Finite positive definite kernels, spherical cone partitions, and explicit identities

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Abstract

This note records a simple geometric mechanism that converts finite positive definite kernels into exact finite-dimensional identities involving

$$\frac{\pi^{n/2}}{\Gamma(n/2)},$$

namely one half of the surface volume of the unit sphere $\mathbb{S}^{n-1}$. The basic input is a positive definite Gram matrix, or more generally the restriction of a positive definite kernel to a finite set. By Gram realization, such a matrix can be represented by a basis of vectors in Euclidean space. The associated 2^n sign cones then induce a partition of $\mathbb{S}^{n-1}$ into simplicial regions, and one representative from each antipodal pair always has total volume $\pi^{n/2}/\Gamma(n/2)$.

In dimension two, this recovers the familiar identity $\alpha + \beta + \gamma = \pi$ in a cone-partition form. In dimension three, one obtains explicit solid-angle identities and corresponding arctangent formulas. As a concrete example, we discuss the kernel

$$K(a, b) = \min(a, b),$$

restricted to prime samples. Its normalized Gram matrix is governed by prime ratios, and the resulting spherical cone volumes admit exact expressions in low dimensions and stable numerical evaluation in higher dimensions via Gaussian orthant probabilities attached to the inverse normalized Gram matrix.

No claim of deep arithmetic consequence is made here. The point is instead that a standard Gram-matrix construction gives a uniform geometric framework in which such identities arise naturally.

Contents

1 Introduction

The classical identity

$$\alpha + \beta + \gamma = \pi$$

for the angles of a Euclidean triangle is often the first place where π appears through a finite geometric decomposition. The present note describes a higher-dimensional analogue based on spherical cone partitions.

The underlying observation is simple. If

$$v_1, \ldots, v_n \in \mathbb{R}^n$$

form a basis, then the 2^n simplicial cones

$$C_\varepsilon = \operatorname{cone}(\varepsilon_1 v_1, \ldots, \varepsilon_n v_n), \qquad \varepsilon \in \{\pm 1\}^n,$$

partition $\mathbb{R}^n$ up to boundaries. Intersecting with the unit sphere yields a partition of $\mathbb{S}^{n-1}$ into spherical simplices. Consequently, the sum of the spherical volumes of one representative from each antipodal pair is always

$$\frac{\pi^{n/2}}{\Gamma(n/2)}.$$

Thus every basis gives an exact finite-dimensional identity of this type.

Positive definite kernels provide a convenient source of such bases. If G is a positive definite Gram matrix, then by Gram realization it comes from linearly independent vectors in Euclidean space; after normalizing the vectors, the geometry is encoded by a correlation matrix. This is the point of view used throughout.

A natural example is the Brownian kernel

$$K(a, b) = \min(a, b).$$

On every strictly increasing sample

$$0 < t_1 < \cdots < t_n$$

it yields a positive definite Gram matrix. If one chooses prime samples, the normalized Gram matrix takes the explicit form

$$R_{ij} = \sqrt{\frac{p_{\min(i,j)}}{p_{\max(i,j)}}}.$$

This makes the cone geometry particularly concrete.

The note has two modest aims. First, it records the kernel-to-spherical-volume mechanism in a way that is convenient for explicit computation. Second, it illustrates the mechanism on the prime–min example, where both exact low-dimensional formulas and numerical higher-dimensional evaluations are available.

2 Fiedler's Gram-matrix framework

The starting point is standard.

Theorem 2.1 (Gram realization). *Let $G = (g_{ij})$ be a real symmetric positive semidefinite matrix of rank r. Then there exist vectors*

$$p_1, \ldots, p_n \in \mathbb{R}^r$$

such that

$$g_{ij} = \langle p_i, p_j \rangle \qquad (1 \le i, j \le n).$$

If G is positive definite, then the vectors $p_1, \ldots, p_n$ are linearly independent.

Definition 2.2. *Let $G = (g_{ij})$ be a positive definite $n \times n$ matrix, and let*

$$D = \operatorname{diag}(g_{11}, \dots, g_{nn}).$$

The normalized Gram matrix of G is

$$R = D^{-1/2} G D^{-1/2}.$$

Then R is positive definite with diagonal entries 1, so R is the Gram matrix of unit vectors

$$u_1, \dots, u_n \in \mathbb{S}^{n-1}.$$

Only the normalized matrix matters for the spherical geometry of the associated cones. In particular,

$$r_{ij} = \langle u_i, u_j \rangle = \frac{g_{ij}}{\sqrt{g_{ii} g_{jj}}}$$

records the relevant pairwise cosines.

3 Simplicial cones and spherical simplices

Let $v_1, \dots, v_n$ be a basis of $\mathbb{R}^n$. For each sign pattern

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$$

define the simplicial cone

$$C_\varepsilon = \operatorname{cone}(\varepsilon_1 v_1, \dots, \varepsilon_n v_n) = \Big\{ \sum_{i=1}^n t_i \varepsilon_i v_i : t_i \geq 0 \Big\}.$$

Its spherical image is

$$\Sigma_\varepsilon = C_\varepsilon \cap \mathbb{S}^{n-1},$$

a spherical $(n-1)$-simplex.

Proposition 3.1 (Sphere partition identity)**.** *The family $\{\Sigma_\varepsilon : \varepsilon \in \{\pm 1\}^n\}$ partitions $\mathbb{S}^{n-1}$ up to boundaries of spherical measure zero. Hence*

$$\sum_{\varepsilon \in \{\pm 1\}^n} \operatorname{vol}_{n-1}(\Sigma_\varepsilon) = \operatorname{vol}(\mathbb{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Moreover, antipodal cones have the same spherical volume, so if one chooses exactly one representative from each antipodal pair, then

$$\sum_{\varepsilon \sim -\varepsilon} \operatorname{vol}_{n-1}(\Sigma_\varepsilon) = \frac{\pi^{n/2}}{\Gamma(n/2)}.$$

Proof. Since $v_1, \dots, v_n$ is a basis, every nonzero vector $x \in \mathbb{R}^n$ has a unique expansion

$$x = \sum_{i=1}^n a_i v_i,$$

and the sign pattern of the coefficients a_i determines exactly one cone C_ε containing x in its relative interior. Thus the cones cover $\mathbb{R}^n$ and overlap only on lower-dimensional boundaries. Intersecting with $\mathbb{S}^{n-1}$ yields the spherical partition. The antipodal map sends Σ_ε isometrically onto $\Sigma_{-\varepsilon}$, so antipodal pairs have equal volume. The formula for $\operatorname{vol}(\mathbb{S}^{n-1})$ is standard. □

This elementary proposition is the basic source of all subsequent identities.

3.1 Dimension two

For a basis u_1, u_2 of $\mathbb{R}^2$, let

$$c = \langle u_1, u_2 \rangle = \cos\theta, \qquad 0 < \theta < \pi.$$

The four cones come in two antipodal pairs, and the two distinct spherical lengths are

$$\theta = \arccos(c), \qquad \pi - \theta = \arccos(-c).$$

Hence

$$\arccos(c) + \arccos(-c) = \pi.$$

So the familiar two-dimensional identity is already a half-sphere identity.

