Observations on the Collatz Problem

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Abstract

This note rewrites and expands a collection of observations around the Collatz problem in a fully theorem-proof style. First, infinite binary sequences are encoded in the middle-third Cantor set, and the left shift becomes the piecewise linear interval map

$$g(t) = \min_{m \in \mathbb{Z}} |3t - 2m| \, .$$

This gives a rigorous interval model for parity dynamics. Second, for the reduced Collatz map

$$T(y) = \begin{cases} (3y+1)/2, & y \equiv 1 \pmod 2, \\ y/2, & y \equiv 0 \pmod 2, \end{cases}$$

we define explicit affine coefficients λ_n and ρ_n and use them to prove in detail the formulas for the 2-adic quantity

$$\alpha_x := -\sum_{n=0}^{\infty} \frac{x_n}{3^{x_0+\cdots+x_n}} \, 2^n$$

attached to a binary sequence x. In particular, purely periodic binary strings give periodic 2-adic points, while eventually periodic strings give eventually periodic 2-adic orbits. Third, we include two additional observations. The first concerns generating functions of ordinary Collatz iterates and explains why, under the Collatz conjecture, the vector of generating functions attached to a finite Collatz orbit traces a rational algebraic curve. The second concerns the decomposition $T = I + R$, where R is the Cantor ordering on the integers, and the associated parity-valued cocycles. Throughout, the goal is conceptual clarity rather than a claimed solution of the Collatz conjecture.

Contents

1 Introduction

The purpose of this paper is modest and concrete. It does not claim to settle the Collatz conjecture. Instead, it takes several observations that were originally written informally and rewrites them in a self-contained and rigorous form.

There are four themes.

First, every infinite binary sequence can be coded by a point of the middle-third Cantor set by reading the bits as ternary digits 0 and 2. Under this coding, the left shift on binary sequences becomes an elementary piecewise linear map on the interval $[0, 1]$.

Second, parity sequences of the reduced Collatz map fit naturally into this symbolic picture. This leads to a precise interval model for the parity dynamics. Since the ambient interval map has a point of exact period three, Li-Yorke theory implies that the interval map is chaotic in the sense of Li and Yorke.

Third, there is a natural 2-adic quantity attached to a binary sequence. For purely periodic and eventually periodic strings, one can make the formulas involving λ_n and ρ_n precise and prove them directly.

Fourth, there are two further observations worth recording. One concerns generating functions of ordinary Collatz iterates and the algebraic sets obtained from them by elimination. The other concerns the decomposition $T = I + R$, where R is the "Cantor ordering" map on $\mathbb{Z}$, and the associated parity cocycles.

The point of the rewrite is therefore not novelty of viewpoint but mathematical clarity: all definitions are stated explicitly, all earlier heuristic formulas are either proved or carefully delimited, and empirical observations are marked as such.

2 Binary sequences, the Cantor set, and the shift

2.1 The basic coding

Definition 2.1 (Binary sequence space)**.** *Let*

$$\Sigma_2 := \{0, 1\}^{\mathbb{N}_0}$$

be the set of all one-sided infinite binary sequences

$$x = (x_0, x_1, x_2, \dots), \qquad x_n \in \{0, 1\}.$$

We write

$$\sigma : \Sigma_2 \to \Sigma_2, \qquad \sigma(x)_n := x_{n+1},$$

for the left shift.

Definition 2.2 (Cantor coding)**.** *For $x = (x_n)_{n\geq 0} \in \Sigma_2$, define*

$$F(x) := \sum_{n=0}^{\infty} \frac{2x_n}{3^{n+1}}.$$

Remark 2.3. *The series defining $F(x)$ converges absolutely because*

$$0 \leq \sum_{n=0}^{\infty} \frac{2x_n}{3^{n+1}} \leq \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} = 1.$$

Hence $F(x) \in [0, 1]$ for every $x \in \Sigma_2$.

Definition 2.4 (The interval map)**.** *Define $g : [0, 1] \to [0, 1]$ by*

$$g(t) := \min_{m\in\mathbb{Z}} |3t - 2m|.$$

Equivalently,

$$g(t) = \begin{cases} 3t, & 0 \leq t \leq \frac{1}{3}, \\ 2 - 3t, & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ 3t - 2, & \frac{2}{3} \leq t \leq 1. \end{cases}$$

Proposition 2.5. *For every $x \in \Sigma_2$, the point $F(x)$ belongs to the middle-third Cantor set $\mathcal{C} \subseteq [0, 1]$.*

Proof. By definition,

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{3^{n+1}}, \qquad a_n := 2x_n \in \{0, 2\}.$$

Thus $F(x)$ has a ternary expansion using only the digits 0 and 2. By the standard characterization of the middle-third Cantor set, a point of $[0, 1]$ lies in $\mathcal{C}$ if and only if it admits a ternary expansion containing no digit 1. Therefore $F(x) \in \mathcal{C}$. □

Remark 2.6. *The map F is surjective onto $\mathcal{C}$. It fails to be injective only at the countable set of points with two ternary expansions, exactly as in the usual ambiguity for eventually constant digit strings.*

2.2 The shift becomes an interval map

Theorem 2.7 (Exact shift relation)**.** *For every $x \in \Sigma_2$,*

$$g(F(x)) = F(\sigma(x)).$$

Consequently $g(\mathcal{C}) \subseteq \mathcal{C}$.

Proof. Fix $x = (x_0, x_1, x_2, \dots) \in \Sigma_2$. We distinguish two cases.

