A Report on Markov Algorithms and Entropy

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1 Markov algorithms and Entropy

Let M be a Markov algorithm with alphabet Σ of terminals and non-terminals. The terminal alphabet is $\{0,1\}$. Let $i \in \{1,\dots,n\}$ be given in its binary representation, so $|i| \approx \log i$. To each Markov algorithm M we associate a function

$$f_M : (\Sigma \cup \{\times\})^* \longrightarrow (\Sigma \cup \{\times\})^*$$

defined as follows. On input

$$i = w_i \times i \quad (w_i \in \Sigma^*, \ \times \notin \Sigma),$$

$f_M(i)$ reads off w_i, applies one Markov replacement to obtain $M(w_i)$, and then re-appends "$\times i$". Hence

$$f_M(i) := M(w_i) \times i.$$

Here $M(w)$ means: apply a single rule of the Markov algorithm to w, or return w if no rule applies.

Proposition 1.1. *For any Markov algorithm M, the map f_M is injective.*

Proof. If $f_M(i) = f_M(j)$ then

$$M(w_i) \times i = M(w_j) \times j$$

and so $i = j$. □

Suppose M decides a recursive set $L \subset \mathbb{N}$, by outputting 1 if $i \in L$ and 0 otherwise. We measure time by the total length of all intermediate strings:

$$\hat{t}_M(i) := \sum_{k=0}^{r} \left|M^{(k)}(i)\right| = \left|f_M^{(r)}(i)\right| - r,$$

where $M^{(0)}(i) = i$, $M^{(k+1)}(i) = M\big(M^{(k)}(i)\big)$, and $M^{(r)}(i) \in \{0,1\}$ is the halting output. The equality follows by a simple induction tracking the appended "$\times$" symbols.

Two Markov algorithms M_1, M_2 deciding the same language L are called *computationally equivalent*,

$$M_1 \equiv M_2 \quad :\Longleftrightarrow \quad \forall i\, \exists t_1, t_2 : \ M_1^{(t_1)}(i) = M_2^{(t_2)}(i).$$

We then compare their time-complexities: define

$$M_1 \leq M_2 \quad :\Longleftrightarrow \quad \forall i\, \exists r_1, r_2 : \ \hat{t}_{M_1}(i) \leq \hat{t}_{M_2}(i).$$

Finally write $M_1 \approx M_2$ if $M_1 \equiv M_2$, $M_1 \leq M_2$, and $M_2 \leq M_1$.

Entropy reduction?

Let

$$X_1 \supset X_2 \supset \cdots \supset X_t$$

be finite topological spaces, each X_{i+1} a continuous image of X_i via a surjection $f_i : X_i \to X_{i+1}$, with $|X_i| > |X_{i+1}|$. Define the *Shannon entropy at step i* by

$$H_i = -\sum_{x \in X_{i+1}} \frac{\left|f_i^{-1}(x)\right|}{|X_i|} \log_2\!\Big(\frac{\left|f_i^{-1}(x)\right|}{|X_i|}\Big).$$

Also let

$$\hat{f} = f_t \circ f_{t-1} \circ \cdots \circ f_1 : X_1 \longrightarrow X_t,$$

and set

$$\hat{H} = H_{\hat{f}} = -\sum_{z \in X_t} \frac{\left|\hat{f}^{-1}(z)\right|}{|X_1|} \log_2\!\Big(\frac{\left|\hat{f}^{-1}(z)\right|}{|X_1|}\Big).$$

Question 1.1. *Can one prove in general that*

$$H_i \;\geq\; \hat{H} \quad \textit{and} \quad H_i \;\geq\; H_{i+1} \quad \textit{for each } i\,?$$

2 Finite topologies and Markov algorithms for decision problems

We now illustrate with several decision problems how one may impose a finite preorder topology on a search space, and how the Markov steps correspond to continuous maps.

2.1 1. Primality test

For $n \in \mathbb{N}$, let

$$X_n = \{\,2, \ldots, \lfloor \sqrt{n} \rfloor\},$$

with preorder $x \leq y :\Leftrightarrow x \mid y$. Opens are

$$O_x = \{\, y : x \mid y\}.$$

The solution set is

$$Y_n^+ = \{\, x \in X_n : x \mid n,\; x \neq 1, n\}.$$

2. Partition problem

Given $S = (s_1, \ldots, s_r)$, let $I = \{1, \ldots, r\}$ and

$$X_S = \mathcal{P}(I), \quad U \leq V :\Leftrightarrow \sum_{u \in U} s_u \leq \sum_{v \in V} s_v, \quad O_U = \{V : U \leq V\}.$$

The target set is

$$Y_S^+ = \{\, U : \sum_{u \in U} s_u = \sum_{v \in U^c} s_v\}.$$

2.2 3. Composite test via gcd-order

For n, set

$$X_n = \{1, \ldots, n{-}1\}, \quad x \leq y :\Leftrightarrow \gcd(n,x) \leq \gcd(n,y), \quad O_x = \{y : x \leq y\}, \quad Y_n^+ = \{x : \gcd(n,x) \notin \{1,n\}\}.$$

2.3 4. Partition-number test

For n, let

$$X_n = \{d : d \mid n,\; \gcd(d, \tfrac{n}{d}) = 1\}, \quad x \leq y :\Leftrightarrow \Omega(x) \leq \Omega(y), \quad Y_n^+ = \{d : \Omega(d) = \Omega(n/d)\},$$

where Ω counts prime-factors with multiplicity.

More generally, given any finite X and $Y \subset X$, the quotient

$$\pi : X \longrightarrow X/Y$$

collapsing Y to one point is monotone and hence continuous. Its Shannon entropy is

$$H(\pi) = -\frac{|X| - |Y|}{|X|} \log_2\!\Big(\frac{|X| - |Y|}{|X|}\Big) - \frac{|Y|}{|X|} \log_2\!\Big(\frac{|Y|}{|X|}\Big).$$

Finally, one may attach to each execution trace

$$(M, x) = (x_1, \ldots, x_t)$$

a finite topological space

$$S_x = \bigcup_{i=1}^{t} \hat{S}_{x_i}, \quad \hat{S}_x = \{y : M^{(j)}(y) = x \text{ for } j = 0, 1\},$$

with preorder $a \leq b \iff b = M^{(j)}(a)$ for $j \leq 1$. Then each single-rule step $M : S_{x_i} \to S_{x_{i+1}}$ is continuous.

Question 2.1. *Is it possible to enforce conditions on these constructions so as to obtain a bijection*

$$\{\textit{Markov algorithms}\} \longleftrightarrow \{\textit{finite topological spaces with extra data}\}?$$