3.2 Dimension three

Now let $u_1, u_2, u_3 \in \mathbb{S}^2$ be linearly independent, and write

$$x = \langle u_2, u_3 \rangle, \qquad y = \langle u_1, u_3 \rangle, \qquad z = \langle u_1, u_2 \rangle.$$

Then the normalized Gram matrix is

$$R = \begin{pmatrix} 1 & z & y \\ z & 1 & x \\ y & x & 1 \end{pmatrix}, \qquad \Delta = \det(R) = 1 + 2xyz - x^2 - y^2 - z^2.$$

For the cone generated by u_1, u_2, u_3, the corresponding solid angle is

$$\Omega_{+++} = 2\,\mathrm{atan2}(\sqrt{\Delta}, 1 + x + y + z).$$

Changing signs changes only the signs of x, y, z in the denominator. Choosing one representative from each antipodal pair gives four solid angles:

$$\begin{aligned}
\Omega_1 &= 2\,\mathrm{atan2}(\sqrt{\Delta}, 1 + x + y + z), \\
\Omega_2 &= 2\,\mathrm{atan2}(\sqrt{\Delta}, 1 + x - y - z), \\
\Omega_3 &= 2\,\mathrm{atan2}(\sqrt{\Delta}, 1 - x + y - z), \\
\Omega_4 &= 2\,\mathrm{atan2}(\sqrt{\Delta}, 1 - x - y + z).
\end{aligned}$$

Since these four cones represent half of the sphere partition, their sum is 2π.

Theorem 3.2 (Three-dimensional identity)**.** *With the notation above,*

$$\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 = 2\pi,$$

or equivalently,

$$\begin{aligned}
&\mathrm{atan2}(\sqrt{\Delta}, 1 + x + y + z) + \mathrm{atan2}(\sqrt{\Delta}, 1 + x - y - z) \\
&\quad + \mathrm{atan2}(\sqrt{\Delta}, 1 - x + y - z) + \mathrm{atan2}(\sqrt{\Delta}, 1 - x - y + z) = \pi.
\end{aligned}$$

4 The general n-dimensional formula

Theorem 4.1 (Kernel-to-half-sphere principle in dimension n)**.** *Let G be a positive definite $n \times n$ matrix, and let $R = D^{-1/2}GD^{-1/2}$ be its normalized Gram matrix. Choose unit vectors $u_1, \dots, u_n \in \mathbb{S}^{n-1}$ with Gram matrix R. For each sign pattern $\varepsilon \in \{\pm 1\}^n$, let $\Omega_\varepsilon(G)$ denote the spherical $(n-1)$-volume of the simplex*

$$\Sigma_\varepsilon = \operatorname{cone}(\varepsilon_1 u_1, \dots, \varepsilon_n u_n) \cap \mathbb{S}^{n-1}.$$

Then

$$\sum_{\varepsilon \in \{\pm 1\}^n} \Omega_\varepsilon(G) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and, after choosing one representative from each antipodal pair,

$$\sum_{\varepsilon \sim -\varepsilon} \Omega_\varepsilon(G) = \frac{\pi^{n/2}}{\Gamma(n/2)}.$$

Remark 4.2. *For $n = 2$ this gives a formula for π. For $n = 3$ it gives a formula for 2π. In general it gives an exact identity for the constant*

$$\frac{\pi^{n/2}}{\Gamma(n/2)},$$

which is one half of the surface volume of $\mathbb{S}^{n-1}$.

5 From positive definite kernels to finite-dimensional formulas

Let X be a set and let

$$K : X \times X \to \mathbb{R}$$

be a positive definite kernel. For any finite set of pairwise distinct points

$$x_1, \dots, x_n \in X,$$

the matrix

$$G = (K(x_i, x_j))_{i,j=1}^n$$

is positive semidefinite. If G is positive definite, then by Gram realization there exist linearly independent vectors $v_1, \dots, v_n$ in $\mathbb{R}^n$ with Gram matrix G. Therefore the sphere-partition identity applies.

Proposition 5.1 (Finite sample from a positive definite kernel)**.** *Let K be a positive definite kernel on X, and let $x_1, \dots, x_n \in X$ be such that the Gram matrix*

$$G_{ij} = K(x_i, x_j)$$

is positive definite. Define

$$r_{ij} = \frac{K(x_i, x_j)}{\sqrt{K(x_i, x_i)K(x_j, x_j)}}.$$

Then there exists a basis of unit vectors $u_1, \dots, u_n \in \mathbb{S}^{n-1}$ such that

$$\langle u_i, u_j \rangle = r_{ij}.$$

Consequently the corresponding spherical simplex volumes satisfy

$$\sum_{\varepsilon \sim -\varepsilon} \Omega_\varepsilon = \frac{\pi^{n/2}}{\Gamma(n/2)}.$$

Thus every strictly positive definite finite sample produces an identity of this type.

The practical message is straightforward: once a finite positive definite kernel sample is nondegenerate, the normalized Gram matrix determines a spherical cone decomposition and hence a finite identity for half the sphere volume.

6 The kernel $K(a,b) = \min(a,b)$ and primes

A particularly natural example is the Brownian kernel

$$K(a,b) = \min(a,b), \qquad a,b > 0.$$

On every finite set of distinct positive numbers this yields a positive definite Gram matrix. Indeed, if

$$0 < t_1 < \cdots < t_n,$$

then one can realize it by the vectors

$$v_1 = (\sqrt{t_1}, 0, \ldots, 0), \qquad v_2 = (\sqrt{t_1}, \sqrt{t_2 - t_1}, 0, \ldots, 0),$$

and so on, so that

$$\langle v_i, v_j \rangle = \min(t_i, t_j).$$

Hence, for primes

$$p_1 < p_2 < \cdots < p_n,$$

the Gram matrix

$$G_{ij} = \min(p_i, p_j)$$

is positive definite and produces an n-dimensional basis cone decomposition.

After normalization,

$$r_{ij} = \frac{\min(p_i, p_j)}{\sqrt{p_i p_j}} = \sqrt{\frac{\min(p_i, p_j)}{\max(p_i, p_j)}}.$$

So the geometry is governed by prime ratios.

6.1 The three-prime case

Take three primes $p_1 < p_2 < p_3$, and define

$$z = \sqrt{\frac{p_1}{p_2}}, \qquad y = \sqrt{\frac{p_1}{p_3}}, \qquad x = \sqrt{\frac{p_2}{p_3}}.$$

Then

$$\Delta = 1 + 2xyz - x^2 - y^2 - z^2$$

and the four distinct solid angles attached to the basis cone decomposition are

$$\begin{aligned}
\Omega_1 &= 2\,\mathrm{atan2}\big(\sqrt{\Delta}, 1 + x + y + z\big),\\
\Omega_2 &= 2\,\mathrm{atan2}\big(\sqrt{\Delta}, 1 + x - y - z\big),\\
\Omega_3 &= 2\,\mathrm{atan2}\big(\sqrt{\Delta}, 1 - x + y - z\big),\\
\Omega_4 &= 2\,\mathrm{atan2}\big(\sqrt{\Delta}, 1 - x - y + z\big),
\end{aligned}$$

with

$$\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 = 2\pi.$$

Equivalently,

$$\begin{aligned}
&\mathrm{atan2}\big(\sqrt{\Delta}, 1 + x + y + z\big) + \mathrm{atan2}\big(\sqrt{\Delta}, 1 + x - y - z\big)\\
&\quad + \mathrm{atan2}\big(\sqrt{\Delta}, 1 - x + y - z\big) + \mathrm{atan2}\big(\sqrt{\Delta}, 1 - x - y + z\big) = \pi.
\end{aligned}$$

Thus the kernel $K(a,b) = \min(a,b)$ yields an explicit three-dimensional identity on every prime triple.

