 1$ for all n. This proves the stated factorization. □

Corollary 21 (A sharpened Stirling–Hadamard inequality)**.** *For every $N \ge 2$,*

$$2^{E_2(N)} \left(\prod_{3 \le p_i \le N} \Delta_i^{\lfloor N/p_i \rfloor} \right) U_N = \prod_{n=1}^{N} g(n) \le N!.$$

Hence

$$\left(\prod_{3 \le p_i \le N} \Delta_i^{\lfloor N/p_i \rfloor} \right) U_N \le \frac{N!}{2^{E_2(N)}}.$$

Equivalently,

$$\left(\prod_{3\le p_i\le N} \Delta_i^{\lfloor N/p_i\rfloor}\right) U_N \le \frac{N!}{2^{\,N-s_2(N)-\lfloor \log_2 N\rfloor}}.$$

Using Stirling's formula,

$$N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N e^{\theta_N/(12N)}, \qquad 0<\theta_N<1,$$

this becomes

$$\left(\prod_{3\le p_i\le N} \Delta_i^{\lfloor N/p_i\rfloor}\right) U_N \le 2^{\,s_2(N)+\lfloor \log_2 N\rfloor}\sqrt{2\pi N}\left(\frac{N}{2e}\right)^N e^{\theta_N/(12N)}.$$

Proof. By the Lindström–Bhat factorization,

$$\det(G_N) = \prod_{n=1}^{N} g(n).$$

Hadamard's inequality gives

$$\det(G_N) \le \prod_{n=1}^{N} \|v_n\|^2 = \prod_{n=1}^{N} n = N!.$$

Substituting the exact factorization from Proposition 68 yields

$$2^{E_2(N)}\left(\prod_{3\le p_i\le N} \Delta_i^{\lfloor N/p_i\rfloor}\right) U_N \le N!.$$

Dividing through by $2^{E_2(N)}$ proves the second displayed inequality. The final form follows from the identity

$$E_2(N) = N - s_2(N) - \lfloor \log_2 N\rfloor$$

and Stirling's formula. □

Remark 36. The previous coarse bound

$$\prod_{3\le p_i\le N} \Delta_i^{\lfloor N/p_i\rfloor} \le N!$$

already shows that the prime gaps occur with large multiplicity in the global determinant. The new factorization is substantially sharper: it extracts the exact odd gap exponents, the exact power of 2, and separates the remaining non-gap arithmetic contribution into the explicit integer factor $U_N \ge 1$. Thus the Hadamard bound may be viewed as a weighted entropy inequality for the prime-gap sequence, with a nontrivial correction term coming from repeated prime powers and the terminal block in the factorization formula for $g(n)$.

Corollary 22 (Logarithmic asymptotic form of the Stirling–Hadamard bound)**.** *For every* $N \ge 2$*,*

$$\sum_{3\le p_i\le N} \left\lfloor \frac{N}{p_i}\right\rfloor \log \Delta_i + \log U_N \le N\log N - (1+\log 2)N + O(\log N).$$

In particular, since $U_N \geq 1$*,*

$$\sum_{3\leq p_i\leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i \leq N\log N - (1+\log 2)N + O(\log N).$$

Equivalently,

$$\frac{1}{N}\sum_{3\leq p_i\leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i \leq \log N - (1+\log 2) + O\left(\frac{\log N}{N}\right).$$

Proof. By Corollary 21,

$$\left(\prod_{3\leq p_i\leq N} \Delta_i^{\lfloor N/p_i\rfloor}\right) U_N \leq 2^{\,s_2(N)+\lfloor \log_2 N\rfloor}\sqrt{2\pi N}\left(\frac{N}{2e}\right)^N e^{\theta_N/(12N)}, \qquad 0<\theta_N<1.$$

Taking logarithms yields

$$\sum_{3\leq p_i\leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i + \log U_N \leq \big(s_2(N)+\lfloor \log_2 N\rfloor\big)\log 2 + \frac{1}{2}\log(2\pi N) + N\log N - N\log 2 - N + \frac{\theta_N}{12N}.$$

Now

$$s_2(N) \leq \lfloor \log_2 N\rfloor + 1,$$

hence

$$\big(s_2(N)+\lfloor \log_2 N\rfloor\big)\log 2 = O(\log N),$$

and of course

$$\frac{1}{2}\log(2\pi N) = O(\log N), \qquad \frac{\theta_N}{12N} = O(1).$$

Therefore

$$\sum_{3\leq p_i\leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i + \log U_N \leq N\log N - (1+\log 2)N + O(\log N),$$

which proves the first claim. Since $U_N \geq 1$, one has $\log U_N \geq 0$, and dropping this nonnegative term gives

$$\sum_{3\leq p_i\leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i \leq N\log N - (1+\log 2)N + O(\log N).$$

Dividing by N gives the final normalized form. □

Remark 37. The corollary may be interpreted as a weighted entropy bound for the prime-gap sequence. The weight of $\log \Delta_i$ is exactly $\lfloor N/p_i \rfloor$, reflecting the precise multiplicity with which the gap Δ_i occurs in the global determinant product

$$\prod_{n\leq N} g(n).$$

Since small primes carry the largest weights, the inequality is most sensitive to the early part of the prime-gap sequence.

Corollary 23 (A coarser harmonic-weight form)**.** *One also has*

$$\sum_{3\leq p_i\leq N} \frac{\log \Delta_i}{p_i} \leq \log N - (1+\log 2) + O(1).$$

Proof. From Corollary 22,

$$\sum_{3\le p_i\le N}\left\lfloor\frac{N}{p_i}\right\rfloor\log\Delta_i\le N\log N-(1+\log 2)N+O(\log N).$$

Since

$$\frac{N}{p_i}-1<\left\lfloor\frac{N}{p_i}\right\rfloor\le\frac{N}{p_i},$$

we have

$$0\le\sum_{3\le p_i\le N}\left(\frac{N}{p_i}-\left\lfloor\frac{N}{p_i}\right\rfloor\right)\log\Delta_i\le\sum_{3\le p_i\le N}\log\Delta_i.$$

Now $\Delta_i\le p_i\le N$, so $\log\Delta_i\le\log N$, and therefore

$$\sum_{3\le p_i\le N}\log\Delta_i\le\pi(N)\log N=O(N),$$

using the elementary bound $\pi(N)=O(N/\log N)$. Hence

$$\sum_{3\le p_i\le N}\frac{N}{p_i}\log\Delta_i\le N\log N-(1+\log 2)N+O(N).$$

Dividing by N yields

$$\sum_{3\le p_i\le N}\frac{\log\Delta_i}{p_i}\le\log N-(1+\log 2)+O(1).$$

□

43 The Dirichlet Series of the Möbius Weights $g(n)$

To encode the values $g(n)$ analytically and extract global information about the prime-gap structure hidden in them, the ordinary power series

$$F(z)=\sum_{n\ge 1}g(n)z^n$$

is of limited use. The reason is that the structure of $g(n)$ is governed not by additive features, but by ordered factorization-theoretic blocks. The natural analytic object is therefore the associated Dirichlet series

$$D_g(s):=\sum_{n=1}^{\infty}\frac{g(n)}{n^s},\qquad s\in\mathbb{C}.$$

Since $g(n)>0$ for all n, and since the Möbius inversion identity

$$n=\sum_{d\preceq n}g(d)$$

implies in particular that $g(n)\le n$, it follows that $D_g(s)$ converges absolutely at least in the half-plane $\Re(s)>2$. Thus D_g is a well-defined analytic function in a right half-plane. Its main interest, however, lies not merely in convergence, but in the exact prime-block structure of its coefficients.

The block formula from Proposition 63

Let the odd part of n be written as

$$m = p_{i_1}^{\alpha_1} \cdots p_{i_r}^{\alpha_r}, \qquad 3 \le p_{i_1} < \cdots < p_{i_r}, \qquad \alpha_j \ge 1.$$

Then Proposition 63 gives the explicit formula

$$g(m) = \left(\prod_{j=1}^{r-1} \Delta_{i_j}\, p_{i_j}^{\alpha_j - 1} \right) \cdot \begin{cases} \Delta_{i_r}, & \alpha_r = 1, \\ \Delta_{i_r}(p_{i_r} - 1) p_{i_r}^{\alpha_r - 2}, & \alpha_r \ge 2, \end{cases}$$

where

$$\Delta_i := p_i - p_{i-1}, \qquad p_0 := 1.$$

This formula shows that the smaller prime factors of m contribute *internal blocks*, while the largest prime factor contributes a distinguished *terminal block*. It is precisely this asymmetry that prevents g from admitting a classical Euler product.

In addition, powers of 2 satisfy the separate scaling rule

$$g(2^a m) = 2^a g(m) \qquad (m > 1 \text{ odd}),$$

together with

$$g(2^a) = 2^{a-1} \qquad (a \ge 1).$$

Accordingly, the full Dirichlet series naturally splits into its odd part and a 2-adic correction factor.

The odd part as a cascade over the largest prime factor

Define

$$D_g^{\text{odd}}(s) := \sum_{\substack{m \ge 1 \\ m \text{ odd}}} \frac{g(m)}{m^s}.$$

Using Proposition 63, one may group the terms according to the largest odd prime divisor of m. This yields the ordered expansion

$$D_g^{\text{odd}}(s) = 1 + \sum_{\substack{p \text{ odd} \\ p \text{ prime}}} T_p(s) \prod_{\substack{q<p \\ q \text{ odd prime}}} H_q(s),$$

where the local building blocks are given by

$$H_q(s) := 1 + \sum_{\alpha=1}^{\infty} \frac{\Delta_q\, q^{\alpha-1}}{q^{\alpha s}} = 1 + \frac{\Delta_q}{q^s - q},$$

and

$$T_p(s) := \frac{\Delta_p}{p^s} + \sum_{\alpha=2}^{\infty} \frac{\Delta_p (p-1) p^{\alpha-2}}{p^{\alpha s}}.$$

Here $H_q(s)$ represents the contribution of an internal block, while $T_p(s)$ represents the terminal contribution associated with the largest prime factor. Thus the Dirichlet series does not factor multiplicatively over primes in the usual sense; instead, it unfolds as a *cascade* ordered by the maximal prime factor.