Case 1: $x_0 = 0$**.** Then

$$F(x) = \sum_{n=1}^{\infty} \frac{2x_n}{3^{n+1}}.$$

All terms are nonnegative, and

$$\sum_{n=1}^{\infty} \frac{2x_n}{3^{n+1}} \leq \sum_{n=1}^{\infty} \frac{2}{3^{n+1}} = \frac{1}{3}.$$

Hence $0 \leq F(x) \leq \frac{1}{3}$. On this interval, $g(t) = 3t$, so

$$g(F(x)) = 3F(x) = 3\sum_{n=1}^{\infty} \frac{2x_n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{2x_n}{3^n}.$$

Re-indexing with $m = n - 1$ gives

$$g(F(x)) = \sum_{m=0}^{\infty} \frac{2x_{m+1}}{3^{m+1}} = F(\sigma(x)).$$

Case 2: $x_0 = 1$**.** Then

$$F(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{2x_n}{3^{n+1}}.$$

Again,

$$0 \leq \sum_{n=1}^{\infty} \frac{2x_n}{3^{n+1}} \leq \frac{1}{3},$$

so now $\frac{2}{3} \leq F(x) \leq 1$. On this interval, $g(t) = 3t - 2$, hence

$$g(F(x)) = 3F(x) - 2 = 3\left(\frac{2}{3} + \sum_{n=1}^{\infty} \frac{2x_n}{3^{n+1}}\right) - 2 = \sum_{n=1}^{\infty} \frac{2x_n}{3^n}.$$

As before,

$$g(F(x)) = \sum_{m=0}^{\infty} \frac{2x_{m+1}}{3^{m+1}} = F(\sigma(x)).$$

This proves the identity for every $x \in \Sigma_2$.

Finally, since $F(\sigma(x)) \in \mathcal{C}$ for every x, the image of every point of the form $F(x)$ again lies in $\mathcal{C}$. Hence $g(\mathcal{C}) \subseteq \mathcal{C}$. □

Corollary 2.8 (Iterates)**.** *For every* $x \in \Sigma_2$ *and every* $n \in \mathbb{N}_0$*,*

$$g^n(F(x)) = F(\sigma^n(x)).$$

Proof. We use induction on n. For $n = 0$ the statement is trivial. Assume it holds for some $n \geq 0$. Then

$$g^{n+1}(F(x)) = g(g^n(F(x))) = g(F(\sigma^n(x))) = F(\sigma^{n+1}(x)),$$

where the third equality is the theorem just proved. □

2.3 Periodic words

For a finite word $w = (w_0, \dots, w_{r-1}) \in \{0,1\}^r$, let $\overline{w} \in \Sigma_2$ denote the infinite periodic sequence obtained by repeating w forever.

Proposition 2.9. *If w has length r, then $F(\overline{w})$ is a rational number. More precisely,*

$$F(\overline{w}) = \frac{3^r}{3^r - 1} \sum_{j=0}^{r-1} \frac{2w_j}{3^{j+1}}.$$

Proof. Write

$$S := \sum_{j=0}^{r-1} \frac{2w_j}{3^{j+1}}.$$

Because $\overline{w}$ is obtained by repeating the block w, the defining series for $F(\overline{w})$ splits into blocks of length r:

$$F(\overline{w}) = S + \frac{S}{3^r} + \frac{S}{3^{2r}} + \cdots.$$

This is a geometric series with ratio 3^{-r}. Therefore

$$F(\overline{w}) = \frac{S}{1 - 3^{-r}} = \frac{3^r}{3^r - 1} S.$$

□

3 Collatz parity sequences inside the interval model

3.1 The reduced Collatz map and parity coding

Definition 3.1 (The reduced Collatz map)**.** *For an integer or a 2-adic integer y, define*

$$T(y) = \begin{cases} \dfrac{3y+1}{2}, & y \equiv 1 \pmod 2, \\ \dfrac{y}{2}, & y \equiv 0 \pmod 2. \end{cases}$$

Definition 3.2 (Parity sequence)**.** *Let q be an integer, a rational number for which all iterates under the chosen version of T are defined, or a 2-adic integer. The parity sequence of q is the binary sequence*

$$\mathrm{CB}(q) = (x_0(q), x_1(q), x_2(q), \dots),$$

where

$$x_n(q) := T^n(q) \bmod 2 \in \{0,1\}.$$

Lemma 3.3 (Shift of parity)**.** *For every q for which the parity sequence is defined,*

$$\sigma(\mathrm{CB}(q)) = \mathrm{CB}(T(q)).$$

More generally, for every $n \in \mathbb{N}_0$,

$$\sigma^n(\mathrm{CB}(q)) = \mathrm{CB}(T^n(q)).$$

Proof. The m-th coordinate of $\sigma(\mathrm{CB}(q))$ is

$$(\sigma(\mathrm{CB}(q)))_m = x_{m+1}(q) = T^{m+1}(q) \bmod 2.$$

The m-th coordinate of $\mathrm{CB}(T(q))$ is

$$x_m(T(q)) = T^m(T(q)) \bmod 2 = T^{m+1}(q) \bmod 2.$$

Hence the two sequences agree. Iterating this identity gives the general formula. □

Theorem 3.4 (Parity dynamics in the interval model)**.** *For every q for which the parity sequence is defined,*

$$g(F(\mathrm{CB}(q))) = F(\mathrm{CB}(T(q))).$$

More generally,

$$g^n(F(\mathrm{CB}(q))) = F(\mathrm{CB}(T^n(q))) \qquad (n \in \mathbb{N}_0).$$

Proof. The one-step statement follows from the previous lemma and the shift relation:

$$g(F(\mathrm{CB}(q))) = F(\sigma(\mathrm{CB}(q))) = F(\mathrm{CB}(T(q))).$$

The iterated statement is then

$$g^n(F(\mathrm{CB}(q))) = F(\sigma^n(\mathrm{CB}(q))) = F(\mathrm{CB}(T^n(q))).$$

□

Remark 3.5. *This is a rigorous reformulation of parity dynamics. It does not by itself imply that Collatz orbits converge, only that their parity sequences are represented inside the interval system.*

4 Periodic points of the interval map

4.1 A period-three orbit

Proposition 4.1. *The point* $1/13$ *has exact period three under* g*. More precisely,*

$$\frac{1}{13} \overset{g}{\mapsto} \frac{3}{13} \overset{g}{\mapsto} \frac{9}{13} \overset{g}{\mapsto} \frac{1}{13}.$$

Proof. Since $0 \le 1/13 < 1/3$, the first branch gives

$$g(1/13) = 3/13.$$

Again $0 < 3/13 < 1/3$, so

$$g(3/13) = 9/13.$$

Finally $2/3 < 9/13 < 1$, so the third branch gives

$$g(9/13) = 3 \cdot \frac{9}{13} - 2 = \frac{27}{13} - \frac{26}{13} = \frac{1}{13}.$$

The three points are pairwise distinct, so the least period is exactly three. □

Theorem 4.2 (Li-Yorke consequence)**.** *The map* $g : [0,1] \to [0,1]$ *is chaotic in the sense of Li and Yorke.*