6.2 An exact formula for Ω_N and its power-series expansion

Let

$$p_N < p_{N+1} < p_{N+2}$$

be three consecutive primes, and write

$$g_N := p_{N+1} - p_N, \qquad g_{N+1} := p_{N+2} - p_{N+1}.$$

In the prime–min case with $n = 3$, consider the normalized correlation matrix

$$C_N = \begin{pmatrix} 1 & \sqrt{p_N/p_{N+1}} & \sqrt{p_N/p_{N+2}} \\ \sqrt{p_N/p_{N+1}} & 1 & \sqrt{p_{N+1}/p_{N+2}} \\ \sqrt{p_N/p_{N+2}} & \sqrt{p_{N+1}/p_{N+2}} & 1 \end{pmatrix}.$$

Set

$$x := \sqrt{\frac{p_{N+1}}{p_{N+2}}}, \qquad z := \sqrt{\frac{p_N}{p_{N+1}}}, \qquad y := \sqrt{\frac{p_N}{p_{N+2}}} = xz.$$

Then

$$\det C_N = 1 + 2xyz - x^2 - y^2 - z^2 = \frac{(p_{N+1} - p_N)(p_{N+2} - p_{N+1})}{p_{N+1}p_{N+2}} = \frac{g_N g_{N+1}}{p_{N+1}p_{N+2}}.$$

Moreover, the explicit spherical-angle formula gives

$$\Omega_N = 2\,\mathrm{atan2}\big(\sqrt{\det C_N},\ 1 + x + y + z\big).$$

Since $y = xz$, the second argument simplifies to

$$1 + x + y + z = 1 + x + xz + z = (1 + x)(1 + z),$$

and hence

$$\Omega_N = 2\,\mathrm{atan2}\left(\sqrt{\frac{g_N g_{N+1}}{p_{N+1}p_{N+2}}}, \left(1 + \sqrt{\frac{p_{N+1}}{p_{N+2}}}\right)\left(1 + \sqrt{\frac{p_N}{p_{N+1}}}\right)\right).$$

Because both arguments are positive, this is equivalently

$$\tan\frac{\Omega_N}{2} = \frac{\sqrt{g_N g_{N+1}/(p_{N+1}p_{N+2})}}{\left(1 + \sqrt{p_{N+1}/p_{N+2}}\right)\left(1 + \sqrt{p_N/p_{N+1}}\right)}.$$

Multiplying numerator and denominator by $\sqrt{p_{N+1}p_{N+2}}$ yields the exact identity

$$\tan\frac{\Omega_N}{2} = \frac{\sqrt{g_N g_{N+1}}}{(\sqrt{p_N} + \sqrt{p_{N+1}})(\sqrt{p_{N+1}} + \sqrt{p_{N+2}})}.$$

Therefore

$$\boxed{\Omega_N = 2\arctan\left(\frac{\sqrt{g_N g_{N+1}}}{(\sqrt{p_N} + \sqrt{p_{N+1}})(\sqrt{p_{N+1}} + \sqrt{p_{N+2}})}\right).}$$

Now define

$$X_N := \frac{\sqrt{g_N g_{N+1}}}{(\sqrt{p_N} + \sqrt{p_{N+1}})(\sqrt{p_{N+1}} + \sqrt{p_{N+2}})}.$$

Since $0 < X_N < 1$, the Taylor series

$$\arctan x = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{2m+1} \qquad (|x| < 1)$$

applies termwise, giving

$$\boxed{\Omega_N = 2\sum_{m=0}^{\infty}(-1)^m \frac{1}{2m+1}\left(\frac{\sqrt{g_N g_{N+1}}}{\big(\sqrt{p_N}+\sqrt{p_{N+1}}\big)\big(\sqrt{p_{N+1}}+\sqrt{p_{N+2}}\big)}\right)^{2m+1}.}$$

Equivalently,

$$\Omega_N = 2\sum_{m=0}^{\infty}(-1)^m \frac{(g_N g_{N+1})^{m+\frac{1}{2}}}{(2m+1)\,\big(\sqrt{p_N}+\sqrt{p_{N+1}}\big)^{2m+1}\big(\sqrt{p_{N+1}}+\sqrt{p_{N+2}}\big)^{2m+1}}.$$

This identity exactly records the product $g_N g_{N+1}$ in geometric form. By itself, however, it should not be interpreted as yielding a new direct bound for a single prime gap.

6.3 General prime samples

For any $n \geq 2$, the sample $p_1, \ldots, p_n$ gives an explicit normalized Gram matrix

$$r_{ij} = \sqrt{\frac{p_{\min(i,j)}}{p_{\max(i,j)}}}.$$

The corresponding antipodal half of the basis cone decomposition satisfies

$$\sum_{\varepsilon\sim-\varepsilon} \Omega_\varepsilon = \frac{\pi^{n/2}}{\Gamma(n/2)}.$$

This gives a family of exact finite-dimensional identities attached to the chosen prime tuple.

7 Relation with the kernel used in *Some formulas for* π

In earlier three-point formulas, the kernel

$$K_2(a,b) = \frac{\min(a,b)^2}{\max(a,b)^2}$$

was used. This kernel already has diagonal entries 1, so it is itself a normalized Gram matrix. Hence it fits the present framework with no extra normalization step.

If one restricts K_2 to a finite set $x_1, \ldots, x_n$, then the matrix

$$R_{ij} = \frac{\min(x_i,x_j)^2}{\max(x_i,x_j)^2}$$

can be treated directly as a Gram matrix of unit vectors (provided it is positive definite on the chosen sample). The same basis-cone construction then yields

$$\sum_{\varepsilon\sim-\varepsilon} \Omega_\varepsilon = \frac{\pi^{n/2}}{\Gamma(n/2)}.$$

Thus both kernels — the Brownian kernel $\min(a,b)$ and the normalized ratio kernel $\min(a,b)^2/\max(a,b)^2$ — fit naturally into the same geometric picture.

8 Power-series expansions and ten explicit prime arctangent identities

In dimension three the half-sphere identity takes the form

$$\operatorname{atan2}(\sqrt{\Delta}, d_1) + \operatorname{atan2}(\sqrt{\Delta}, d_2) + \operatorname{atan2}(\sqrt{\Delta}, d_3) + \operatorname{atan2}(\sqrt{\Delta}, d_4) = \pi,$$

where

$$d_1 = 1 + x + y + z, \quad d_2 = 1 + x - y - z, \quad d_3 = 1 - x + y - z, \quad d_4 = 1 - x - y + z,$$

and

$$\Delta = 1 + 2xyz - x^2 - y^2 - z^2.$$

Whenever $d_j > 0$, one has

$$\operatorname{atan2}(\sqrt{\Delta}, d_j) = \arctan\left(\frac{\sqrt{\Delta}}{d_j}\right).$$

Hence each three-dimensional identity can be converted into an arctangent formula

$$\arctan(t_1) + \arctan(t_2) + \arctan(t_3) + \arctan(t_4) = \pi, \qquad t_j = \frac{\sqrt{\Delta}}{d_j}.$$

For $|u| \leq 1$ one may further expand

$$\arctan(u) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} u^{2m+1}.$$

If $u > 1$, one uses

$$\arctan(u) = \frac{\pi}{2} - \arctan\left(\frac{1}{u}\right) = \frac{\pi}{2} - \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} u^{-(2m+1)}.$$

Thus every explicit three-dimensional prime example below yields a convergent infinite series representation.