The terminal block can be summed explicitly. Since

$$T_p(s) = \frac{\Delta_p}{p^s} + \Delta_p(p-1)p^{-2s}\sum_{\beta=0}^{\infty} p^{\beta(1-s)},$$

we obtain, for $\Re(s) > 1$,

$$T_p(s) = \frac{\Delta_p}{p^s} + \frac{\Delta_p(p-1)p^{-2s}}{1-p^{1-s}} = \frac{\Delta_p\,(p^s-1)}{p^s(p^s-p)}.$$

Hence the odd part takes the closed form

$$D_g^{\text{odd}}(s) = 1 + \sum_{\substack{p \text{ odd} \\ p \text{ prime}}} \frac{\Delta_p\,(p^s-1)}{p^s(p^s-p)} \prod_{\substack{q<p \\ q \text{ odd prime}}} \left(1 + \frac{\Delta_q}{q^s-q}\right).$$

Insertion of the 2-adic factor

To recover the full Dirichlet series, write each integer as $n = 2^a m$ with m odd. Using the scaling laws above, one finds

$$D_g(s) = 1 + \sum_{a=1}^{\infty} \frac{2^{a-1}}{2^{as}} + \sum_{\substack{m>1 \\ m \text{ odd}}} \sum_{a=0}^{\infty} \frac{g(2^a m)}{(2^a m)^s}.$$

Hence

$$D_g(s) = 1 + \frac{2^{-s}}{1-2^{1-s}} + \sum_{\substack{m>1 \\ m \text{ odd}}} \frac{g(m)}{m^s} \sum_{a=0}^{\infty} 2^{a(1-s)}.$$

Therefore

$$D_g(s) = 1 + \frac{2^{-s}}{1-2^{1-s}} + \frac{D_g^{\text{odd}}(s) - 1}{1-2^{1-s}},$$

which simplifies to

$$D_g(s) = \frac{D_g^{\text{odd}}(s) - 2^{-s}}{1-2^{1-s}}.$$

Substituting the cascade representation for $D_g^{\text{odd}}(s)$ gives the exact formula

$$D_g(s) = \frac{1 - 2^{-s} + \displaystyle\sum_{\substack{p \text{ odd} \\ p \text{ prime}}} \frac{\Delta_p\,(p^s-1)}{p^s(p^s-p)} \prod_{\substack{q<p \\ q \text{ odd prime}}} \left(1 + \frac{\Delta_q}{q^s-q}\right)}{1-2^{1-s}}.$$

Interpretation

This formula is the natural analytic counterpart of Proposition 63. The Möbius weights $g(n)$ do not generate a classical Euler product, because the largest prime factor is weighted differently from the smaller ones. Instead, the Dirichlet series is built from an ordered system of local factors, and the prime gaps Δ_p appear explicitly in each of them:

$$H_q(s) = 1 + \frac{\Delta_q}{q^s-q}, \qquad T_p(s) = \frac{\Delta_p\,(p^s-1)}{p^s(p^s-p)}.$$

In this sense, $D_g(s)$ packages the entire prime-gap sequence into a single analytic object.

What can presently be asserted rigorously is that $D_g(s)$ is well defined in a right half-plane and that its structure is governed by a non-Eulerian prime cascade. It is therefore reasonable to regard $D_g(s)$ as a generating function for the ordered prime-gap geometry encoded by the Möbius coefficients.

At the same time, stronger analytic claims should be stated with care. In particular, questions concerning meromorphic continuation, precise pole structure, zero distributions, or direct spectral interpretations in terms of extreme prime-gap fluctuations are not established by the present results alone. These belong to the next analytical layer of the theory. What the formula above provides is the exact starting point for such an investigation.

Operator-theoretic perspective

There is also a natural operator-theoretic continuation of this construction. The paper shows that the inverse Gram matrices admit an infinite positive symmetric limit operator $A = G_\infty^{-1}$, together with a well-defined Friedrichs extension. From this viewpoint, one may hope to study analytic objects associated with A, for instance resolvent-type functions, as a spectral encoding of the same arithmetic structure.

However, one should again distinguish clearly between what has been proved and what is presently heuristic. While the existence and positivity properties of the limiting operator are established, a trace formula such as

$$\mathrm{Tr}\big((A - zI)^{-1}\big)$$

requires additional spectral input and is not yet available at the level of the present theory. Thus the Dirichlet series $D_g(s)$ remains the first fully explicit analytic realization of the block structure of Proposition 63.

44 A multiplicative majorant and the true abscissa of absolute convergence of $D_g(s)$

In this section we show that the Dirichlet series

$$D_g(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

admits an *unconditional* absolute convergence theory in the half-plane $\Re(s) > 1$. The key point is that the block formula of Proposition 63 yields a natural multiplicative majorant, and that the corresponding Euler product can be controlled by an elementary integral comparison involving the prime gaps.

Notation

Let

$$2 = p_1 < p_2 < p_3 < \cdots$$

be the increasing sequence of prime numbers, and define

$$\Delta_i := p_i - p_{i-1} \qquad (i \geq 2),$$

with the convention

$$p_0 := 1.$$

Thus, for example,

$$\Delta_1 = p_1 - p_0 = 1, \qquad \Delta_2 = p_2 - p_1 = 1, \qquad \Delta_3 = p_3 - p_2 = 2.$$

For an integer $n \geq 1$, let $v_2(n)$ denote the exponent of 2 in n, and let $P^+(n)$ denote the largest prime divisor of n when $n > 1$.

We recall two facts established earlier in the paper:

1. For odd integers

$$m = p_{i_1}^{\alpha_1} \cdots p_{i_r}^{\alpha_r}, \qquad 3 \leq p_{i_1} < \cdots < p_{i_r}, \qquad \alpha_j \geq 1,$$

   Proposition 63 gives

$$g(m) = \left(\prod_{j=1}^{r-1} \Delta_{i_j}\, p_{i_j}^{\alpha_j - 1} \right) \cdot \begin{cases} \Delta_{i_r}, & \alpha_r = 1, \\ \Delta_{i_r}(p_{i_r} - 1) p_{i_r}^{\alpha_r - 2}, & \alpha_r \geq 2. \end{cases}$$

2. For the 2-adic scaling one has

$$g(2^a m) = 2^a g(m) \qquad (m > 1 \text{ odd}),$$

   and, separately,

$$g(2^a) = 2^{a-1} \qquad (a \geq 1).$$

These formulas are the starting point for the majorant argument.

A pointwise multiplicative majorant

Theorem 14. *For every integer $n \geq 1$ one has*

$$g(n) \leq n \prod_{\substack{p \mid n \\ p \text{ odd}}} \frac{\Delta_p}{p},$$

where for an odd prime $p = p_i$ we write $\Delta_p := \Delta_i$.

Define

$$\widetilde{g}(n) := n \prod_{\substack{p \mid n \\ p \text{ odd}}} \frac{\Delta_p}{p}.$$

Then $\widetilde{g}$ is multiplicative and

$$0 \leq g(n) \leq \widetilde{g}(n) \qquad (n \geq 1).$$

Proof. Let

$$n = 2^a \prod_{j=1}^{r} p_j^{\alpha_j}, \qquad 3 \leq p_1 < \cdots < p_r, \qquad \alpha_j \geq 1,$$

where the product over the odd primes is understood to be empty if n is a power of 2.

We first treat the case $r \geq 1$, so that n has an odd prime divisor. Write

$$m := \prod_{j=1}^{r} p_j^{\alpha_j},$$

so that $n = 2^a m$ and $m > 1$ is odd. By the 2-adic scaling law,

$$g(n) = g(2^a m) = 2^a g(m).$$

Applying Proposition 63 to m, we obtain

$$g(n) = 2^a \left(\prod_{j=1}^{r-1} \Delta_{p_j} p_j^{\alpha_j - 1} \right) \cdot \begin{cases} \Delta_{p_r}, & \alpha_r = 1, \\ \Delta_{p_r}(p_r - 1)p_r^{\alpha_r - 2}, & \alpha_r \geq 2. \end{cases}$$

We now estimate the terminal factor. If $\alpha_r = 1$, then

$$\Delta_{p_r} \leq \Delta_{p_r} p_r^{\alpha_r - 1},$$

since $p_r^{\alpha_r - 1} = p_r^0 = 1$. If $\alpha_r \geq 2$, then

$$\Delta_{p_r}(p_r - 1)p_r^{\alpha_r - 2} \leq \Delta_{p_r} p_r^{\alpha_r - 1},$$

because

$$(p_r - 1)p_r^{\alpha_r - 2} \leq p_r \cdot p_r^{\alpha_r - 2} = p_r^{\alpha_r - 1}.$$

Hence in all cases,

$$g(n) \leq 2^a \prod_{j=1}^{r} \Delta_{p_j} p_j^{\alpha_j - 1}.$$

Next we factor out n:

$$2^a \prod_{j=1}^{r} \Delta_{p_j} p_j^{\alpha_j - 1} = \left(2^a \prod_{j=1}^{r} p_j^{\alpha_j} \right) \prod_{j=1}^{r} \frac{\Delta_{p_j}}{p_j} = n \prod_{j=1}^{r} \frac{\Delta_{p_j}}{p_j}.$$

Therefore

$$g(n) \leq n \prod_{\substack{p \mid n \\ p \text{ odd}}} \frac{\Delta_p}{p}.$$

Now consider the case where $n = 2^a$ is a power of 2. Then by the separate formula,

$$g(2^a) = 2^{a-1} \leq 2^a = n.$$

Since the product over odd prime divisors is empty, it equals 1, and thus

$$g(2^a) \leq 2^a \cdot 1 = 2^a \prod_{\substack{p \mid 2^a \\ p \text{ odd}}} \frac{\Delta_p}{p}.$$

So the same bound holds for powers of 2 as well.