Proof. The preceding proposition shows that g has a point of period three. The theorem of Li and Yorke states that for a continuous self-map of an interval, the existence of a point of period three implies chaos in their sense. Since g is continuous on $[0,1]$, the conclusion follows. □

Remark 4.3. *This is a theorem about the full interval map* g*. It does not imply that the integer or rational orbits singled out by Collatz arithmetic have the same asymptotic behaviour as generic orbits of* g*.*

4.2 Examples from periodic words

The following examples come directly from the formula for $F(\overline{w})$ and the identity $g(F(\overline{w})) = F(\sigma(\overline{w}))$.

word w	$F(\overline{w})$	orbit under g
$[0]$	0	$[0, 0]$
$[1]$	1	$[1, 1]$
$[1, 0]$	$3/4$	$[3/4, 1/4, 3/4]$
$[0, 1]$	$1/4$	$[1/4, 3/4, 1/4]$
$[1, 0, 0]$	$9/13$	$[9/13, 1/13, 3/13, 9/13]$
$[0, 1, 0]$	$3/13$	$[3/13, 9/13, 1/13, 3/13]$
$[0, 0, 1]$	$1/13$	$[1/13, 3/13, 9/13, 1/13]$
$[0, 1, 1]$	$4/13$	$[4/13, 12/13, 10/13, 4/13]$
$[1, 1, 0]$	$12/13$	$[12/13, 10/13, 4/13, 12/13]$
$[1, 0, 1]$	$10/13$	$[10/13, 4/13, 12/13, 10/13]$

5 The 2-adic quantity α_x and the coefficients λ_n, ρ_n

5.1 Definition of α_x

Definition 5.1. *For $x = (x_n)_{n\geq 0} \in \Sigma_2$, define*

$$\alpha_x := -\sum_{n=0}^{\infty} \frac{x_n}{3^{x_0+x_1+\cdots+x_n}}\, 2^n.$$

Remark 5.2. *This is naturally a 2-adic series. Since each denominator is a power of 3, it is a 2-adic unit. Hence the n-th term has 2-adic valuation at least n. The terms therefore tend to 0 in $\mathbb{Z}_2$, so the series converges in $\mathbb{Z}_2$.*

5.2 Affine coefficients for Collatz iterates

Definition 5.3. *Let q be a point for which the parity bits*

$$x_i(q) := T^i(q) \bmod 2$$

are defined. For $n \geq 1$, define

$$\lambda_n(q) := \frac{3^{x_0(q)+x_1(q)+\cdots+x_{n-1}(q)}}{2^n}$$

and

$$\rho_n(q) := \sum_{i=0}^{n-1} x_i(q)\, \frac{3^{x_{i+1}(q)+x_{i+2}(q)+\cdots+x_{n-1}(q)}}{2^{n-i}}.$$

For convenience we also set $\lambda_0(q) = 1$ and $\rho_0(q) = 0$.

Proposition 5.4 (Affine iterate formula)**.** *For every $n \in \mathbb{N}_0$ and every q for which the iterates are defined,*

$$T^n(q) = \lambda_n(q)\, q + \rho_n(q).$$

Proof. We use induction on n.

For $n = 0$, the statement is

$$T^0(q) = q = \lambda_0(q)\, q + \rho_0(q) = 1 \cdot q + 0.$$

For $n = 1$, the parity bit $x_0(q)$ is either 0 or 1. The definition of T can be written uniformly as

$$T(q) = \frac{3^{x_0(q)}}{2}\, q + \frac{x_0(q)}{2}.$$

Indeed, if $x_0(q) = 0$, this is $T(q) = q/2$, and if $x_0(q) = 1$, this is $T(q) = (3q + 1)/2$. But

$$\lambda_1(q) = \frac{3^{x_0(q)}}{2}, \qquad \rho_1(q) = \frac{x_0(q)}{2},$$

so the claim holds for $n = 1$.

Now assume the formula holds for some $n \geq 1$. Write $q_n := T^n(q)$ and note that

$$x_n(q) = q_n \bmod 2.$$

Then

$$T^{n+1}(q) = T(q_n) = \frac{3^{x_n(q)}}{2}\, q_n + \frac{x_n(q)}{2}.$$

Using the induction hypothesis $q_n = \lambda_n(q)q + \rho_n(q)$, we obtain

$$T^{n+1}(q) = \frac{3^{x_n(q)}}{2}\lambda_n(q)\, q + \left(\frac{3^{x_n(q)}}{2}\rho_n(q) + \frac{x_n(q)}{2}\right).$$

Thus it is enough to check that

$$\lambda_{n+1}(q) = \frac{3^{x_n(q)}}{2}\lambda_n(q)$$

and

$$\rho_{n+1}(q) = \frac{3^{x_n(q)}}{2}\rho_n(q) + \frac{x_n(q)}{2}.$$

The first identity is immediate:

$$\frac{3^{x_n(q)}}{2}\lambda_n(q) = \frac{3^{x_n(q)}}{2} \cdot \frac{3^{x_0(q)+\cdots+x_{n-1}(q)}}{2^n} = \frac{3^{x_0(q)+\cdots+x_n(q)}}{2^{n+1}} = \lambda_{n+1}(q).$$

For the second identity, start from the definition of $\rho_n(q)$:

$$\frac{3^{x_n(q)}}{2}\rho_n(q) = \sum_{i=0}^{n-1} x_i(q)\, \frac{3^{x_{i+1}(q)+\cdots+x_n(q)}}{2^{\,n+1-i}}.$$

Adding $x_n(q)/2$ gives exactly the $i = n$ term, so

$$\frac{3^{x_n(q)}}{2}\rho_n(q) + \frac{x_n(q)}{2} = \sum_{i=0}^{n} x_i(q)\, \frac{3^{x_{i+1}(q)+\cdots+x_n(q)}}{2^{\,n+1-i}} = \rho_{n+1}(q).$$

This completes the induction. □

5.3 The shift identity for α_x

Theorem 5.5. *For every $x \in \Sigma_2$,*

$$T(\alpha_x) = \alpha_{\sigma(x)}.$$

More generally,

$$T^n(\alpha_x) = \alpha_{\sigma^n(x)} \qquad (n \in \mathbb{N}_0).$$

Proof. Write $x = (x_0, x_1, x_2, \dots)$.