For the kernel $K(a, b) = \min(a, b)$ and a prime triple $p_1 < p_2 < p_3$, we have

$$x = \sqrt{\frac{p_2}{p_3}}, \qquad y = \sqrt{\frac{p_1}{p_3}}, \qquad z = \sqrt{\frac{p_1}{p_2}}, \qquad \Delta = 1 + 2xyz - x^2 - y^2 - z^2.$$

The following ten prime triples produce explicit algebraic arctangent identities.

Example $(2, 3, 5)$**.**

$$\arctan\left(\frac{\sqrt{30}}{3\sqrt{10} + 3\sqrt{15} + 5\sqrt{6} + 15}\right) + \arctan\left(\frac{\sqrt{30}}{-5\sqrt{6} - 3\sqrt{10} + 3\sqrt{15} + 15}\right)$$
$$+ \arctan\left(\frac{\sqrt{30}}{-5\sqrt{6} - 3\sqrt{15} + 3\sqrt{10} + 15}\right) + \arctan\left(\frac{\sqrt{30}}{-3\sqrt{15} - 3\sqrt{10} + 5\sqrt{6} + 15}\right) = \pi.$$

Example $(3, 5, 7)$**.**

$$\arctan\left(\frac{2\sqrt{35}}{5\sqrt{21} + 7\sqrt{15} + 5\sqrt{35} + 35}\right) + \arctan\left(\frac{2\sqrt{35}}{-7\sqrt{15} - 5\sqrt{21} + 5\sqrt{35} + 35}\right)$$
$$+ \arctan\left(\frac{2\sqrt{35}}{-5\sqrt{35} - 7\sqrt{15} + 5\sqrt{21} + 35}\right) + \arctan\left(\frac{2\sqrt{35}}{-5\sqrt{35} - 5\sqrt{21} + 7\sqrt{15} + 35}\right) = \pi.$$

Example $(5,7,11)$.

$$\arctan\left(\frac{2\sqrt{154}}{7\sqrt{55}+7\sqrt{77}+11\sqrt{35}+77}\right)+\arctan\left(\frac{2\sqrt{154}}{-11\sqrt{35}-7\sqrt{55}+7\sqrt{77}+77}\right) + \arctan\left(\frac{2\sqrt{154}}{-11\sqrt{35}-7\sqrt{77}+7\sqrt{55}+77}\right)+\arctan\left(\frac{2\sqrt{154}}{-7\sqrt{77}-7\sqrt{55}+11\sqrt{35}+77}\right)=\pi.$$

Example $(7,11,13)$.

$$\arctan\left(\frac{2\sqrt{286}}{11\sqrt{91}+13\sqrt{77}+11\sqrt{143}+143}\right)+\arctan\left(\frac{2\sqrt{286}}{-13\sqrt{77}-11\sqrt{91}+11\sqrt{143}+143}\right) + \arctan\left(\frac{2\sqrt{286}}{-11\sqrt{143}-13\sqrt{77}+11\sqrt{91}+143}\right)+\arctan\left(\frac{2\sqrt{286}}{-11\sqrt{143}-11\sqrt{91}+13\sqrt{77}+143}\right)=\pi.$$

Example $(11,13,17)$.

$$\arctan\left(\frac{2\sqrt{442}}{13\sqrt{187}+13\sqrt{221}+17\sqrt{143}+221}\right)+\arctan\left(\frac{2\sqrt{442}}{-17\sqrt{143}-13\sqrt{187}+13\sqrt{221}+221}\right) + \arctan\left(\frac{2\sqrt{442}}{-17\sqrt{143}-13\sqrt{221}+13\sqrt{187}+221}\right)+\arctan\left(\frac{2\sqrt{442}}{-13\sqrt{221}-13\sqrt{187}+17\sqrt{143}+221}\right)=\pi.$$

Example $(13,17,19)$.

$$\arctan\left(\frac{2\sqrt{646}}{17\sqrt{247}+19\sqrt{221}+17\sqrt{323}+323}\right)+\arctan\left(\frac{2\sqrt{646}}{-19\sqrt{221}-17\sqrt{247}+17\sqrt{323}+323}\right) + \arctan\left(\frac{2\sqrt{646}}{-17\sqrt{323}-19\sqrt{221}+17\sqrt{247}+323}\right)+\arctan\left(\frac{2\sqrt{646}}{-17\sqrt{323}-17\sqrt{247}+19\sqrt{221}+323}\right)=\pi.$$

Example $(17,19,23)$.

$$\arctan\left(\frac{2\sqrt{874}}{19\sqrt{391}+19\sqrt{437}+23\sqrt{323}+437}\right)+\arctan\left(\frac{2\sqrt{874}}{-23\sqrt{323}-19\sqrt{391}+19\sqrt{437}+437}\right) + \arctan\left(\frac{2\sqrt{874}}{-23\sqrt{323}-19\sqrt{437}+19\sqrt{391}+437}\right)+\arctan\left(\frac{2\sqrt{874}}{-19\sqrt{437}-19\sqrt{391}+23\sqrt{323}+437}\right)=\pi.$$

Example $(19,23,29)$.

$$\arctan\left(\frac{2\sqrt{4002}}{23\sqrt{551}+23\sqrt{667}+29\sqrt{437}+667}\right)+\arctan\left(\frac{2\sqrt{4002}}{-29\sqrt{437}-23\sqrt{551}+23\sqrt{667}+667}\right) + \arctan\left(\frac{2\sqrt{4002}}{-29\sqrt{437}-23\sqrt{667}+23\sqrt{551}+667}\right)+\arctan\left(\frac{2\sqrt{4002}}{-23\sqrt{667}-23\sqrt{551}+29\sqrt{437}+667}\right)=\pi.$$

Example $(23,29,31)$.

$$\arctan\left(\frac{2\sqrt{2697}}{29\sqrt{713}+31\sqrt{667}+29\sqrt{899}+899}\right)+\arctan\left(\frac{2\sqrt{2697}}{-31\sqrt{667}-29\sqrt{713}+29\sqrt{899}+899}\right) + \arctan\left(\frac{2\sqrt{2697}}{-29\sqrt{899}-31\sqrt{667}+29\sqrt{713}+899}\right)+\arctan\left(\frac{2\sqrt{2697}}{-29\sqrt{899}-29\sqrt{713}+31\sqrt{667}+899}\right)=\pi.$$

Example $(29,31,37)$.

$$\arctan\left(\frac{2\sqrt{3441}}{31\sqrt{1073}+31\sqrt{1147}+37\sqrt{899}+1147}\right)+\arctan\left(\frac{2\sqrt{3441}}{-37\sqrt{899}-31\sqrt{1073}+31\sqrt{1147}+1147}\right) + \arctan\left(\frac{2\sqrt{3441}}{-37\sqrt{899}-31\sqrt{1147}+31\sqrt{1073}+1147}\right)+\arctan\left(\frac{2\sqrt{3441}}{-31\sqrt{1147}-31\sqrt{1073}+37\sqrt{899}+1147}\right)=\pi.$$

9 A concise summary of the mechanism

The argument used throughout can be stated briefly.

1. A positive definite kernel on a finite set gives a Gram matrix.
2. By Gram realization, this matrix comes from vectors in Euclidean space.
3. If the matrix is positive definite of size n, these vectors form a basis of $\mathbb{R}^n$.
4. The associated 2^n simplicial cones partition space, hence their spherical images partition $\mathbb{S}^{n-1}$.
5. Therefore the sum of one representative from each antipodal pair is always

$$\frac{\pi^{n/2}}{\Gamma(n/2)}.$$

This is the central observation; the prime–min example is one explicit instance of it.