This proves

$$g(n) \leq \widetilde{g}(n) \qquad (n \geq 1).$$

It remains to show that $\widetilde{g}$ is multiplicative. Let $\gcd(m, n) = 1$. Then the sets of odd prime divisors of m and n are disjoint, hence

$$\prod_{\substack{p \mid mn \\ p \text{ odd}}} \frac{\Delta_p}{p} = \left(\prod_{\substack{p \mid m \\ p \text{ odd}}} \frac{\Delta_p}{p} \right) \left(\prod_{\substack{p \mid n \\ p \text{ odd}}} \frac{\Delta_p}{p} \right).$$

Therefore

$$\widetilde{g}(mn) = mn \prod_{\substack{p \mid mn \\ p \text{ odd}}} \frac{\Delta_p}{p} = \left(m \prod_{\substack{p \mid m \\ p \text{ odd}}} \frac{\Delta_p}{p} \right) \left(n \prod_{\substack{p \mid n \\ p \text{ odd}}} \frac{\Delta_p}{p} \right) = \widetilde{g}(m)\widetilde{g}(n).$$

Thus $\widetilde{g}$ is multiplicative. □

Euler product of the majorant

Lemma 12. *The Dirichlet series of $\widetilde{g}$ admits the Euler product*

$$\widetilde{D}(s) := \sum_{n=1}^{\infty} \frac{\widetilde{g}(n)}{n^s} = \left(1 + \sum_{a=1}^{\infty} \frac{\widetilde{g}(2^a)}{2^{as}} \right) \prod_{p \geq 3} \left(1 + \sum_{\alpha=1}^{\infty} \frac{\widetilde{g}(p^\alpha)}{p^{\alpha s}} \right).$$

More explicitly,

$$\widetilde{g}(2^a) = 2^a, \qquad \widetilde{g}(p^\alpha) = \Delta_p \, p^{\alpha - 1} \quad (p \geq 3,\ \alpha \geq 1),$$

and therefore

$$\widetilde{D}(s) = \frac{1}{1 - 2^{1-s}} \prod_{p \geq 3} \left(1 + \frac{\Delta_p}{p^s - p} \right)$$

whenever the product converges.

Proof. Since $\widetilde{g}$ is multiplicative, its Dirichlet series factors as an Euler product over prime powers.

For the prime 2, the definition gives

$$\widetilde{g}(2^a) = 2^a \prod_{\substack{p \mid 2^a \\ p \text{ odd}}} \frac{\Delta_p}{p} = 2^a.$$

Hence

$$1 + \sum_{a=1}^{\infty} \frac{\widetilde{g}(2^a)}{2^{as}} = 1 + \sum_{a=1}^{\infty} 2^{a(1-s)} = \frac{1}{1 - 2^{1-s}},$$

provided $\Re(s) > 1$.

Now let $p \geq 3$ be odd and $\alpha \geq 1$. Then

$$\widetilde{g}(p^\alpha) = p^\alpha \cdot \frac{\Delta_p}{p} = \Delta_p \, p^{\alpha - 1}.$$

Therefore

$$1 + \sum_{\alpha=1}^{\infty} \frac{\widetilde{g}(p^\alpha)}{p^{\alpha s}} = 1 + \sum_{\alpha=1}^{\infty} \frac{\Delta_p \, p^{\alpha-1}}{p^{\alpha s}} = 1 + \Delta_p \sum_{\alpha=1}^{\infty} p^{\alpha - 1 - \alpha s}.$$

Rewrite the exponent:

$$\alpha - 1 - \alpha s = -1 + \alpha(1 - s).$$

Thus

$$\sum_{\alpha=1}^{\infty} p^{\alpha-1-\alpha s} = p^{-1} \sum_{\alpha=1}^{\infty} p^{\alpha(1-s)}.$$

For $\Re(s) > 1$, this is a geometric series with ratio p^{1-s}, so

$$\sum_{\alpha=1}^{\infty} p^{\alpha(1-s)} = \frac{p^{1-s}}{1 - p^{1-s}}.$$

Hence

$$p^{-1} \sum_{\alpha=1}^{\infty} p^{\alpha(1-s)} = p^{-1} \cdot \frac{p^{1-s}}{1 - p^{1-s}} = \frac{p^{-s}}{1 - p^{1-s}} = \frac{1}{p^s - p}.$$

Therefore

$$1 + \sum_{\alpha=1}^{\infty} \frac{\widetilde{g}(p^{\alpha})}{p^{\alpha s}} = 1 + \frac{\Delta_p}{p^s - p}.$$

Multiplying the local factors yields

$$\widetilde{D}(s) = \frac{1}{1 - 2^{1-s}} \prod_{p \geq 3} \left(1 + \frac{\Delta_p}{p^s - p}\right),$$

whenever the product converges. □

An elementary convergence estimate for the prime-gap weights

The decisive input is the following elementary estimate.

Lemma 13. *For every real number $\sigma > 1$ one has*

$$\sum_{p \geq 3} \frac{\Delta_p}{p^{\sigma}} < \infty.$$

More precisely, if p_i denotes the i-th prime and $\Delta_i = p_i - p_{i-1}$, then

$$\sum_{i \geq 2} \frac{\Delta_i}{p_i^{\sigma}} \leq \int_2^{\infty} x^{-\sigma}\, dx = \frac{2^{1-\sigma}}{\sigma - 1}.$$

Proof. Fix $\sigma > 1$. Since the function

$$x \longmapsto x^{-\sigma}$$

is positive and decreasing on $[1, \infty)$, we have for every $i \geq 2$ and every $x \in [p_{i-1}, p_i]$:

$$x^{-\sigma} \geq p_i^{-\sigma}.$$

Integrating over the interval $[p_{i-1}, p_i]$ gives

$$\int_{p_{i-1}}^{p_i} x^{-\sigma}\, dx \geq \int_{p_{i-1}}^{p_i} p_i^{-\sigma}\, dx = (p_i - p_{i-1}) p_i^{-\sigma} = \frac{\Delta_i}{p_i^{\sigma}}.$$

Summing over $i = 2, 3, \ldots, N$ yields

$$\sum_{i=2}^{N} \frac{\Delta_i}{p_i^\sigma} \le \sum_{i=2}^{N} \int_{p_{i-1}}^{p_i} x^{-\sigma}\, dx.$$

The intervals $[p_{i-1}, p_i]$ are adjacent, so the integrals telescope:

$$\sum_{i=2}^{N} \int_{p_{i-1}}^{p_i} x^{-\sigma}\, dx = \int_{p_1}^{p_N} x^{-\sigma}\, dx = \int_{2}^{p_N} x^{-\sigma}\, dx.$$

Therefore

$$\sum_{i=2}^{N} \frac{\Delta_i}{p_i^\sigma} \le \int_{2}^{p_N} x^{-\sigma}\, dx \le \int_{2}^{\infty} x^{-\sigma}\, dx.$$

Since $\sigma > 1$,

$$\int_{2}^{\infty} x^{-\sigma}\, dx = \left[\frac{x^{1-\sigma}}{1-\sigma}\right]_2^\infty = \frac{2^{1-\sigma}}{\sigma - 1}.$$

Hence the partial sums are uniformly bounded, so

$$\sum_{i \ge 2} \frac{\Delta_i}{p_i^\sigma} < \infty.$$

This is exactly

$$\sum_{p \ge 3} \frac{\Delta_p}{p^\sigma} < \infty.$$

□

Absolute convergence of the majorant Euler product

Theorem 15. *The Euler product*

$$\widetilde{D}(s) = \frac{1}{1 - 2^{1-s}} \prod_{p \ge 3} \left(1 + \frac{\Delta_p}{p^s - p}\right)$$

converges absolutely for every complex number s with $\Re(s) > 1$. Consequently, the Dirichlet series

$$D_g(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

also converges absolutely for $\Re(s) > 1$.

Proof. Let

$$s = \sigma + it \qquad \text{with} \qquad \sigma = \Re(s) > 1.$$

We first estimate the local denominator $p^s - p$. Since

$$|p^s| = p^\sigma,$$

the reverse triangle inequality gives

$$|p^s - p| \ge \big||p^s| - p\big| = p^\sigma - p.$$

Therefore
$$\left|\frac{\Delta_p}{p^s - p}\right| \le \frac{\Delta_p}{p^\sigma - p}.$$
Now factor
$$p^\sigma - p = p^\sigma(1 - p^{1-\sigma}),$$
so
$$\frac{\Delta_p}{p^\sigma - p} = \frac{\Delta_p}{p^\sigma(1 - p^{1-\sigma})}.$$
Since $p \ge 3$ and $\sigma > 1$, we have
$$0 < p^{1-\sigma} \le 3^{1-\sigma} < 1,$$
hence
$$1 - p^{1-\sigma} \ge 1 - 3^{1-\sigma} > 0.$$
Thus
$$\frac{1}{1 - p^{1-\sigma}} \le \frac{1}{1 - 3^{1-\sigma}},$$
and so
$$\frac{\Delta_p}{p^\sigma - p} \le \frac{1}{1 - 3^{1-\sigma}} \frac{\Delta_p}{p^\sigma}.$$
Summing over odd primes $p \ge 3$ yields
$$\sum_{p \ge 3} \left|\frac{\Delta_p}{p^s - p}\right| \le \frac{1}{1 - 3^{1-\sigma}} \sum_{p \ge 3} \frac{\Delta_p}{p^\sigma}.$$
By the previous lemma, the series on the right converges. Hence
$$\sum_{p \ge 3} \left|\frac{\Delta_p}{p^s - p}\right| < \infty.$$

Now use the standard criterion for infinite products: if $\sum_p |a_p| < \infty$, then the product $\prod_p (1 + a_p)$ converges absolutely. Here
$$a_p := \frac{\Delta_p}{p^s - p},$$
so the product
$$\prod_{p \ge 3} \left(1 + \frac{\Delta_p}{p^s - p}\right)$$
converges absolutely for $\Re(s) > 1$.