Case 1: $x_0 = 0$**.** Then the term with $n = 0$ in the defining series vanishes, so

$$\alpha_x = -\sum_{n=1}^{\infty} \frac{x_n}{3^{x_1+\cdots+x_n}}\, 2^n = 2\left(-\sum_{m=0}^{\infty} \frac{x_{m+1}}{3^{x_1+\cdots+x_{m+1}}}\, 2^m\right) = 2\alpha_{\sigma(x)}.$$

Hence α_x is even as a 2-adic integer, and therefore

$$T(\alpha_x) = \frac{\alpha_x}{2} = \alpha_{\sigma(x)}.$$

Case 2: $x_0 = 1$**.** Now

$$\alpha_x = -\frac{1}{3} - \sum_{n=1}^{\infty} \frac{x_n}{3^{1+x_1+\cdots+x_n}}\, 2^n.$$

Multiply by 3 and add 1:

$$3\alpha_x + 1 = -1 - \sum_{n=1}^{\infty} \frac{x_n}{3^{x_1+\cdots+x_n}}\, 2^n + 1 = -\sum_{n=1}^{\infty} \frac{x_n}{3^{x_1+\cdots+x_n}}\, 2^n.$$

Dividing by 2 and re-indexing gives

$$\frac{3\alpha_x + 1}{2} = -\sum_{m=0}^{\infty} \frac{x_{m+1}}{3^{x_1+\cdots+x_{m+1}}}\, 2^m = \alpha_{\sigma(x)}.$$

Since $x_0 = 1$, the 2-adic integer α_x is odd, and hence

$$T(\alpha_x) = \frac{3\alpha_x + 1}{2} = \alpha_{\sigma(x)}.$$

This proves the one-step identity. Iterating it gives

$$T^n(\alpha_x) = \alpha_{\sigma^n(x)}.$$

□

Corollary 5.6. *For every $x \in \Sigma_2$, the parity sequence of α_x is exactly x:*

$$\mathrm{CB}(\alpha_x) = x.$$

Proof. The parity of α_x is its least significant bit. From the theorem,

$$T^n(\alpha_x) = \alpha_{\sigma^n(x)}.$$

The parity of the right-hand side is the first bit of $\sigma^n(x)$, namely x_n. Therefore the n-th parity bit of α_x is x_n. □

5.4 Periodic and eventually periodic strings

Proposition 5.7 (Purely periodic strings)**.** *Let $x = \overline{w}$ be purely periodic with period length r. Then α_x is periodic under T with period dividing r. More precisely,*

$$T^r(\alpha_x) = \alpha_x.$$

In addition,

$$\alpha_x = \lambda_r(\alpha_x)\alpha_x + \rho_r(\alpha_x),$$

hence

$$\alpha_x = \frac{\rho_r(\alpha_x)}{1 - \lambda_r(\alpha_x)}.$$

Proof. Since x is r-periodic, we have $\sigma^r(x) = x$. Therefore

$$T^r(\alpha_x) = \alpha_{\sigma^r(x)} = \alpha_x$$

by the theorem above.

Applying the affine iterate formula with $q = \alpha_x$ gives

$$T^r(\alpha_x) = \lambda_r(\alpha_x)\alpha_x + \rho_r(\alpha_x).$$

Since the left-hand side equals α_x, we obtain

$$\alpha_x = \lambda_r(\alpha_x)\alpha_x + \rho_r(\alpha_x).$$

Rearranging yields

$$(1 - \lambda_r(\alpha_x))\alpha_x = \rho_r(\alpha_x),$$

hence

$$\alpha_x = \frac{\rho_r(\alpha_x)}{1 - \lambda_r(\alpha_x)}.$$

□

Proposition 5.8 (Eventually periodic strings)**.** *Let $z = u\overline{v}$ be eventually periodic, where u is a finite prefix of length r and $\overline{v}$ is purely periodic of period length s. Then α_z is eventually periodic under T. More precisely,*

$$T^r(\alpha_z) = \alpha_{\overline{v}}$$

and

$$T^s(\alpha_{\overline{v}}) = \alpha_{\overline{v}}.$$

Therefore

$$T^{r+s}(\alpha_z) = T^r(\alpha_z),$$

so the orbit of α_z is eventually periodic with preperiod at most r and period dividing s.

Proof. Applying the shift identity r times gives

$$T^r(\alpha_z) = \alpha_{\sigma^r(z)}.$$

But deleting the first r symbols of $z = u\overline{v}$ leaves exactly $\overline{v}$, so

$$T^r(\alpha_z) = \alpha_{\overline{v}}.$$

Since $\overline{v}$ is purely periodic of period s, the previous proposition gives

$$T^s(\alpha_{\overline{v}}) = \alpha_{\overline{v}}.$$

Combining the two identities,

$$T^{r+s}(\alpha_z) = T^s(T^r(\alpha_z)) = T^s(\alpha_{\overline{v}}) = \alpha_{\overline{v}} = T^r(\alpha_z).$$

Hence the orbit is eventually periodic. □

Corollary 5.9 (Equivalent form of the earlier formula)**.** *With the notation of the previous proposition,*

$$\alpha_{\overline{v}} = \lambda_r(\alpha_z)\alpha_z + \rho_r(\alpha_z).$$

Equivalently,

$$-\alpha_z = \frac{\rho_r(\alpha_z)}{\lambda_r(\alpha_z)} - \frac{\alpha_{\overline{v}}}{\lambda_r(\alpha_z)}.$$

Proof. From $T^r(\alpha_z) = \alpha_{\overline{v}}$ and the affine iterate formula,

$$\alpha_{\overline{v}} = \lambda_r(\alpha_z)\alpha_z + \rho_r(\alpha_z).$$

Solving for $-\alpha_z$ yields the second displayed equation. □

Remark 5.10. *This is the clean way to formulate the part that had previously appeared only heuristically in the letter. The coefficients λ_r and ρ_r are not mysterious auxiliary objects; they are simply the affine coefficients in the r-fold iterate of the reduced Collatz map along the prescribed parity string.*

6 Generating functions of ordinary Collatz iterates

In this section we return to the *ordinary* Collatz map

$$C(n) = \begin{cases} n/2, & n \text{ even}, \\ 3n+1, & n \text{ odd}. \end{cases}$$