10 Archimedean bracketing in two dimensions

Let $\theta \in (0, \pi)$ and consider the sector of the unit disk with central angle θ. Its area is

$$A(\theta) = \frac{\theta}{2}.$$

The inscribed triangle has area

$$A_{\text{in}}(\theta) = \frac{1}{2}\sin\theta,$$

while the circumscribed tangent triangle has area

$$A_{\text{out}}(\theta) = \frac{1}{2}\tan\theta.$$

Hence

$$\frac{1}{2}\sin\theta \le \frac{\theta}{2} \le \frac{1}{2}\tan\theta, \qquad \text{equivalently} \qquad \sin\theta \le \theta \le \tan\theta.$$

This is the classical Archimedean picture: the curved sector is squeezed between a linear inscribed object and a linear circumscribed object.

Now subdivide the arc into m equal parts, so that $\theta = m\delta$ with $\delta = \theta/m$. Summing the areas of the m inscribed and circumscribed triangles gives

$$\frac{m}{2}\sin\frac{\theta}{m} \le \frac{\theta}{2} \le \frac{m}{2}\tan\frac{\theta}{m}.$$

Thus

$$m\sin\frac{\theta}{m} \le \theta \le m\tan\frac{\theta}{m},$$

and both bounds converge to θ as $m \to \infty$.

11 An n-dimensional generalization

Let $u_1, \dots, u_n \in S^{n-1} \subset \mathbb{R}^n$ be linearly independent, let

$$C = \mathrm{cone}(u_1, \dots, u_n), \qquad \Sigma = C \cap S^{n-1},$$

and let

$$G = (\langle u_i, u_j \rangle)_{i,j=1}^n$$

be the normalized Gram matrix. The set Σ is a spherical $(n-1)$-simplex, and the radial sector over it is

$$C \cap B^n.$$

By polar integration,

$$\mathrm{vol}_n(C \cap B^n) = \frac{1}{n}\,\mathrm{vol}_{n-1}(\Sigma).$$

On the other hand, the Euclidean simplex

$$T = \mathrm{conv}(0, u_1, \dots, u_n)$$

is contained in $C \cap B^n$, and its volume equals

$$\mathrm{vol}_n(T) = \frac{\sqrt{\mathrm{Det}\, G}}{n!}.$$

Therefore

$$\boxed{\mathrm{vol}_{n-1}(\Sigma) \geq \frac{\sqrt{\mathrm{Det}\, G}}{(n-1)!}.}$$

This is a first-order lower bound.

To refine it, one triangulates the spherical simplex Σ into smaller spherical simplices. For each triangulation level m one may form an inscribed chordal sum I_m and a circumscribed tangent sum O_m such that

$$I_m \leq \mathrm{vol}_{n-1}(\Sigma) \leq O_m, \qquad I_m \to \mathrm{vol}_{n-1}(\Sigma), \quad O_m \to \mathrm{vol}_{n-1}(\Sigma) \quad (m \to \infty).$$

This is the higher-dimensional analogue of Archimedes' polygonal approximation.

12 A proposition for the prime–min kernel in dimension three

Let

$$K(a,b) = \min(a,b)$$

and let $p < q < r$ be three consecutive primes. Consider the normalized Gram matrix

$$C(p,q,r) = \begin{pmatrix} 1 & \sqrt{p/q} & \sqrt{p/r} \\ \sqrt{p/q} & 1 & \sqrt{q/r} \\ \sqrt{p/r} & \sqrt{q/r} & 1 \end{pmatrix}.$$

Let $\Omega(p,q,r)$ denote the spherical area of the distinguished small spherical triangle corresponding to the sign cone $(+,+,+)$.

For each Archimedean refinement level m, let $I_m(p,q,r)$ and $O_m(p,q,r)$ be the inscribed and circumscribed polygonal approximations of this spherical triangle. Then

$$I_m(p,q,r) \leq \Omega(p,q,r) \leq O_m(p,q,r), \qquad I_m(p,q,r),\, O_m(p,q,r) \to \Omega(p,q,r) \quad (m \to \infty).$$

Moreover, the determinant simplifies exactly to

$$\boxed{\operatorname{Det} C(p,q,r) = \frac{(q-p)(r-q)}{qr}.}$$

Hence the first-order lower bound becomes

$$\boxed{\frac{1}{2}\sqrt{\frac{(q-p)(r-q)}{qr}} \le \Omega(p,q,r).}$$

Proposition 12.1. *Let $p_N < p_{N+1} < p_{N+2}$ be three consecutive primes, and write*

$$g_N = p_{N+1} - p_N, \qquad g_{N+1} = p_{N+2} - p_{N+1}.$$

Let $\Omega_N = \Omega(p_N, p_{N+1}, p_{N+2})$ be the spherical area of the corresponding distinguished prime–min triangle. Then for every fixed N there exist Archimedean sums $I_m(N)$ and $O_m(N)$ such that

$$I_m(N) \le \Omega_N \le O_m(N), \qquad I_m(N), O_m(N) \to \Omega_N \quad (m \to \infty),$$

and

$$\frac{1}{2}\sqrt{\frac{g_N g_{N+1}}{p_{N+1}p_{N+2}}} \le \Omega_N.$$

Furthermore,

$$\Omega_N \sim \frac{1}{2}\sqrt{\frac{g_N g_{N+1}}{p_{N+1}p_{N+2}}} \qquad (N \to \infty).$$

Proof. The Archimedean bracketing for fixed N is the spherical-triangle version of the construction above. The determinant identity is a direct expansion of $\operatorname{Det} C(p,q,r)$. Since p_N, p_{N+1}, p_{N+2} are consecutive primes, the prime number theorem implies

$$\frac{p_{N+1}}{p_N} \to 1, \qquad \frac{p_{N+2}}{p_N} \to 1.$$

Hence the three normalized inner products in $C(p_N, p_{N+1}, p_{N+2})$ tend to 1, so the spherical triangle becomes small. For small spherical triangles, spherical area is asymptotic to Euclidean chordal area. That flat area is exactly

$$\frac{1}{2}\sqrt{\operatorname{Det} C(p_N, p_{N+1}, p_{N+2})} = \frac{1}{2}\sqrt{\frac{g_N g_{N+1}}{p_{N+1}p_{N+2}}}.$$

Therefore

$$\Omega_N \sim \frac{1}{2}\sqrt{\frac{g_N g_{N+1}}{p_{N+1}p_{N+2}}}.$$ □

13 Prime number theorem simplification

By the prime number theorem,

$$p_N \sim N \log N.$$

Since also $p_{N+1} \sim p_N$ and $p_{N+2} \sim p_N$, the proposition yields the asymptotic form

$$\boxed{\Omega_N \sim \frac{\sqrt{g_N g_{N+1}}}{2p_N} \sim \frac{\sqrt{g_N g_{N+1}}}{2N \log N}.}$$

Thus the spherical area of the distinguished prime–min triangle is controlled, to first order, by the geometric mean of two consecutive prime gaps divided by the scale $p_N \sim N \log N$.

Remark 13.1. *This should be read as an asymptotic encoding of geometric size, not as a new theorem about prime gaps.*

14 Twin primes and fast arctangent formulas

In this section we record a family of explicit three-dimensional formulas in which the arctangent arguments become small. The construction uses triples of the form

$$(p, p+2, c),$$

where p and $p+2$ are twin primes and $c > p+2$ is even.