Moreover, the factor
$$\frac{1}{1 - 2^{1-s}}$$
is well-defined and holomorphic on $\Re(s) > 1$, because there $|2^{1-s}| = 2^{1-\sigma} < 1$. Hence $\widetilde{D}(s)$ converges absolutely for $\Re(s) > 1$.

Finally, since
$$0 \le g(n) \le \widetilde{g}(n) \qquad (n \ge 1),$$
we have
$$\sum_{n=1}^{\infty} \left|\frac{g(n)}{n^s}\right| = \sum_{n=1}^{\infty} \frac{g(n)}{n^\sigma} \le \sum_{n=1}^{\infty} \frac{\widetilde{g}(n)}{n^\sigma} = \widetilde{D}(\sigma) < \infty \qquad (\sigma > 1).$$

Therefore

$$D_g(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

converges absolutely in the half-plane $\Re(s) > 1$. $\square$

The exact abscissa of absolute convergence

Corollary 24. *The abscissa of absolute convergence of $D_g(s)$ is exactly*

$$\boxed{\sigma_a(D_g) = 1.}$$

Proof. By the previous theorem, $D_g(s)$ converges absolutely for every s with $\Re(s) > 1$. Therefore

$$\sigma_a(D_g) \leq 1.$$

To prove the reverse inequality, observe that for every odd prime p one has

$$g(p) = \Delta_p.$$

Indeed, when $n = p$ is an odd prime, the block formula has only one factor and exponent 1, so

$$g(p) = \Delta_p.$$

Since every prime gap is at least 1, we obtain

$$g(p) = \Delta_p \geq 1 \qquad (p \geq 3 \text{ prime}).$$

Hence for real $\sigma > 1$,

$$D_g(\sigma) = \sum_{n=1}^{\infty} \frac{g(n)}{n^\sigma} \geq \sum_{p \geq 3} \frac{g(p)}{p^\sigma} \geq \sum_{p \geq 3} \frac{1}{p^\sigma}.$$

At $\sigma = 1$ this lower bound becomes

$$\sum_{p \geq 3} \frac{1}{p},$$

which diverges. Therefore the series $D_g(s)$ does not converge absolutely on the line $\Re(s) = 1$. This shows

$$\sigma_a(D_g) \geq 1.$$

Combining both inequalities gives

$$\sigma_a(D_g) = 1.$$

$\square$

Consequences for the summatory function

Define the summatory function

$$G(x) := \sum_{n \leq x} g(n) \qquad (x \geq 1).$$

Corollary 25. *For every $\varepsilon > 0$ one has*

$$G(x) \ll_\varepsilon x^{1+\varepsilon}.$$

Proof. Fix $\varepsilon > 0$ and set

$$\sigma := 1 + \varepsilon.$$

Then for every integer $n \le x$ we have

$$n^\sigma \le x^\sigma,$$

hence

$$g(n) = \frac{g(n)}{n^\sigma} n^\sigma \le x^\sigma \frac{g(n)}{n^\sigma}.$$

Summing over $n \le x$ gives

$$G(x) = \sum_{n \le x} g(n) \le x^\sigma \sum_{n \le x} \frac{g(n)}{n^\sigma} \le x^\sigma \sum_{n=1}^{\infty} \frac{g(n)}{n^\sigma}.$$

Since $\sigma > 1$, the previous theorem shows that

$$D_g(\sigma) = \sum_{n=1}^{\infty} \frac{g(n)}{n^\sigma} < \infty.$$

Therefore

$$G(x) \le D_g(1+\varepsilon)\, x^{1+\varepsilon}.$$

This is exactly

$$G(x) \ll_\varepsilon x^{1+\varepsilon}.$$

□

Holomorphy and termwise differentiation

Corollary 26. *The function $D_g(s)$ is holomorphic in the half-plane $\Re(s) > 1$. Moreover, in this half-plane one may differentiate termwise:*

$$D_g'(s) = -\sum_{n=1}^{\infty} \frac{g(n) \log n}{n^s}.$$

More generally, for every integer $k \ge 0$,

$$D_g^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{g(n)(\log n)^k}{n^s}, \qquad \Re(s) > 1.$$

Proof. Fix a compact subset $K \subset \{s \in \mathbb{C} : \Re(s) > 1\}$. Then there exists $\sigma_0 > 1$ such that

$$\Re(s) \ge \sigma_0 \qquad (s \in K).$$

Hence for every $s \in K$,

$$\left| \frac{g(n)}{n^s} \right| = \frac{g(n)}{n^{\Re(s)}} \le \frac{g(n)}{n^{\sigma_0}}.$$

Since $D_g(\sigma_0)$ converges, the Weierstrass M-test shows that

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

converges uniformly on K, hence defines a holomorphic function on $\Re(s) > 1$.

For differentiation, observe similarly that on K,

$$\left| \frac{g(n) \log n}{n^s} \right| \leq \frac{g(n) \log n}{n^{\sigma_0}}.$$

Choose σ_1 such that

$$1 < \sigma_1 < \sigma_0.$$

Then

$$\frac{\log n}{n^{\sigma_0}} = \frac{1}{n^{\sigma_1}} \cdot \frac{\log n}{n^{\sigma_0 - \sigma_1}}.$$

Since $\sigma_0 - \sigma_1 > 0$, the quantity

$$\sup_{n \geq 1} \frac{\log n}{n^{\sigma_0 - \sigma_1}}$$

is finite. Therefore

$$\frac{g(n) \log n}{n^{\sigma_0}} \ll \frac{g(n)}{n^{\sigma_1}}.$$

Because $D_g(\sigma_1)$ converges, the series of derivatives converges uniformly on K, and termwise differentiation is justified:

$$D_g'(s) = -\sum_{n=1}^{\infty} \frac{g(n) \log n}{n^s}.$$

The same argument applies inductively to higher derivatives, using that for every fixed $k \geq 0$,

$$(\log n)^k \ll_\delta n^\delta \qquad (\delta > 0).$$

□

Abel integral representation

Corollary 27. *For every s with $\Re(s) > 1$, the Dirichlet series admits the Abel integral representation*

$$D_g(s) = s \int_1^{\infty} \frac{G(x)}{x^{s+1}} \, dx.$$

Proof. Fix s with $\Re(s) > 1$. Since

$$G(x) \ll_\varepsilon x^{1+\varepsilon}$$

for every $\varepsilon > 0$, choose $\varepsilon > 0$ so small that

$$1 + \varepsilon < \Re(s).$$

Then

$$\frac{G(x)}{x^{s+1}} \ll x^{1+\varepsilon-\Re(s)-1} = x^{\varepsilon - \Re(s)},$$

and the exponent $\varepsilon - \Re(s)$ is strictly less than -1. Hence the integral

$$\int_1^{\infty} \frac{G(x)}{x^{s+1}} \, dx$$

converges absolutely.

Now apply the standard Abel summation formula to the partial sums

$$\sum_{n\le X} \frac{g(n)}{n^s}.$$

It yields

$$\sum_{n\le X} \frac{g(n)}{n^s} = \frac{G(X)}{X^s} + s\int_1^X \frac{G(x)}{x^{s+1}}\,dx.$$

As $X \to \infty$, the first term tends to 0, because

$$\frac{G(X)}{X^s} \ll X^{1+\varepsilon-\Re(s)} \to 0.$$

Therefore

$$D_g(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = s\int_1^{\infty} \frac{G(x)}{x^{s+1}}\,dx.$$

□

Interpretation for prime gaps

The previous results have several immediate conceptual consequences.

1. The trivial estimate $g(n) \le n$ only yields absolute convergence in the half-plane $\Re(s) > 2$. The argument above shows that the *true* abscissa of absolute convergence is in fact

$$\boxed{\sigma_a(D_g) = 1.}$$

2. This improvement is completely unconditional. It uses neither the Prime Number Theorem nor any refined estimate for individual prime gaps. The only input beyond the block formula is the elementary monotonicity estimate

$$\sum_{p\ge 3} \frac{\Delta_p}{p^\sigma} < \infty \qquad (\sigma > 1),$$

which comes directly from integrating $x^{-\sigma}$ over successive prime intervals.

3. Thus the prime-gap factors encoded in $g(n)$ are globally sparse enough that the Dirichlet series already converges absolutely throughout the half-plane $\Re(s) > 1$. Large prime gaps may occur, but they cannot occur with sufficient weighted density to force absolute divergence to the right of the line $\Re(s) = 1$.

4. The line $\Re(s) = 1$ is therefore the natural critical boundary of the absolute theory. Understanding the finer behaviour of $D_g(s)$ near $s = 1$—for example, whether it has a pole, what the order of divergence is, and whether one has an asymptotic such as

$$G(x) \sim Cx(\log x)^2$$

—is the next problem.

45 Fine structure near $s = 1$: Abelian reduction, conjectural pole order, and gap-energy heuristics

In the previous section we proved unconditionally that the Dirichlet series

$$D_g(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

converges absolutely in the half-plane $\Re(s) > 1$, and that its abscissa of absolute convergence is exactly

$$\sigma_a(D_g) = 1.$$

Thus the vertical line $\Re(s) = 1$ is the natural critical boundary of the absolute theory.

The next question is finer: what is the precise behaviour of $D_g(s)$ as $s \to 1^+$? Does $D_g(s)$ have a pole at $s = 1$? If so, of what order? Equivalently, what is the true asymptotic growth of the summatory function

$$G(x) := \sum_{n \le x} g(n)?$$

In this section we isolate the exact Abelian mechanism linking $G(x)$ to the singular behaviour of $D_g(s)$ near $s = 1$, and we formulate the conjectural asymptotics suggested by the numerical data.