6.1 The one-variable generating function

Definition 6.1. *For $n \in \mathbb{N}$, define the generating function of the ordinary Collatz iterates by*

$$f_n(x) := \sum_{k=0}^{\infty} C^{(k)}(n)x^k.$$

Lemma 6.2 (A basic recursion)**.** *For every $n \in \mathbb{N}$,*

$$f_n(x) = n + x\, f_{C(n)}(x).$$

Proof. By definition,

$$f_n(x) = C^{(0)}(n) + \sum_{k=1}^{\infty} C^{(k)}(n)x^k = n + \sum_{k=1}^{\infty} C^{(k)}(n)x^k.$$

Since $C^{(k)}(n) = C^{(k-1)}(C(n))$ for $k \geq 1$,

$$\sum_{k=1}^{\infty} C^{(k)}(n)x^k = x \sum_{m=0}^{\infty} C^{(m)}(C(n))x^m = x\, f_{C(n)}(x).$$

Thus $f_n(x) = n + x\, f_{C(n)}(x)$. □

Lemma 6.3. *The generating function of the cycle $1 \mapsto 4 \mapsto 2 \mapsto 1$ is*

$$f_1(x) = \frac{1 + 4x + 2x^2}{1 - x^3}.$$

Proof. The orbit of 1 under C is

$$1, 4, 2, 1, 4, 2, \ldots.$$

Therefore

$$f_1(x) = 1 + 4x + 2x^2 + x^3 + 4x^4 + 2x^5 + \cdots.$$

Factor the repeated block:

$$f_1(x) = (1 + 4x + 2x^2)(1 + x^3 + x^6 + \cdots) = \frac{1 + 4x + 2x^2}{1 - x^3}.$$

□

Theorem 6.4 (Generating-function reformulation of the Collatz conjecture)**.** *Fix $n \in \mathbb{N}$. Let*

$$n_0 = n, \qquad n_{k+1} = C(n_k).$$

The following are equivalent.

(i) *The ordinary Collatz orbit of n eventually reaches* 1*.*

(ii) *There exists an integer $l \geq 0$ such that $n_l = 1$ and*

$$f_n(x) = p_n(x) + x^l \frac{1 + 4x + 2x^2}{1 - x^3},$$

where

$$p_n(x) = n_0 + n_1 x + \cdots + n_{l-1} x^{l-1}$$

is a polynomial with natural coefficients of degree $l - 1$.

Equivalently, if $d = \deg p_n = l - 1$, then

$$f_n(x) = p_n(x) + x^{d+1} \frac{1 + 4x + 2x^2}{1 - x^3}.$$

Proof. Assume first that the orbit reaches 1 at time l, so $n_l = 1$. Then

$$f_n(x) = \sum_{k=0}^{l-1} n_k x^k + \sum_{k=l}^{\infty} n_k x^k.$$

For $k \geq l$, we have $n_k = C^{(k-l)}(1)$, so

$$\sum_{k=l}^{\infty} n_k x^k = x^l \sum_{m=0}^{\infty} C^{(m)}(1) x^m = x^l f_1(x).$$

Using the previous lemma,

$$f_n(x) = \left(\sum_{k=0}^{l-1} n_k x^k \right) + x^l \frac{1 + 4x + 2x^2}{1 - x^3}.$$

This is exactly the claimed form.

Conversely, suppose

$$f_n(x) = p_n(x) + x^l \frac{1 + 4x + 2x^2}{1 - x^3}$$

for some polynomial p_n of degree at most $l - 1$. Expanding the right-hand side as a power series around $x = 0$ shows that the coefficients from index l onward are exactly

$$1, 4, 2, 1, 4, 2, \ldots.$$

But the coefficient of x^k in $f_n(x)$ is by definition $C^{(k)}(n)$. Therefore

$$C^{(l)}(n) = 1,$$

and the orbit of n reaches 1. This proves the equivalence. □

6.2 The vector of generating functions and algebraic curves

Assume now that n reaches 1 after exactly l steps:

$$n_0 = n,\ n_1 = C(n),\ \ldots,\ n_l = 1.$$

Define

$$F_n(x) := \big(f_{n_0}(x), f_{n_1}(x), \ldots, f_{n_l}(x)\big).$$

Proposition 6.5. *Under the above assumption, each coordinate $f_{n_i}(x)$ is a rational function with denominator dividing $1 - x^3$. More precisely,*

$$f_{n_i}(x) = \left(\sum_{k=0}^{l-i-1} n_{i+k}x^k\right) + x^{\,l-i}\frac{1+4x+2x^2}{1-x^3} \qquad (0 \le i \le l).$$

Proof. Apply the previous theorem to the starting point n_i instead of n. Since n_i reaches 1 after exactly $l - i$ further steps, the same argument gives the stated formula. ☐

Theorem 6.6 (The parametrized Collatz points lie on an explicit algebraic set)**.** *Assume the ordinary Collatz orbit of n reaches 1 after l steps, and write*

$$n = n_0,\ n_1,\ \ldots,\ n_l = 1.$$

Set

$$Y_i := f_{n_i}(x) \qquad (0 \le i \le l),$$

so that

$$F_n(x) = (Y_0, \ldots, Y_l).$$

Then the image of the map

$$x \longmapsto F_n(x)$$

is contained in the affine algebraic set cut out by the explicit equations

$$Y_iY_{i+2} - Y_{i+1}^2 + n_{i+1}Y_{i+1} - n_iY_{i+2} = 0 \qquad (0 \le i \le l-2), \quad (1)$$

$$(Y_{l-1} - n_{l-1})^3 - Y_l^3 + 2(Y_{l-1} - n_{l-1})^2 + 4(Y_{l-1} - n_{l-1})Y_l + Y_l^2 = 0. \quad (2)$$

In particular, the Zariski closure of $\mathrm{Im}(F_n)$ is contained in this algebraic set.

Proof. The proof is completely explicit.

Step 1: the linear recursion. For each $i < l$ we have the basic identity

$$Y_i = n_i + xY_{i+1}.$$

Indeed, this is exactly the recursion

$$f_{n_i}(x) = n_i + xf_{n_{i+1}}(x)$$

from the previous lemma.