We work with the meet kernel attached to ordered prime factorizations. If

$$m = p_1 \cdots p_r, \qquad n = q_1 \cdots q_s,$$

where the prime factors are listed in nondecreasing order, then define

$$K_\wedge(m,n) = \prod_{i=1}^{\min(r,s)} \min(p_i, q_i).$$

Theorem 14.1 (Twin-prime family)**.** *Let p and $p+2$ be twin primes, and let $c > p+2$ be an even integer. Set*

$$a = p, \qquad b = p+2, \qquad c = c,$$

and consider the normalized three-point data

$$x = \frac{K_\wedge(a,b)}{\sqrt{K_\wedge(a,a)K_\wedge(b,b)}}, \qquad y = \frac{K_\wedge(a,c)}{\sqrt{K_\wedge(a,a)K_\wedge(c,c)}}, \qquad z = \frac{K_\wedge(b,c)}{\sqrt{K_\wedge(b,b)K_\wedge(c,c)}}.$$

Then the three-dimensional half-sphere identity becomes

$$\arctan(t_1) + \arctan(t_2) + \arctan(t_3) + \arctan(t_4) = \pi,$$

where

$$t_j = \frac{\sqrt{\Delta}}{d_j}, \qquad \Delta = 1 + 2xyz - x^2 - y^2 - z^2,$$

and

$$d_1 = 1+x+y+z, \quad d_2 = 1+x-y-z, \quad d_3 = 1-x+y-z, \quad d_4 = 1-x-y+z.$$

Moreover:

(i)
$$x = \sqrt{\frac{p}{p+2}}, \qquad y = \frac{2}{\sqrt{pc}}, \qquad z = \frac{2}{\sqrt{(p+2)c}} = xy.$$

(ii)
$$\Delta = (1-x^2)(1-y^2),$$
$$d_1 = (1+x)(1+y), \quad d_2 = (1+x)(1-y), \quad d_3 = (1-x)(1+y), \quad d_4 = (1-x)(1-y).$$

(iii) *Consequently,*

$$t_1 = \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}}, \qquad t_2 = \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+y}{1-y}},$$
$$t_3 = \sqrt{\frac{1+x}{1-x}}\sqrt{\frac{1-y}{1+y}} = \frac{1}{t_2}, \qquad t_4 = \sqrt{\frac{1+x}{1-x}}\sqrt{\frac{1+y}{1-y}} = \frac{1}{t_1}.$$

(iv) *If we define*

$$\nu_j = \min(t_j, t_j^{-1}) \qquad (j = 1, 2, 3, 4),$$

and

$$\rho = \max(\nu_1, \nu_2, \nu_3, \nu_4),$$

then

$$\rho = \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+y}{1-y}}.$$

(v) *If $c = c(p)$ tends to infinity with p, then*

$$\rho \sim \frac{1}{\sqrt{2p}}, \qquad p \to \infty.$$

In particular,

$$\rho \to 0.$$

Proof. Since $a = p$ and $b = p + 2$ are prime, their ordered prime-factor strings each have length one. Because c is even, its smallest prime factor is 2, so

$$K_\wedge(p, c) = 2, \qquad K_\wedge(p + 2, c) = 2.$$

Also,

$$K_\wedge(p, p + 2) = p, \qquad K_\wedge(p, p) = p, \qquad K_\wedge(p + 2, p + 2) = p + 2, \qquad K_\wedge(c, c) = c.$$

Hence

$$x = \sqrt{\frac{p}{p+2}}, \qquad y = \frac{2}{\sqrt{pc}}, \qquad z = \frac{2}{\sqrt{(p+2)c}} = xy,$$

which proves (i).

Using $z = xy$,

$$\begin{aligned}\Delta &= 1 + 2xyz - x^2 - y^2 - z^2\\ &= 1 + 2x^2y^2 - x^2 - y^2 - x^2y^2\\ &= 1 - x^2 - y^2 + x^2y^2\\ &= (1 - x^2)(1 - y^2),\end{aligned}$$

and similarly

$$\begin{aligned}d_1 &= 1 + x + y + xy = (1 + x)(1 + y),\\ d_2 &= 1 + x - y - xy = (1 + x)(1 - y),\\ d_3 &= 1 - x + y - xy = (1 - x)(1 + y),\\ d_4 &= 1 - x - y + xy = (1 - x)(1 - y),\end{aligned}$$

proving (ii).

Dividing $\sqrt{\Delta} = \sqrt{(1 - x)(1 + x)(1 - y)(1 + y)}$ by the factored denominators gives the formulas in (iii). They imply $t_1t_4 = 1$ and $t_2t_3 = 1$.

Because $0 < x < 1$ and $0 < y < 1$, one has $0 < t_1 < 1$, $0 < t_2 < 1$, $t_3 > 1$, and $t_4 > 1$, hence

$$\nu_1 = t_1, \qquad \nu_2 = t_2, \qquad \nu_3 = t_2, \qquad \nu_4 = t_1.$$

Since

$$\sqrt{\frac{1+y}{1-y}} > \sqrt{\frac{1-y}{1+y}},$$

we have $t_2 > t_1$, so

$$\rho = t_2 = \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+y}{1-y}},$$

which proves (iv).

Finally,

$$1 - x^2 = 1 - \frac{p}{p+2} = \frac{2}{p+2}.$$

Using $1 - x^2 = (1-x)(1+x)$ and $x \to 1$ gives

$$\frac{1-x}{1+x} \sim \frac{1}{2p}, \qquad \sqrt{\frac{1-x}{1+x}} \sim \frac{1}{\sqrt{2p}}.$$

Also,

$$y = \frac{2}{\sqrt{pc}} \to 0$$

if $c = c(p) \to \infty$, hence

$$\sqrt{\frac{1+y}{1-y}} = 1 + O\left(\frac{1}{\sqrt{pc}}\right).$$

Combining the two estimates yields

$$\rho \sim \frac{1}{\sqrt{2p}}.$$

□

Remark 14.2 (Why the parameter ρ matters)**.** *If $0 < u < 1$, then*

$$\arctan(u) = u - \frac{u^3}{3} + \frac{u^5}{5} - \cdots .$$

The speed of convergence is controlled by the size of u. Thus ρ measures the worst effective argument among the four arctangent series after one is allowed to use

$$\arctan(t) = \frac{\pi}{2} - \arctan\left(\frac{1}{t}\right) \qquad (t > 1).$$

Therefore $\rho \to 0$ means that this family gives asymptotically faster and faster arctangent-series expansions.

Remark 14.3 (Why large even c is favorable)**.** *For fixed twin primes p and $p+2$, the quantity $x = \sqrt{p/(p+2)}$ is fixed, while*

$$y = \frac{2}{\sqrt{pc}}$$

decreases as c increases through even integers. Since

$$\rho = \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+y}{1-y}}$$

is increasing in y, the best convergence among even values of c is obtained by taking c as large as allowed.

14.1 A short asymptotic corollary and a numerical remark

The theorem above has an immediate consequence for the family

$$(a,b,c) = (p, p+2, c), \qquad p, p+2 \text{ twin primes}, \qquad c > p+2 \text{ even}.$$

Indeed,

$$\rho(p,c) = \sqrt{\frac{1-\sqrt{p/(p+2)}}{1+\sqrt{p/(p+2)}}}\sqrt{\frac{1+2/\sqrt{pc}}{1-2/\sqrt{pc}}}.$$

Hence, whenever $c = c(p) \to \infty$, one has

$$\rho(p,c) \sim \frac{1}{\sqrt{2p}} \qquad (p \to \infty),$$

and therefore

$$\rho(p,c) \to 0.$$

So this family yields an infinite sequence of arctangent formulas for π with improving convergence.