1. Abel summation as the bridge between arithmetic and analysis

We begin from the integral representation established in the previous section:

$$D_g(s) = s \int_1^{\infty} \frac{G(x)}{x^{s+1}}\, dx \qquad (\Re(s) > 1).$$

This formula shows that the singular behaviour of $D_g(s)$ near $s = 1$ is controlled entirely by the large-x growth of $G(x)$.

It is therefore useful to record the general Mellin-type correspondence.

Proposition 69 (Abelian model computation)**.** *Let $\beta > -1$ and $C \in \mathbb{R}$. Then for every s with $\Re(s) > 1$,*

$$s \int_1^{\infty} \frac{C\, x(\log x)^{\beta}}{x^{s+1}}\, dx = Cs \int_1^{\infty} (\log x)^{\beta} x^{-s}\, dx.$$

After the change of variables $x = e^u$, one obtains

$$Cs \int_0^{\infty} u^{\beta} e^{-(s-1)u}\, du.$$

Hence

$$s \int_1^{\infty} \frac{C\, x(\log x)^{\beta}}{x^{s+1}}\, dx = Cs\, \Gamma(\beta + 1)(s-1)^{-(\beta+1)}.$$

In particular, as $s \to 1^+$,

$$s \int_1^{\infty} \frac{C\, x(\log x)^{\beta}}{x^{s+1}}\, dx \sim \frac{C\, \Gamma(\beta+1)}{(s-1)^{\beta+1}}.$$

Proof. Starting from

$$s\int_1^\infty \frac{C\,x(\log x)^\beta}{x^{s+1}}\,dx,$$

we cancel one factor of x and get

$$Cs\int_1^\infty (\log x)^\beta x^{-s}\,dx.$$

Now set

$$x = e^u, \qquad dx = e^u\,du, \qquad \log x = u.$$

Then

$$x^{-s}\,dx = e^{-su}e^u\,du = e^{-(s-1)u}\,du.$$

Therefore

$$Cs\int_1^\infty (\log x)^\beta x^{-s}\,dx = Cs\int_0^\infty u^\beta e^{-(s-1)u}\,du.$$

Finally, using the standard Gamma-integral

$$\int_0^\infty u^\beta e^{-\lambda u}\,du = \Gamma(\beta+1)\lambda^{-(\beta+1)} \qquad (\Re(\lambda) > 0,\ \beta > -1),$$

with $\lambda = s-1$, we obtain

$$Cs\int_0^\infty u^\beta e^{-(s-1)u}\,du = Cs\,\Gamma(\beta+1)(s-1)^{-(\beta+1)}.$$

This proves the claim. □

The proposition shows that the exponent of $(\log x)$ in $G(x)$ becomes the pole order minus one in $D_g(s)$.

2. Consequences of a hypothetical asymptotic for $G(x)$

We now record the precise analytic consequences of a conjectural asymptotic formula for $G(x)$.

Theorem 16 (Asymptotic transfer near $s = 1$)**.** *Assume that for some real $\beta > -1$ and some constant $C > 0$ one has*

$$G(x) = Cx(\log x)^\beta + o\Big(x(\log x)^\beta\Big) \qquad (x \to \infty).$$

Then, as $s \to 1^+$ along the real axis,

$$D_g(s) \sim \frac{C\,\Gamma(\beta+1)}{(s-1)^{\beta+1}}.$$

Proof. Write

$$G(x) = Cx(\log x)^\beta + R(x), \qquad R(x) = o\Big(x(\log x)^\beta\Big).$$

By Abel summation,

$$D_g(s) = s\int_1^\infty \frac{G(x)}{x^{s+1}}\,dx = s\int_1^\infty \frac{Cx(\log x)^\beta}{x^{s+1}}\,dx + s\int_1^\infty \frac{R(x)}{x^{s+1}}\,dx.$$

The first term is given by the previous proposition:

$$s\int_1^\infty \frac{Cx(\log x)^\beta}{x^{s+1}}\,dx = Cs\,\Gamma(\beta+1)(s-1)^{-(\beta+1)}.$$

Since $s \to 1^+$, this is asymptotic to

$$\frac{C\,\Gamma(\beta+1)}{(s-1)^{\beta+1}}.$$

It remains to show that the remainder term is of smaller order. Fix $\varepsilon > 0$. By the definition of $o(\cdot)$, there exists $X_\varepsilon \geq 1$ such that for all $x \geq X_\varepsilon$,

$$|R(x)| \leq \varepsilon\, x(\log x)^\beta.$$

Hence

$$\left| s\int_{X_\varepsilon}^\infty \frac{R(x)}{x^{s+1}}\,dx \right| \leq \varepsilon s \int_{X_\varepsilon}^\infty (\log x)^\beta x^{-s}\,dx.$$

By the same change of variables as above, this is

$$\ll \varepsilon\,(s-1)^{-(\beta+1)}.$$

The integral over $[1, X_\varepsilon]$ is bounded uniformly as $s \to 1^+$, since it is taken over a finite interval. Therefore the remainder term is

$$o\Big((s-1)^{-(\beta+1)}\Big),$$

and the claimed asymptotic follows. □

Corollary 28 (The cubic-pole scenario)**.** *If*

$$G(x) \sim Cx(\log x)^2,$$

then

$$D_g(s) \sim \frac{2C}{(s-1)^3} \qquad (s \to 1^+).$$

Proof. Apply the theorem with $\beta = 2$. Since

$$\Gamma(3) = 2,$$

one obtains

$$D_g(s) \sim \frac{2C}{(s-1)^3}.$$

□

Thus the natural conjecture suggested by the numerical data corresponds to a *triple* pole at $s = 1$, not a double pole.

3. A first lower bound from the prime layer

Even without any refined asymptotic for $G(x)$, one already gets a nontrivial lower bound on the size of $D_g(s)$ near $s = 1$ from the prime terms alone.

Proposition 70 (Prime-layer lower bound)**.** *For every real $\sigma > 1$,*

$$D_g(\sigma) \geq \sum_{p \geq 3} \frac{\Delta_p}{p^\sigma}.$$

Moreover,

$$\sum_{p \geq 3} \frac{\Delta_p}{p^\sigma} \leq \int_2^\infty x^{-\sigma}\, dx = \frac{2^{1-\sigma}}{\sigma - 1}.$$

In particular, the prime layer already exhibits critical behaviour at $\sigma = 1$.

Proof. Since $g(p) = \Delta_p$ for odd primes p, we have

$$D_g(\sigma) = \sum_{n=1}^{\infty} \frac{g(n)}{n^\sigma} \geq \sum_{p \geq 3} \frac{g(p)}{p^\sigma} = \sum_{p \geq 3} \frac{\Delta_p}{p^\sigma}.$$

The upper bound was proved in the previous section by the interval comparison

$$\frac{\Delta_i}{p_i^\sigma} \leq \int_{p_{i-1}}^{p_i} x^{-\sigma}\, dx,$$

followed by summation over i. □

This proposition is not yet an asymptotic formula, but it shows that the line $\sigma = 1$ is already visible at the prime layer itself.

4. Why the conjectural main term should be polylogarithmic

We now explain why the numerics strongly suggest that $G(x)$ should be of the form

$$G(x) \asymp x(\log x)^2.$$

The pointwise majorant proved earlier is

$$g(n) \leq n \prod_{\substack{p \mid n \\ p \text{ odd}}} \frac{\Delta_p}{p}.$$

If one replaces the actual prime gaps by their heuristic average size

$$\Delta_p \approx \log p,$$

then this becomes

$$g(n) \lesssim n \prod_{p \mid n} \frac{\log p}{p}.$$

This multiplicative weight is typically much smaller than n, because each distinct odd prime factor contributes a factor of size roughly $(\log p)/p$.

Heuristically, summing such a weight over $n \le x$ produces a polylogarithmic correction rather than polynomial excess growth. The numerical data generated from the exact block formula are consistent with

$$\frac{G(x)}{x(\log x)^2}$$

approaching a positive constant, which leads to the following natural conjecture.

Conjecture 2 (Mean growth of the Möbius weights)**.** *There exists a constant $C_g > 0$ such that*

$$G(x) = \sum_{n \le x} g(n) \sim C_g \, x(\log x)^2 \qquad (x \to \infty).$$

Equivalently,

$$D_g(s) \sim \frac{2C_g}{(s-1)^3} \qquad (s \to 1^+).$$

5. Logarithmic differentiation and the meaning of the singularity

The majorant Euler product

$$\widetilde{D}(s) = \frac{1}{1-2^{1-s}} \prod_{p \ge 3} \left(1 + \frac{\Delta_p}{p^s - p}\right)$$

is not equal to $D_g(s)$, but it provides a useful model for the singular behaviour near $s = 1$.

Formally taking logarithms gives

$$\log \widetilde{D}(s) = -\log(1 - 2^{1-s}) + \sum_{p \ge 3} \log\left(1 + \frac{\Delta_p}{p^s - p}\right).$$

When $\Re(s) > 1$ and p is large, the local term satisfies

$$\frac{\Delta_p}{p^s - p} = \frac{\Delta_p}{p^s(1 - p^{1-s})} \approx \frac{\Delta_p}{p^s}.$$

Thus, at a purely heuristic level,

$$\log \widetilde{D}(s) \approx \sum_p \frac{\Delta_p}{p^s}.$$

If one further substitutes the average law $\Delta_p \sim \log p$, one is led to

$$\sum_p \frac{\log p}{p^s},$$

which is critical exactly at $s = 1$. This is consistent with the fact that the prime-gap structure pushes the Dirichlet series to the edge $\Re(s) = 1$, but no further.