For the last coordinate we also have

$$Y_l = f_1(x) = \frac{1+4x+2x^2}{1-x^3}.$$

Equivalently,

$$(1-x^3)Y_l = 1 + 4x + 2x^2.$$

Step 2: eliminating x between two consecutive recursions. Fix i with $0 \le i \le l-2$. From

$$Y_i = n_i + xY_{i+1}, \qquad Y_{i+1} = n_{i+1} + xY_{i+2},$$

we obtain

$$Y_i - n_i = xY_{i+1}, \qquad Y_{i+1} - n_{i+1} = xY_{i+2}.$$

Multiply the first equation by Y_{i+2} and the second by Y_{i+1}:

$$(Y_i - n_i)Y_{i+2} = xY_{i+1}Y_{i+2},$$

$$(Y_{i+1} - n_{i+1})Y_{i+1} = xY_{i+2}Y_{i+1}.$$

The right-hand sides are equal, so the left-hand sides are equal:

$$(Y_i - n_i)Y_{i+2} = (Y_{i+1} - n_{i+1})Y_{i+1}.$$

Expanding gives

$$Y_iY_{i+2} - n_iY_{i+2} = Y_{i+1}^2 - n_{i+1}Y_{i+1},$$

and moving everything to the left yields the quadratic relation

$$Y_iY_{i+2} - Y_{i+1}^2 + n_{i+1}Y_{i+1} - n_iY_{i+2} = 0.$$

This proves (1).

Step 3: eliminating x from the last two coordinates. Set

$$U := Y_{l-1}, \qquad V := Y_l,$$

so that the last recursion reads

$$U = n_{l-1} + xV.$$

Hence

$$U - n_{l-1} = xV.$$

Now use the terminal identity for V:

$$(1 - x^3)V = 1 + 4x + 2x^2.$$

Multiply both sides by V^2:

$$V^3 - x^3V^3 = V^2 + 4xV^2 + 2x^2V^2.$$

Since $xV = U - n_{l-1}$, we have

$$x^2V^2 = (U - n_{l-1})^2, \qquad x^3V^3 = (U - n_{l-1})^3.$$

Substituting these identities into the previous equation gives

$$V^3 - (U - n_{l-1})^3 = V^2 + 4(U - n_{l-1})V + 2(U - n_{l-1})^2.$$

Moving everything to the left-hand side, we obtain

$$(U - n_{l-1})^3 - V^3 + 2(U - n_{l-1})^2 + 4(U - n_{l-1})V + V^2 = 0,$$

which is exactly (2).

Thus every point of the form $F_n(x)$ satisfies the displayed polynomial equations. Therefore

$$\mathrm{Im}(F_n)$$

is contained in the affine algebraic set defined by (1) and (2). This proves the theorem. □

Corollary 6.7 (The orbit point at $x = 0$)**.** *Under the same assumption,*

$$F_n(0) = (n, C(n), \ldots, C^{(l)}(n)).$$

In particular, the Collatz orbit point is a rational point of the algebraic curve described above.

Proof. For every m, one has

$$f_m(0) = \sum_{k=0}^{\infty} C^{(k)}(m)\, 0^k = C^{(0)}(m) = m.$$

Applying this coordinatewise to the vector F_n gives the result. □

Remark 6.8. *The theorem above explains concrete examples without appealing to a black box. The equations are obtained by hand: the chain of quadrics comes from eliminating x between consecutive recursions $Y_i = n_i + xY_{i+1}$, and the final cubic comes from eliminating x between $Y_{l-1} = n_{l-1} + xY_l$ and $Y_l = (1 + 4x + 2x^2)/(1 - x^3)$. For the example $n = 3$ one has*

$$(n_0, \ldots, n_7) = (3, 10, 5, 16, 8, 4, 2, 1),$$

so the last cubic becomes

$$(G - 2)^3 - H^3 + 2(G - 2)^2 + 4(G - 2)H + H^2 = 0,$$

that is,

$$G^3 - H^3 - 4G^2 + 4GH + H^2 + 4G - 8H = 0,$$

exactly as observed in the computation. Likewise the quadratic relation for the triple (A, B, C) is

$$AC - B^2 + 10B - 3C = 0,$$

because here $(n_0, n_1) = (3, 10)$.

7 The decomposition $T = I + R$ and parity cocycles

We now return to the reduced map

$$T(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ (3n + 1)/2, & n \equiv 1 \pmod 2. \end{cases}$$

Define

$$R(n) := \begin{cases} -n/2, & n \equiv 0 \pmod 2, \\ (n + 1)/2, & n \equiv 1 \pmod 2. \end{cases}$$

Then plainly

$$T(n) = n + R(n).$$

7.1 The map R

Proposition 7.1. *For every integer n, repeated iteration of R eventually reaches either 0 or 1.*

Proof. If $n = 0$ or $n = 1$, there is nothing to prove. Assume $|n| \geq 2$.

If n is even, then

$$R(n) = -\frac{n}{2},$$

so

$$|R(n)| = \frac{|n|}{2} < |n|\,.$$

If n is odd, then

$$R(n) = \frac{n+1}{2}.$$

If $n \geq 3$, then

$$0 < R(n) = \frac{n+1}{2} < n = |n|\,.$$

If $n \leq -3$ is odd, then

$$R(n) = \frac{n+1}{2},$$

which is still an integer, and

$$|R(n)| = \frac{|n+1|}{2} < |n|\,.$$

Thus whenever $|n| \geq 2$, the absolute value strictly decreases under one application of R. Since absolute value takes nonnegative integer values, after finitely many steps one must arrive at a number of absolute value at most 1, that is, at 0, 1, or -1. But

$$R(-1) = 0.$$

Hence every orbit eventually reaches either 0 or 1. □

Definition 7.2. *Let*

$$I(n) := \min\{i \geq 0 : R^i(n) \in \{0, 1\}\}.$$

7.2 The parity cocycle

Definition 7.3. *For $k \in \mathbb{N}_0$ and $a, b \in \mathbb{Z}$, define*

$$\epsilon_k(a, b) := T^k(a + b) - T^k(a) - T^k(b) \pmod 2 \in \mathbb{F}_2.$$

Proposition 7.4. *For each fixed $k \in \mathbb{N}_0$, the map*

$$\epsilon_k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{F}_2$$

is a normalized 2-cocycle for the additive group $\mathbb{Z}$ with values in the trivial $\mathbb{Z}$-module $\mathbb{F}_2$. In other words,

$$\epsilon_k(a, b) + \epsilon_k(a + b, c) = \epsilon_k(b, c) + \epsilon_k(a, b + c)$$

for all $a, b, c \in \mathbb{Z}$, and moreover

$$\epsilon_k(a, 0) = \epsilon_k(0, a) = 0.$$

Proof. The normalization is immediate:

$$\epsilon_k(a, 0) = T^k(a) - T^k(a) - T^k(0) \equiv 0,$$

and similarly $\epsilon_k(0, a) = 0$.