This also explains the numerical pattern in a restricted search over

$$(p, p+2, c), \qquad c \le N, \qquad c \text{ even}.$$

For fixed p, the quantity $\rho(p,c)$ decreases as c increases, so the best choice of c under a cutoff is the largest even integer allowed. Moreover, since the dominant factor is asymptotically $1/\sqrt{2p}$, the best triple under a cutoff N is expected to come from the largest twin-prime pair below N together with the largest admissible even value of c.

For example, a SageMath computation with $N = 10000$ yields the triple

$$(9929, 9931, 10000),$$

with

$$\rho \approx 0.007097371649198394.$$

This agrees well with the leading estimate

$$\frac{1}{\sqrt{2 \cdot 9929}} \approx 0.0070969.$$

In that example, the worst effective arctangent parameter is therefore about 7.1×10^{-3}, and the corresponding arctangent series already provide roughly 500 decimal digits using about 117 terms in the slowest summand.

15 Background on spherical simplices and Schläfli functions

The following sections summarize standard background on spherical simplices, Schläfli-type volume formulas, and related reduction identities. They are included because they provide context for the orthoscheme discussion below. They are not claimed as original here.

Source note. Several formulas and viewpoints in this background section follow the literature on Schläfli functions and, in particular, expository material associated with Warren D. Smith's Appendix C on spherical simplices and related formulas. The purpose of the present inclusion is organizational rather than original.

15.1 Introduction: From normal distributions to spherical simplices

In probabilistic applications, integrals over N-dimensional normal distributions with zero mean frequently appear. If the integration domain is bounded by N hyperplanes through the origin, a linear change of variables reduces the problem to the geometry of a spherical simplex. One basic identity is

$$\frac{1}{2}K^{-N/2}\Gamma\left(\frac{N}{2}\right)\int_{\text{spherical }(N-1)\text{-simplex}} 1\, d^{N-1}\text{Area} = \int_{\text{infinite flat }N\text{-cone}} \exp(-K|x|^2)\, dx_1 \ldots dx_N \tag{1}$$

for any $K > 0$.

If the integrand contains additional polynomials, one obtains moments of spherical simplices:

$$\frac{1}{2}K^{-(N+A+B+\ldots)/2}\Gamma\left(\frac{N+A+B+\ldots}{2}\right)\int_{\text{spherical }(N-1)\text{-simplex}} \left[x_1^A x_2^B \ldots\right] d^{N-1}\text{Area}$$
$$= \int_{\text{infinite flat }N\text{-cone}} \left[x_1^A x_2^B \ldots\right] \exp(-K|x|^2)\, dx_1 \ldots dx_N. \tag{2}$$

15.2 Reduction of moments to simple volumes

Even powers can be generated by differentiation, for example

$$\frac{\partial^k}{\partial a^k}\exp(-ax^2 - by^2 - cz^2) = (-x^2)^k \exp(-ax^2 - by^2 - cz^2). \tag{3}$$

Dimension can also be reduced by exact differential identities such as

$$\frac{\partial}{\partial x}\exp(-x^2 - y^2 - z^2) = -2x\exp(-x^2 - y^2 - z^2). \tag{4}$$

Applying Gauss's divergence theorem yields, for a cone S with faces F and outward unit normal f,

$$\int_S (a\cdot x)\exp\left(-\frac{|x|^2}{2}\right) d^N x = -\sum_{\text{faces }F\text{ of }S}\int_F (a\cdot f)\exp\left(-\frac{|x|^2}{2}\right) d^{N-1}x. \tag{5}$$

15.3 Non-Euclidean geometry: definitions and matrices

We view $(n-1)$-dimensional spherical geometry as the surface of the sphere $x_1^2+x_2^2+\cdots+x_n^2 = 1$ in $\mathbb{R}^n$. Analogously, $(n-1)$-dimensional hyperbolic geometry is defined by the upper sheet of the hyperboloid $x_1 = \sqrt{1 + x_2^2 + \cdots + x_n^2}$.

For a unified notation, define the signature matrix

$$D = \text{diag}(1, \pm 1, \pm 1, \ldots, \pm 1), \tag{6}$$

where the plus sign indicates spherical geometry and the minus sign indicates hyperbolic geometry. The characteristic surface is thus defined by $x \cdot Dx = 1$.

A simplex in this geometry is formed by taking the convex hull of n points lying on the surface together with the origin, extending the resulting cone to infinity, and intersecting with the surface.

Let X be an $n \times n$ matrix whose rows are the vectors representing the vertices of the simplex. The boundary hyperplanes take the form $h \cdot x \geq 0$, where h points outward. The normal vectors correspond to the rows of

$$H = -X^{-T}. \tag{7}$$

To determine the dihedral angles θ_{jk}, define

$$C = (XDX^T)^{-1} = HDH^T. \tag{8}$$

Then

$$\theta_{jk} = \pi - \arccos\left(\pm\frac{C_{jk}}{\sqrt{C_{jj}C_{kk}}}\right), \quad \text{for } j \neq k. \tag{9}$$

The non-Euclidean edge length E_{jk} between vertices j and k is obtained from

$$E_{jk} = (\pm)^{-1/2} \arccos\Big((X^T DX)_{jk}\Big). \tag{10}$$

15.4 Sphere volume and surface area

The N-dimensional volume $\mathrm{Vol}_N(r)$ and the $(N-1)$-dimensional surface area $\mathrm{Surf}_N(r)$ of a sphere of radius r in flat space are

$$\mathrm{Vol}_N(r) = \frac{\pi^{N/2} r^N}{(N/2)!}, \qquad \mathrm{Surf}_N(r) = O_N r^{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} r^{N-1}. \tag{11}$$

The sequence O_N satisfies the recurrence relation $O_N = O_{N-2}\frac{2\pi}{N-2}$.

15.5 Analytic continuation and the rational angle conjecture

A standard analytic-continuation principle says that any closed-form analytic formula describing the volume of an n-dimensional spherical simplex in terms of its dihedral angles also yields, subject to branch choices, the volume of the corresponding hyperbolic simplex after multiplication by i^n.

A related conjectural theme, due to J. Cheeger and J. Simons, is that the volume of a spherical simplex whose dihedral angles are rational multiples of 2π is generally not a rational multiple of the total spherical volume. One example discussed in the literature is the tetrahedron with dihedral-angle matrix

$$\begin{pmatrix} * & 2/5 & 19/47 & 7/17 \\ 2/5 & * & 15/37 & 12/29 \\ 19/47 & 15/37 & * & 17/43 \\ 7/17 & 12/29 & 17/43 & * \end{pmatrix} \pi.$$

Its fractional volume $s \approx 0.00338665\ldots$ has a nonterminating continued fraction, which is often cited as heuristic evidence for irrationality.

15.6 Schläfli functions: a compact formula list

Let $S_N(X)$ be the $(N-1)$-dimensional volume of the simplex defined by the rows of X. Then

$$S_1(X) = 1, \tag{12}$$

$$S_2(X) = E_{12} = \arccos(XX^T). \tag{13}$$

(For hyperbolic geometry, arccos is replaced by arccosh.)

For a triangle ($N = 3$) with edge lengths $a = E_{12}$, $b = E_{23}$, $c = E_{13}$ and semiperimeter $s = (a+b+c)/2$, one has the angle-defect formula

$$S_3(X) = \pm(\theta_{12} + \theta_{13} + \theta_{23} - \pi). \tag{14}$$

Equivalent formulas include:

- **Eriksson (1990):** $2\arctan\left(\frac{\det(X)}{1+G_{12}+G_{23}+G_{13}}\right)$, where $G = XX^T$.