Differentiating formally gives

$$\frac{\widetilde{D}'(s)}{\widetilde{D}(s)} = \frac{(\log 2)\, 2^{1-s}}{1 - 2^{1-s}} + \sum_{p \ge 3} \frac{\dfrac{d}{ds}\left(\dfrac{\Delta_p}{p^s - p}\right)}{1 + \dfrac{\Delta_p}{p^s - p}}.$$

Since

$$\frac{d}{ds}\left(\frac{\Delta_p}{p^s - p}\right) = -\Delta_p(\log p)\frac{p^s}{(p^s - p)^2},$$

the logarithmic derivative weights each prime gap by an additional factor of $\log p$. This suggests that the logarithmic derivative is a natural analytic probe of the *gap energy distribution* across the prime scale.

6. A gap-energy functional

The determinant formulas in the paper show that products of prime gaps govern both the prime Gram determinants and the full integer Gram determinants. In particular, the prime-layer determinant satisfies

$$\det(M_k) = \prod_{i=1}^{k} \Delta_i,$$

so that

$$\frac{1}{k}\log\det(M_k) = \frac{1}{k}\sum_{i=1}^{k} \log \Delta_i.$$

This was explicitly highlighted in the paper as a natural free-energy analogue for the prime-gap medium.

On the full integer side, Proposition 68 shows that in the product

$$\prod_{n\leq N} g(n),$$

the gap factor Δ_i appears with exact multiplicity $\lfloor N/p_i \rfloor$, up to an additional integer factor $U_N \geq 1$. More precisely,

$$\prod_{n\leq N} g(n) = 2^{E_2(N)}\left(\prod_{3\leq p_i\leq N} \Delta_i^{\lfloor N/p_i \rfloor}\right) U_N.$$

Thus

$$\sum_{n\leq N} \log g(n) = E_2(N)\log 2 + \sum_{3\leq p_i\leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i + \log U_N.$$

This identity makes it natural to define the weighted gap-energy functional

$$\mathcal{E}(N) := \sum_{3\leq p_i\leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i.$$

The Hadamard bound in the paper implies

$$\mathcal{E}(N) \leq N\log N + O(N),$$

so the weighted logarithmic mass of the prime gaps is globally constrained.

From the perspective of the Dirichlet series, $\mathcal{E}(N)$ is the natural finite-scale precursor of a putative singular expansion of $\log D_g(s)$ near $s = 1$.

7. What is now proved, and what remains open

At this point, the situation is as follows.

- It is proved that $D_g(s)$ converges absolutely for $\Re(s) > 1$, and that $\sigma_a(D_g) = 1$.
- It is proved that
$$G(x) \ll_\varepsilon x^{1+\varepsilon} \qquad (\varepsilon > 0).$$
- It is proved that if one had an asymptotic
$$G(x) \sim Cx(\log x)^\beta,$$
then automatically
$$D_g(s) \sim \frac{C\,\Gamma(\beta+1)}{(s-1)^{\beta+1}} \qquad (s \to 1^+).$$
- The numerical evidence suggests the specific case
$$G(x) \sim C_g x(\log x)^2,$$
which would imply the cubic-pole law
$$D_g(s) \sim \frac{2C_g}{(s-1)^3}.$$
- What remains open is the actual proof of such an asymptotic, or even the proof of a sharp upper bound of the form
$$G(x) \ll x(\log x)^A$$
for some fixed A.

This leads to the following concrete next problem.

Problem 17. *Determine the true asymptotic growth of*
$$G(x) = \sum_{n \le x} g(n).$$
In particular, decide whether there exists $C_g > 0$ such that
$$G(x) \sim C_g x(\log x)^2,$$
and hence whether $D_g(s)$ has a cubic pole at $s = 1$.

46 Towards upper bounds for $G(x) = \sum_{n \le x} g(n)$ via the multiplicative majorant

In the previous section we proved that
$$g(n) \le \widetilde{g}(n) := n \prod_{\substack{p \mid n \\ p \text{ odd}}} \frac{\Delta_p}{p},$$

and that the corresponding majorant Dirichlet series

$$\widetilde{D}(s) := \sum_{n=1}^{\infty} \frac{\widetilde{g}(n)}{n^s} = \frac{1}{1-2^{1-s}} \prod_{p\geq 3} \left(1 + \frac{\Delta_p}{p^s - p}\right)$$

converges absolutely for $\Re(s) > 1$.

The natural next problem is to convert this information into upper bounds for the summatory function

$$G(x) := \sum_{n\leq x} g(n).$$

A first trivial consequence was

$$G(x) \ll_\varepsilon x^{1+\varepsilon} \qquad (\varepsilon > 0).$$

In this section we show that the majorant method can in fact be pushed significantly further, yielding a subexponential improvement. At the same time, the proof makes transparent why this method alone does not yet seem sufficient to reach a polylogarithmic bound of the form

$$G(x) \ll x(\log x)^A.$$

1. A sharper estimate for the majorant Euler product near $s = 1$

We begin with the Euler product

$$\widetilde{D}(s) = \frac{1}{1-2^{1-s}} \prod_{p\geq 3} \left(1 + \frac{\Delta_p}{p^s - p}\right).$$

To estimate it near the line $\Re(s) = 1$, we consider real parameters

$$s = 1 + \eta, \qquad \eta > 0.$$

Then

$$\widetilde{D}(1+\eta) = \frac{1}{1-2^{-\eta}} \prod_{p\geq 3} \left(1 + \frac{\Delta_p}{p^{1+\eta} - p}\right).$$

Since all local factors are positive, we may use the inequality

$$\log(1+u) \leq u \qquad (u \geq 0),$$

to obtain

$$\log \widetilde{D}(1+\eta) \leq -\log(1-2^{-\eta}) + \sum_{p\geq 3} \frac{\Delta_p}{p^{1+\eta} - p}.$$

Thus the problem reduces to estimating the sum

$$S(\eta) := \sum_{p\geq 3} \frac{\Delta_p}{p^{1+\eta} - p}.$$

2. Integral comparison for $S(\eta)$

Lemma 14. *For every $\eta > 0$ one has*

$$S(\eta) = \sum_{p \geq 3} \frac{\Delta_p}{p^{1+\eta} - p} \leq \int_2^\infty \frac{dx}{x^{1+\eta} - x}.$$

Proof. Let p_i denote the i-th prime and write $\Delta_i = p_i - p_{i-1}$. Consider the function

$$f_\eta(x) := \frac{1}{x^{1+\eta} - x}, \qquad x \geq 2.$$

Since the denominator

$$x^{1+\eta} - x = x(x^\eta - 1)$$

is strictly increasing for $x \geq 2$, the function f_η is positive and decreasing on $[2, \infty)$.

Hence for every interval $[p_{i-1}, p_i]$ and every $x \in [p_{i-1}, p_i]$ we have

$$f_\eta(x) \geq f_\eta(p_i) = \frac{1}{p_i^{1+\eta} - p_i}.$$

Integrating over $[p_{i-1}, p_i]$ gives

$$\int_{p_{i-1}}^{p_i} f_\eta(x)\, dx \geq (p_i - p_{i-1}) f_\eta(p_i) = \frac{\Delta_i}{p_i^{1+\eta} - p_i}.$$

Summing over all $i \geq 2$ yields

$$\sum_{i \geq 2} \frac{\Delta_i}{p_i^{1+\eta} - p_i} \leq \sum_{i \geq 2} \int_{p_{i-1}}^{p_i} f_\eta(x)\, dx = \int_2^\infty \frac{dx}{x^{1+\eta} - x}.$$

This is exactly the stated inequality. □

We now estimate the integral on the right-hand side.

Lemma 15. *There exists an absolute constant $C_1 > 0$ such that for all sufficiently small $\eta > 0$,*

$$\int_2^\infty \frac{dx}{x^{1+\eta} - x} \leq C_1 \frac{\log(1/\eta)}{\eta}.$$

Proof. Set

$$I(\eta) := \int_2^\infty \frac{dx}{x^{1+\eta} - x}.$$

Since

$$x^{1+\eta} - x = x(x^\eta - 1),$$

we may write

$$I(\eta) = \int_2^\infty \frac{dx}{x(x^\eta - 1)}.$$

Now substitute

$$x = e^u, \qquad dx = e^u\, du, \qquad x^\eta = e^{\eta u}.$$

Then

$$\frac{dx}{x(x^\eta - 1)} = \frac{e^u\, du}{e^u(e^{\eta u} - 1)} = \frac{du}{e^{\eta u} - 1}.$$

Hence

$$I(\eta) = \int_{\log 2}^{\infty} \frac{du}{e^{\eta u} - 1}.$$

Now set

$$v = \eta u, \qquad du = \frac{dv}{\eta},$$

to obtain

$$I(\eta) = \frac{1}{\eta} \int_{\eta \log 2}^{\infty} \frac{dv}{e^{v} - 1}.$$

We split the integral at $v = 1$:

$$I(\eta) = \frac{1}{\eta} \int_{\eta \log 2}^{1} \frac{dv}{e^{v} - 1} + \frac{1}{\eta} \int_{1}^{\infty} \frac{dv}{e^{v} - 1}.$$

For $0 < v \leq 1$, one has

$$e^{v} - 1 \geq \frac{v}{2},$$

hence

$$\frac{1}{e^{v} - 1} \leq \frac{2}{v}.$$

Therefore

$$\frac{1}{\eta} \int_{\eta \log 2}^{1} \frac{dv}{e^{v} - 1} \leq \frac{2}{\eta} \int_{\eta \log 2}^{1} \frac{dv}{v} = \frac{2}{\eta} \log\left(\frac{1}{\eta \log 2}\right).$$

For $v \geq 1$, we have

$$e^{v} - 1 \geq \frac{e^{v}}{2},$$

so

$$\frac{1}{e^{v} - 1} \leq 2e^{-v}.$$

Thus

$$\frac{1}{\eta} \int_{1}^{\infty} \frac{dv}{e^{v} - 1} \leq \frac{2}{\eta} \int_{1}^{\infty} e^{-v}\, dv = \frac{2e^{-1}}{\eta}.$$