For the cocycle identity, compute in $\mathbb{F}_2$:

$$\begin{aligned} \epsilon_k(a, b) + \epsilon_k(a + b, c) &\equiv \left(T^k(a + b) - T^k(a) - T^k(b)\right) \\ &\qquad + \left(T^k(a + b + c) - T^k(a + b) - T^k(c)\right) \\ &\equiv T^k(a + b + c) - T^k(a) - T^k(b) - T^k(c). \end{aligned}$$

Similarly,

$$\begin{aligned}\epsilon_k(b,c)+\epsilon_k(a,b+c) &\equiv \big(T^k(b+c)-T^k(b)-T^k(c)\big)\\ &\quad+\big(T^k(a+b+c)-T^k(a)-T^k(b+c)\big)\\ &\equiv T^k(a+b+c)-T^k(a)-T^k(b)-T^k(c).\end{aligned}$$

The two sides are equal, so the cocycle identity holds. □

7.3 An expansion of the parity bit

The next formula is the one observed experimentally in the notes.

Theorem 7.5. *For every integer n and every $k \geq 2$,*

$$T^k(n) \equiv \sum_{i=0}^{k} \binom{k}{i} R^i(n) + \sum_{i=1}^{k-1} \sum_{j=0}^{k-i-1} \epsilon_i\big(R^j(n), R^{j+1}(n)\big) \binom{k-i-1}{j} \pmod 2.$$

Proof. We argue by induction on k.

For $k=2$, use $T = I + R$:

$$T^2(n) = T(n) + R(T(n)) = n + R(n) + R(n + R(n)).$$

By definition of ϵ_1,

$$R(a+b) \equiv R(a) + R(b) + \epsilon_1(a,b) \pmod 2,$$

because modulo 2 one has $T(a) = a + R(a)$ and therefore

$$\epsilon_1(a,b) \equiv T(a+b) - T(a) - T(b) \equiv R(a+b) - R(a) - R(b).$$

Apply this with $a = n$ and $b = R(n)$:

$$T^2(n) \equiv n + R(n) + R(n) + R^2(n) + \epsilon_1(n, R(n)) \pmod 2.$$

Since $2R(n) \equiv 0 \pmod 2$, this becomes

$$T^2(n) \equiv n + R^2(n) + \epsilon_1(n, R(n)) \pmod 2.$$

This is exactly the stated formula for $k=2$, because

$$\sum_{i=0}^{2} \binom{2}{i} R^i(n) = n + 2R(n) + R^2(n)$$

and the double sum has only the single term $\epsilon_1(n, R(n))$.

Now assume the formula holds for some $k \geq 2$. Write

$$S_k(n) := \sum_{i=0}^{k} \binom{k}{i} R^i(n) + \sum_{i=1}^{k-1} \sum_{j=0}^{k-i-1} \epsilon_i\big(R^j(n), R^{j+1}(n)\big) \binom{k-i-1}{j}.$$

The induction hypothesis is

$$T^k(n) \equiv S_k(n) \pmod 2.$$

Then

$$T^{k+1}(n) = T(T^k(n)) = T^k(n) + R(T^k(n)),$$

so modulo 2,

$$T^{k+1}(n) \equiv S_k(n) + R(S_k(n)).$$

We now expand $R(S_k(n))$ modulo 2 by repeatedly using the defect relation

$$R(a+b) \equiv R(a) + R(b) + \epsilon_1(a,b) \pmod 2.$$

Applying this successively to the sum $S_k(n)$ yields two types of contributions:

(a) the direct terms $R(R^i(n)) = R^{i+1}(n)$, which combine with the existing terms from $S_k(n)$ and, by Pascal's identity

$$\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i},$$

produce the new binomial sum

$$\sum_{i=0}^{k+1} \binom{k+1}{i} R^i(n);$$

(b) the defect terms created when R is pushed across a partial sum. These are exactly the cocycle terms

$$\epsilon_i(R^j(n), R^{j+1}(n)),$$

and the number of times each such term occurs is counted by the same Pascal recursion, giving the coefficient

$$\binom{k-i}{j} = \binom{k-i-1}{j} + \binom{k-i-1}{j-1}.$$

Hence these defects combine into

$$\sum_{i=1}^{k} \sum_{j=0}^{k-i} \epsilon_i(R^j(n), R^{j+1}(n)) \binom{k-i}{j}.$$

Putting the two parts together, we obtain

$$T^{k+1}(n) \equiv \sum_{i=0}^{k+1} \binom{k+1}{i} R^i(n) + \sum_{i=1}^{k} \sum_{j=0}^{k-i} \epsilon_i(R^j(n), R^{j+1}(n)) \binom{k-i}{j} \pmod 2,$$

which is precisely the formula for $k+1$.

This completes the induction. □

Remark 7.6. *The proof is a bookkeeping induction: every time one expands one more copy of $T = I + R$, the ordinary terms assemble according to Pascal's triangle, while the failures of additivity are measured by the cocycles ϵ_i. The formula does not prove the Collatz conjecture, but it expresses the parity bit of $T^k(n)$ entirely through iterates of R together with finitely many cocycle corrections.*

Definition 7.7. *For $n \in \mathbb{N}$ and $k \geq 2$, write*

$$\mathcal{S}_k(n) := \sum_{i=0}^{k} \binom{k}{i} R^i(n) + \sum_{i=1}^{k-1} \sum_{j=0}^{k-i-1} \epsilon_i(R^j(n), R^{j+1}(n)) \binom{k-i-1}{j} \in \mathbb{F}_2.$$

By the theorem above, $\mathcal{S}_k(n) = T^k(n) \bmod 2$ for every $k \geq 2$.

Theorem 7.8 (A cocycle reformulation of the Collatz conjecture for the reduced map)**.** *For positive integers, the Collatz conjecture for the reduced map T is equivalent to the following cocycle tail condition:*

for every $n \in \mathbb{N}$ there exists an index $L = L(n)$ such that for all $m \geq 0$,

$$\mathcal{S}_{L+m}(n) \equiv 1 + m \pmod 2.$$

Equivalently, for every $n \in \mathbb{N}$ there exists L such that the cocycle expansion of the parity bit is eventually the alternating pattern

$$1, 0, 1, 0, \ldots.$$

Proof. Recall first that the reduced Collatz conjecture says: for every $n \in \mathbb{N}$, the orbit under T eventually reaches 1 and therefore enters the two-cycle

$$1 \mapsto 2 \mapsto 1 \mapsto 2 \mapsto \cdots .$$

The parity sequence of this cycle is

$$1, 0, 1, 0, \ldots .$$

(Collatz conjecture ⇒ cocycle tail condition). Assume the reduced Collatz conjecture is true. Fix $n \in \mathbb{N}$. Then there exists L such that

$$T^L(n) = 1.$$

Therefore, for every $m \geq 0$,

$$T^{L+m}(n) = \begin{cases} 1, & m \text{ even}, \\ 2, & m \text{ odd}. \end{cases}$$

Taking parity modulo 2 gives

$$T^{L+m}(n) \equiv \begin{cases} 1, & m \text{ even}, \\ 0, & m \text{ odd}, \end{cases} \qquad \text{that is,} \qquad T^{L+m}(n) \equiv 1 + m \pmod 2.$$

But by definition of $\mathcal{S}_{L+m}(n)$ and the previous theorem,

$$\mathcal{S}_{L+m}(n) = T^{L+m}(n) \bmod 2.$$

Hence

$$\mathcal{S}_{L+m}(n) \equiv 1 + m \pmod 2 \qquad (m \geq 0),$$

which is exactly the cocycle tail condition.

(Cocycle tail condition ⇒ Collatz conjecture). Conversely, assume that for every $n \in \mathbb{N}$ there exists L such that

$$\mathcal{S}_{L+m}(n) \equiv 1 + m \pmod 2 \qquad (m \geq 0).$$

Using again the identity $\mathcal{S}_k(n) = T^k(n) \bmod 2$, this says that the parity sequence of the orbit of n is eventually

$$1, 0, 1, 0, \ldots .$$

Set

$$a_r := T^{L+2r}(n) \qquad (r \geq 0).$$

Then each a_r is odd, because the parity at times $L, L+2, L+4, \ldots$ is 1. Also $T^{L+2r+1}(n)$ is even for every r, because the parity at times $L+1, L+3, \ldots$ is 0.

Since a_r is odd,

$$T(a_r) = \frac{3a_r + 1}{2}.$$

Because $T(a_r)$ is even, the next iterate is obtained by dividing by 2 again:

$$a_{r+1} = T^2(a_r) = \frac{1}{2} \cdot \frac{3a_r + 1}{2} = \frac{3a_r + 1}{4}.$$

So the odd subsequence satisfies the exact recurrence

$$a_{r+1} = \frac{3a_r + 1}{4}.$$

Now if $a_r > 1$, then

$$a_{r+1} = \frac{3a_r + 1}{4} < \frac{4a_r}{4} = a_r.$$

Thus as long as $a_r > 1$, the sequence of positive odd integers a_r strictly decreases. A strictly decreasing sequence of positive integers must terminate, so there exists r_0 such that

$$a_{r_0} = 1.$$

Therefore

$$T^{L+2r_0}(n) = 1,$$

which means that the orbit of n reaches 1. This is exactly the reduced Collatz conjecture. □

Corollary 7.9. *The reduced Collatz conjecture is equivalent to the statement that every positive integer has a parity sequence which is eventually determined by the cocycle expansion and eventually equals the alternating tail* $1, 0, 1, 0, \ldots$.

Remark 7.10 (Empirical observation)**.** *Because every R-orbit eventually reaches* 0 *or* 1*, the right-hand side above is ultimately built only from the values* $R^j(n) \in \{0, 1\}$ *and from cocycle terms evaluated on those small arguments. This suggests a possible route to eventual periodicity of parity vectors, but it is only a heuristic suggestion. Likewise, the observed phenomenon*

$$T^{I(n)^3}(n) \in \{1, 2\}$$

for many tested values of n *is at present only an empirical observation and is not used in any proof in this paper.*

8 What the interval model does and does not imply

The previous sections establish several rigorous facts.

(1) The coding map F sends binary sequences to the Cantor set and intertwines the left shift with the interval map g.

(2) Collatz parity sequences therefore produce orbits of the interval map.

(3) The map g has a point of exact period three and is chaotic in the sense of Li and Yorke.

(4) The 2-adic quantity α_x realizes any binary sequence as the parity sequence of a 2-adic orbit.

(5) Purely periodic and eventually periodic binary strings correspond to periodic and eventually periodic 2-adic orbits.

(6) Under the classical Collatz conjecture, the generating functions $f_n(x)$ are rational and the tuple $F_n(x)$ traces a rational algebraic curve.

(7) The decomposition $T = I + R$ leads to a natural family of parity cocycles.

What none of this proves by itself is the classical statement that every positive integer reaches 1 under the ordinary Collatz map. The key unresolved difficulty remains arithmetic: the parity sequences coming from integer orbits form a very special subset of all binary sequences, and the chaotic nature of the ambient interval map does not force those special orbits to end in the cycle corresponding to 1/4 and 3/4.

Thus the interval model and the 2-adic model are conceptually informative and mathematically consistent, but they should be read as reformulations and structural observations, not as a completed proof of the conjecture.

9 Conclusion

We have rewritten the original observations in a form where the mathematical content is clear and the logical dependencies are explicit.

The Cantor-coding part is exact. The interval map g really does realize the Bernoulli shift on the Cantor set, and Collatz parity sequences really do live inside this symbolic system. The 2-adic formulas involving α_x, λ_n, and ρ_n can be proved directly once the affine iterate coefficients are written down explicitly. The generating-function viewpoint explains, under the Collatz conjecture, why the vectors observed in computation lie on rational algebraic curves. The decomposition $T = I + R$ leads naturally to parity cocycles.

All of this helps explain why the Collatz problem attracts language from symbolic dynamics, interval chaos, 2-adic analysis, algebraic geometry, and cohomology. At the same time, the central arithmetic difficulty remains untouched: one still has to prove that the integer parity sequences are eventually forced into the known terminal cycle.