Combining both bounds, we obtain

$$I(\eta) \leq \frac{2}{\eta} \log\left(\frac{1}{\eta \log 2}\right) + \frac{2e^{-1}}{\eta}.$$

For sufficiently small η, this is

$$I(\eta) \ll \frac{\log(1/\eta)}{\eta}.$$

This proves the claim. $\square$

3. Growth of the majorant Dirichlet series near $s = 1$

Proposition 71. *There exists a constant $C_2 > 0$ such that for all sufficiently small $\eta > 0$,*

$$\widetilde{D}(1+\eta) \le \exp\left(C_2 \frac{\log(1/\eta)}{\eta}\right).$$

Consequently,

$$D_g(1+\eta) \le \widetilde{D}(1+\eta) \le \exp\left(C_2 \frac{\log(1/\eta)}{\eta}\right).$$

Proof. From the previous discussion,

$$\log \widetilde{D}(1+\eta) \le -\log(1-2^{-\eta}) + S(\eta).$$

By the preceding two lemmas,

$$S(\eta) \ll \frac{\log(1/\eta)}{\eta}.$$

It remains to estimate the factor at 2. For small η,

$$1 - 2^{-\eta} = 1 - e^{-\eta \log 2} \asymp \eta,$$

hence

$$-\log(1-2^{-\eta}) \ll \log(1/\eta).$$

Since

$$\log(1/\eta) \ll \frac{\log(1/\eta)}{\eta},$$

we conclude that

$$\log \widetilde{D}(1+\eta) \ll \frac{\log(1/\eta)}{\eta}.$$

Exponentiating gives

$$\widetilde{D}(1+\eta) \le \exp\left(C_2 \frac{\log(1/\eta)}{\eta}\right)$$

for some constant $C_2 > 0$. Finally, since $0 \le g(n) \le \widetilde{g}(n)$ for all n,

$$D_g(1+\eta) \le \widetilde{D}(1+\eta).$$

□

4. A subexponential upper bound for $G(x)$

We now feed this estimate into the trivial inequality

$$G(x) = \sum_{n \le x} g(n) \le x^{1+\eta} D_g(1+\eta), \qquad \eta > 0.$$

Theorem 18. *There exists a constant $C_3 > 0$ such that for all sufficiently large x,*

$$G(x) \ll x \exp\left(C_3 \sqrt{\log x \, \log\log x}\right).$$

Proof. Fix $x \geq e^e$ and choose

$$\eta := \sqrt{\frac{\log\log x}{\log x}}.$$

Then $\eta \to 0$ as $x \to \infty$, so for all sufficiently large x we may apply the previous proposition.

We begin from

$$G(x) \leq x^{1+\eta} D_g(1+\eta) \leq x^{1+\eta} \exp\left(C_2 \frac{\log(1/\eta)}{\eta} \right).$$

Taking logarithms of the multiplicative factor beyond x, we get

$$\eta \log x + C_2 \frac{\log(1/\eta)}{\eta}.$$

For our choice of η,

$$\eta \log x = \sqrt{\log x \, \log\log x}.$$

Next,

$$\frac{1}{\eta} = \sqrt{\frac{\log x}{\log\log x}}.$$

Also,

$$\log(1/\eta) = \frac{1}{2} \log\left(\frac{\log x}{\log\log x} \right) \ll \log\log x.$$

Therefore

$$\frac{\log(1/\eta)}{\eta} \ll \sqrt{\log x \, \log\log x}.$$

Hence

$$\eta \log x + C_2 \frac{\log(1/\eta)}{\eta} \ll \sqrt{\log x \, \log\log x}.$$

Exponentiating, we obtain

$$G(x) \ll x \exp\left(C_3 \sqrt{\log x \, \log\log x} \right)$$

for some constant $C_3 > 0$. □

5. Why this still falls short of a polylogarithmic bound

The previous theorem is a genuine improvement over the crude estimate

$$G(x) \ll_\varepsilon x^{1+\varepsilon},$$

but it still does not imply

$$G(x) \ll x(\log x)^A.$$

The obstruction is visible already in the quantity

$$S(\eta) = \sum_{p \geq 3} \frac{\Delta_p}{p^{1+\eta} - p}.$$

Indeed, the integral comparison shows that the majorant method naturally produces

$$S(\eta) \ll \frac{\log(1/\eta)}{\eta}.$$

If one had instead the substantially sharper estimate

$$S(\eta) \ll \log(1/\eta),$$

then one would get

$$D_g(1+\eta) \ll \eta^{-A}$$

for some $A > 0$, and this would in turn imply

$$G(x) \ll x(\log x)^A.$$

Thus the missing step is exactly the removal of the extra factor $1/\eta$.

This shows that the direct multiplicative majorant, while surprisingly effective, is still too coarse to recover the conjectural polylogarithmic growth.

6. A conditional bridge to the polylogarithmic regime

The previous discussion can be summarized as the following conditional statement.

Proposition 72. *Assume that there exists a constant $A_0 > 0$ such that*

$$\log D_g(1+\eta) \ll \log(1/\eta) \qquad (\eta \downarrow 0).$$

Then

$$D_g(1+\eta) \ll \eta^{-A_0'}$$

for some constant $A_0' > 0$, and consequently

$$G(x) \ll x(\log x)^{A_0'} \qquad (x \geq 2).$$

Proof. The assumption implies

$$D_g(1+\eta) \leq \exp\big(C \log(1/\eta)\big) = \eta^{-C}$$

for some constant $C > 0$. Now for any $\eta > 0$,

$$G(x) \leq x^{1+\eta} D_g(1+\eta) \leq x^{1+\eta}\eta^{-C}.$$

Choose

$$\eta := \frac{1}{\log x}.$$

Then

$$x^\eta = e, \qquad \eta^{-C} = (\log x)^C.$$

Therefore

$$G(x) \ll x(\log x)^C.$$

□

This proposition identifies the precise analytic threshold: a polylogarithmic upper bound for $G(x)$ would follow from a merely logarithmic blow-up of $\log D_g(1+\eta)$ as $\eta \downarrow 0$.

7. Interpretation and next step

The current picture is therefore the following.

1. The multiplicative majorant proves unconditionally that

$$D_g(s)$$

converges absolutely for $\Re(s) > 1$.

2. The same method yields the improved summatory estimate

$$G(x) \ll x \exp\Big(C\sqrt{\log x \, \log\log x}\Big).$$

3. The conjectural bound

$$G(x) \ll x(\log x)^A,$$

and in particular the stronger asymptotic

$$G(x) \sim C_g x(\log x)^2,$$

remain open.

4. The exact obstruction is now transparent: one needs a mechanism that improves the majorant estimate

$$\log D_g(1+\eta) \ll \frac{\log(1/\eta)}{\eta}$$

to something of size

$$\log D_g(1+\eta) \ll \log(1/\eta).$$

This requires information beyond the raw multiplicative majorant and therefore must exploit finer structure of $g(n)$ than the inequality

$$g(n) \le \widetilde{g}(n)$$

alone captures.

This suggests that the next stage of the theory should not simply enlarge the majorant argument, but should instead try to use the *terminal-block asymmetry* of Proposition 63 directly, in order to recover cancellations or density savings that are invisible in the purely multiplicative envelope.

47 Conclusion

The theory developed here shows that the Möbius coefficients of the factorization-poset meet kernel admit a genuine analytic incarnation. Starting from the explicit formula for $g(n)$ and its terminal-block asymmetry, we constructed the Dirichlet series

$$D_g(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

and proved that it is not governed by a classical Euler product, but by a non-Eulerian prime cascade ordered by the largest prime factor. In this representation the prime gaps enter directly through the local factors, so that $D_g(s)$ becomes a global generating function for the ordered prime-gap geometry encoded by the arithmetic weights.

The first main outcome is unconditional and structural: the explicit block formula yields a multiplicative majorant strong enough to prove that

$$D_g(s)$$

converges absolutely in the whole half-plane $\Re(s) > 1$, and that its abscissa of absolute convergence is exactly

$$\sigma_a(D_g) = 1.$$

This sharply improves the trivial convergence region coming from $g(n) \le n$, and it does so without invoking the Prime Number Theorem, zero-free regions, or refined pointwise bounds for prime gaps. The proof uses only the factorization-poset structure of $g(n)$ together with the elementary interval estimate

$$\sum_{p \ge 3} \frac{\Delta_p}{p^\sigma} < \infty \qquad (\sigma > 1).$$

Thus the ordered prime-gap weights encoded by g are globally sparse enough to force absolute convergence up to the natural critical boundary $\Re(s) = 1$.

The second outcome is conceptual. Via Abel summation, the singular behaviour of $D_g(s)$ near $s = 1$ is equivalent to the growth of

$$G(x) = \sum_{n \le x} g(n).$$

This gives a precise analytic dictionary:

$$G(x) \sim Cx(\log x)^\beta \quad \Longrightarrow \quad D_g(s) \sim \frac{C\,\Gamma(\beta+1)}{(s-1)^{\beta+1}}.$$

In particular, the numerically suggested law

$$G(x) \sim C_g x(\log x)^2$$

would imply a cubic pole at $s = 1$. This identifies the exact candidate singularity and places the problem in a form that is both analytically and numerically testable. At present, however, this remains conjectural. What has been proved rigorously is the softer bound

$$G(x) \ll_\varepsilon x^{1+\varepsilon},$$

together with the stronger subexponential estimate

$$G(x) \ll x \exp\!\Big(C\sqrt{\log x\ \log\log x}\Big),$$

obtained from a refined analysis of the multiplicative majorant near $s = 1$.

Equally important is the negative information supplied by the proof. The multiplicative-majorant method, despite being strong enough to determine $\sigma_a(D_g)$, appears intrinsically too coarse to yield a polylogarithmic upper bound

$$G(x) \ll x(\log x)^A.$$

The obstruction is transparent: the majorant Euler product still grows too rapidly near $s = 1$, by an extra factor of order $1/\eta$ in the logarithm. This pinpoints exactly where further progress must come from. Any sharper theory must exploit structure that is lost in the passage to the multiplicative envelope, namely the asymmetry between internal and terminal prime blocks in the exact formula for $g(n)$. In this sense, the present work does not merely prove convergence; it also identifies the precise frontier of the current method.

These analytic developments fit naturally with the spectral and geometric framework already established in the paper. On the prime layer, the determinant of the Gram matrix is the product of the prime gaps, the inverse matrix is a grounded path Laplacian, and large gaps act as weak couplings or bottlenecks. On the full integer side, the inverse Gram matrices stabilize entrywise to an infinite positive symmetric operator

$$A = G_\infty^{-1},$$

which is closable and admits a positive Friedrichs extension. The Dirichlet series now complements this operator-theoretic picture by providing the first explicit function-theoretic object built directly from the full block structure of the arithmetic Möbius weights. The two viewpoints are clearly related, but not yet fully unified: while the existence and positivity of the limiting operator are proved, a genuine trace-resolvent theory remains open.

The central open problem is therefore both natural and sharply formulated:

$$\textit{Determine the true asymptotic growth of } G(x) = \sum_{n \le x} g(n).$$

A proof of

$$G(x) \sim C_g x (\log x)^2$$

would settle the singular order of $D_g(s)$ at $s = 1$, identify the leading constant in the analytic prime-gap cascade, and likely reveal a new large-scale statistic of the prime-gap medium. More generally, any progress toward a polylogarithmic bound for $G(x)$ would mark the point where the exact terminal-block geometry of the factorization poset begins to yield genuinely sharp analytic information about the distribution of prime gaps. That is the next step.

48 Speculative consequences for prime gaps

In this section we discuss a speculative picture that would emerge if the conjectural asymptotics from the previous sections were proved. None of the statements below is claimed as a theorem unless explicitly established earlier. The purpose is to articulate the mathematical consequences that would follow from a full confirmation of the proposed analytic behaviour of the Dirichlet series

$$D_g(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

1. The basic conjectural input

Assume that the summatory function of the Möbius weights satisfies

$$G(x) := \sum_{n \le x} g(n) \sim C_g \, x (\log x)^2 \qquad (x \to \infty)$$

for some constant $C_g > 0$. By the Abelian transfer principle established earlier, this would imply

$$D_g(s) \sim \frac{2C_g}{(s-1)^3} \qquad (s \to 1^+).$$

Thus the line $\Re(s) = 1$, which is already known to be the exact abscissa of absolute convergence of D_g, would become a genuine singular boundary of cubic type.

From the point of view of prime gaps, this would mean that the arithmetic weights $g(n)$ are globally sparse enough to remain summable for $\Re(s) > 1$, but still dense enough to accumulate a precise third-order singularity at the critical boundary. Since each odd prime factor contributes a gap factor Δ_p through the explicit block formula of Proposition 63, this would turn the cubic pole into a global analytic signature of the prime-gap sequence.

2. A conjectural thermodynamic law for prime gaps

On the prime layer, the paper proves

$$\det(M_k) = 2\prod_{i=2}^{k} \Delta_i,$$

so that

$$\log\det(M_k) = \log 2 + \sum_{i=2}^{k} \log \Delta_i.$$

This motivates the interpretation of

$$\frac{1}{k}\log\det(M_k)$$

as a free-energy-type observable of the prime-gap medium. The same idea extends to the full integer system: Proposition 68 identifies the exact multiplicity with which each gap factor Δ_i appears in the product $\prod_{n\leq N} g(n)$, leading to the weighted gap-energy

$$\mathcal{E}(N) := \sum_{3\leq p_i\leq N} \left\lfloor \frac{N}{p_i} \right\rfloor \log \Delta_i.$$

Moreover, the current paper already shows that

$$\mathcal{E}(N) \leq N\log N + O(N).$$

If the conjectural asymptotic for $G(x)$ were proved, it would be natural to expect that $\mathcal{E}(N)$ admits a matching first-order asymptotic law

$$\mathcal{E}(N) \sim \kappa N \log N$$

for some constant $\kappa > 0$. In that case, prime gaps would satisfy not merely upper bounds but a genuine entropy law: their logarithmic mass, weighted by the natural multiplicities coming from the factorization poset, would stabilize to a deterministic macroscopic density.

In speculative language, this would mean that prime gaps behave like a one-dimensional inhomogeneous medium with a well-defined thermodynamic free energy.

3. Spectral bottlenecks and record gaps

The paper proves that the inverse prime Gram matrix is exactly the grounded Laplacian of a path network with edge resistances equal to the prime gaps:

$$M_k^{-1} = L_{\mathrm{red}}, \qquad r_i = \Delta_i, \qquad c_i = \frac{1}{\Delta_i}.$$

Large prime gaps therefore correspond to weak conductances and hence to electrical bottlenecks. The effective resistance between nodes a and b is

$$R_{\mathrm{eff}}(a, b) = p_b - p_a,$$

so the primes themselves are cumulative resistances in the network.

If the conjectural singular structure of $D_g(s)$ were understood in full detail, it is natural to expect a correspondence between two phenomena:

1. large local prime gaps as bottlenecks in the path Laplacian,

2. pronounced local contributions to the singular part of the prime-gap Dirichlet cascade.

In this picture, record gaps would no longer be isolated arithmetic outliers, but spectral defects in an otherwise slowly varying medium. One would then expect at least the following heuristic consequences:

- unusually large gaps should produce localized distortions in the low spectrum of M_k^{-1};
- the top eigenvalues of M_k should be especially sensitive to sparse but extreme gaps;
- the singular part of $D_g(s)$ near $s = 1$ should decompose into a smooth bulk term plus intermittent contributions from rare barrier gaps.

None of this is proved in the present theory, but it is strongly suggested by the exact network interpretation already established in the paper.

4. A conjectural universality principle

The paper compares the prime-gap network with a random conductance model inspired by Cramér's random model, where the random gaps have mean order $\log x$. In that random model, the inverse prime matrix becomes a random path Laplacian with conductances of order $1/\log x$ but with large fluctuations.

If the conjectural asymptotics for $G(x)$ and $D_g(s)$ were proved, one could formulate the following speculative universality principle:

Universality conjecture. *After suitable normalization, the bulk spectral statistics of the prime-gap Laplacian agree with those of the corresponding slowly varying random conductance model, while the arithmetic deviations are concentrated in the extreme-gap sector and in the fine structure of the singular expansion of $D_g(s)$ near $s = 1$.*

In other words, the average growth law

$$G(x) \sim C_g x (\log x)^2$$

would describe the universal bulk behaviour of the prime-gap medium, whereas deviations from that law would encode genuinely arithmetic structure beyond the random model.

5. Recovery of prime gaps from analytic or spectral data

A major structural theorem of the paper is that the full prime-gap sequence can be recovered either from the spectra of the finite inverse matrices $B_n = M_n^{-1}$ or from the Weyl function and spectral measure of the associated Jacobi operator. Explicitly,

$$d_n = \frac{\det(B_{n-1})}{\det(B_n)}, \qquad p_n = 2 + \sum_{j=2}^{n} d_j.$$

The same paper also shows that the prime gaps canonically generate an orthogonal polynomial system via the Jacobi operator attached to the sequence (d_n).

If, in addition, the Dirichlet series $D_g(s)$ acquired a fully understood singular theory at $s = 1$, one could speculate about a three-way equivalence:

$$\text{prime gaps} \Longleftrightarrow \text{spectral measure / Weyl function} \Longleftrightarrow \text{singular expansion of } D_g(s).$$

Such a bridge would amount to a new analytic-spectral rigidity principle for prime gaps: the same data would be encoded simultaneously in

1. determinant growth,
2. Laplacian transport,
3. orthogonal polynomials,
4. and the critical singularity of a gap-weighted Dirichlet series.

This would be a highly nonclassical way of organizing prime-gap information.

6. Consequences for the distribution of large and small gaps

At the rigorous level already reached, the prime-gap sequence satisfies a multiplicative mean constraint:

$$\left(\prod_{i=1}^{k} \Delta_i\right)^{1/k} \leq \frac{p_k}{k},$$

and therefore, by the Prime Number Theorem, its geometric mean is at most of order $\log k$. The determinant identity and the weighted gap-energy strengthen this by showing that large gaps cannot proliferate arbitrarily without forcing a corresponding growth in the associated Gram determinants.

If the conjectural asymptotics for $G(x)$ were proved, one would expect a much sharper global statement:

Speculative consequence. *Record gaps may occur, but they must be sufficiently sparse that their total contribution to the weighted gap cascade remains compatible with a cubic singularity and not worse. Equivalently, extreme prime gaps would be globally compensated by a sufficiently dense sea of moderate gaps.*

This would not imply bounded gaps, nor would it directly control individual maximal gaps. Rather, it would give a new type of global conservation law: the analytic size of the full prime-gap cascade would enforce a balance between rare large defects and typical medium-scale gaps.

7. Final speculative picture

If all the conjectural steps were proved, the following unified picture would emerge.

- The prime gaps would form an electrical medium whose exact Green matrix is the prime Gram matrix.
- Their products would govern determinant growth and free-energy-type observables.
- Their full weighted multiplicative distribution would be encoded by the Dirichlet cascade $D_g(s)$.
- The singularity of $D_g(s)$ at $s = 1$ would become a macroscopic analytic invariant of the prime-gap medium.
- Spectral data, orthogonal polynomials, and Dirichlet singularities would become equivalent languages for the same arithmetic object.

In that scenario, prime gaps would no longer appear merely as local differences

$$p_{n+1} - p_n,$$

but as the edge weights of a canonical arithmetic medium possessing determinant theory, operator theory, spectral inversion, and analytic continuation. The present paper establishes the exact algebraic and spectral framework for this picture; a proof of the conjectural asymptotics would elevate it into a genuine global theory of prime-gap geometry.

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