- **Lhuillier:** $4\arctan\left(\sqrt{\tan\left(\frac{s}{2}\right)\tan\left(\frac{s-a}{2}\right)\tan\left(\frac{s-b}{2}\right)\tan\left(\frac{s-c}{2}\right)}\right)$.

- **Cagnoli:** $2\arcsin\left(\frac{\sqrt{\sin(s)\sin(s-a)\sin(s-b)\sin(s-c)}}{2\cos(a/2)\cos(b/2)\cos(c/2)}\right)$.

- **Todhunter:**

$$4\arcsin\left(\sqrt{\frac{\sin(s/2)\sin([s-a]/2)\sin([s-b]/2)\sin([s-c]/2)}{\cos(a/2)\cos(b/2)\cos(c/2)}}\right),$$
$$4\arccos\left(\sqrt{\frac{\cos(s/2)\cos([s-a]/2)\cos([s-b]/2)\cos([s-c]/2)}{\cos(a/2)\cos(b/2)\cos(c/2)}}\right).$$

- **Euler:** $2\arccos\left(\frac{1+\cos(a)+\cos(b)+\cos(c)}{4\cos(a/2)\cos(b/2)\cos(c/2)}\right)$.

- **Lagrange:** $2\arctan\left(\frac{2\sqrt{\sin(s/2)\sin([s-a]/2)\sin([s-b]/2)\sin([s-c]/2)}}{1+\cos(a)+\cos(b)+\cos(c)}\right)$.

- **Gosper (2016):** $\arccos\left(-1+\frac{(1+\cos(a)+\cos(b)+\cos(c))^2}{(1+\cos(a))(1+\cos(b))(1+\cos(c))}\right)$.

15.7 Dimension $N = 4$: the Murakami–Yano–Ushijima formula

For $N = 4$, let $L(z) = \mathrm{Li}_2(z)$ denote the dilogarithm. With the exponentiated angles $\xi_{jk} = \exp(i\theta_{jk})$, define

$$\begin{aligned} U(z) =&L(z) + L(\xi_{12}\xi_{13}\xi_{34}\xi_{24}z) + L(\xi_{12}\xi_{23}\xi_{34}\xi_{14}z) + L(\xi_{13}\xi_{23}\xi_{24}\xi_{14}z) \\ &- L(-\xi_{12}\xi_{13}\xi_{23}z) - L(-\xi_{12}\xi_{24}\xi_{14}z) - L(-\xi_{13}\xi_{34}\xi_{14}z) - L(-\xi_{23}\xi_{34}\xi_{24}z). \end{aligned} \tag{15}$$

The Gram matrix is $G_{jk} = -\cos(\theta_{jk})$ for $j \neq k$ and $G_{jj} = 1$. There are two critical points

$$z_\pm = -2\frac{\sin\theta_{12}\sin\theta_{34} + \sin\theta_{13}\sin\theta_{24} + \sin\theta_{14}\sin\theta_{23} \pm \sqrt{\det(G)}}{\xi_{12}\xi_{34} + \xi_{13}\xi_{24} + \xi_{23}\xi_{14} + \xi_{12}\xi_{13}\xi_{14} + \xi_{12}\xi_{23}\xi_{24} + \xi_{13}\xi_{23}\xi_{34} + \xi_{14}\xi_{24}\xi_{34} + \xi_{12}\xi_{13}\xi_{14}\xi_{23}\xi_{24}\xi_{34}}. \tag{16}$$

Defining $\mu(\theta, z) = U(z) - zU'(z)\ln(z)$, the volume is

$$S_4(X) = F_4(\theta) = \frac{(\pm)^{1/2}}{4}\Big[\mu(\theta, z_+) - \mu(\theta, z_-)\Big]. \tag{17}$$

15.8 Dimension $N = 5$: Dehn's invariant

Using F_4, one may write S_5 as

$$S_5(X) = \frac{2}{3}\sum_{a=1}^{5} F_4\left(\theta^{\text{without row/col } a}\right) - \frac{\pi}{3}\sum_{1\le a<b\le 5}\theta_{ab} + \frac{4\pi^2}{3}. \tag{18}$$

15.9 Dimension reduction and orthoschemes

If a D-dimensional spherical simplex is defined by fewer than $D+1$ hyperplanes through the origin in $(D+1)$-dimensional flat space, its relative volume is equal to that of the lower-dimensional simplex having the same dihedral angles.

A splitting identity also holds for adjacent simplices sharing a boundary face. Let Y be an $(n+1)\times n$ matrix; then, with suitable signs,

$$S_n(Y^{(1)}) \pm S_n(Y^{(n+1)}) = \pm S_n(Y^{(2)}) \pm S_n(Y^{(3)}) \cdots \pm S_n(Y^{(n)}), \tag{19}$$

where $Y^{(b)}$ denotes the matrix Y with the b-th row omitted.

Of particular importance are doubly-asymptotic orthoschemes (DAO). For $n \geq 3$, this leads to the continued-fraction identity

$$\cos^2\alpha_0 = 1 - \frac{\cos^2\alpha_1}{1 - \frac{\cos^2\alpha_2}{1 - \dots \frac{\cos^2\alpha_{n-3}}{1-\cos^2\alpha_{n-2}}}}, \tag{20}$$

with its cyclic permutations.

Lobachevsky's formula for the 3-volume, where $\alpha = \alpha_1 = \alpha_3 = \pi/2 - \alpha_2$, is

$$\mathrm{DAO}_3 = \frac{1}{2}\Lambda_2(\alpha), \qquad \text{with } \Lambda_2(\alpha) = -\frac{1}{2}\int_0^\alpha \ln(4\sin^2 u)\,du. \tag{21}$$

Kellerhals's 5-dimensional formula may be written as

$$\begin{aligned}
32\ \mathrm{DAO}_5 = &- 4K(1/\lambda, 0; \alpha_1) - 2K(\lambda, 0; \alpha_2) + 4K(1/\lambda, 0; \pi/2 - \alpha_0) - 2K(\lambda, 0; \alpha_4) - 4K(1/\lambda, 0; \alpha_5)\\
&+ K(\lambda, -\pi/2 - \alpha_1; \pi/2 + \alpha_1 + \alpha_2) + K(\lambda, -\pi/2 + \alpha_1; \pi/2 - \alpha_1 + \alpha_2)\\
&- K(\lambda, -\pi/2 - \alpha_1; \pi + \alpha_1) - K(\lambda, -\pi/2 + \alpha_1; \pi - \alpha_1)\\
&- K(\lambda, -\pi/2 - \alpha_5; \pi + \alpha_5) - K(\lambda, -\pi/2 + \alpha_5; \pi - \alpha_5)\\
&+ K(\lambda, -\pi/2 - \alpha_5; \pi/2 + \alpha_5 + \alpha_4) + K(\lambda, -\pi/2 + \alpha_5; \pi/2 - \alpha_5 + \alpha_4),
\end{aligned} \tag{22}$$

where

$$K(a, b; x) = \int_{\pi/2}^{x} \Lambda_2(\arctan(\tan(v)/a) - b)\,dv. \tag{23}$$

15.10 Schläfli volume differential and reduction theorems

A central identity is the Schläfli differential:

$$dS_N(X) = \frac{\pm 1}{N-2}\sum_{b=1}^{N}\sum_{a=1}^{b-1} S_{N-2}\left(X^{\{ab\}}\right) d\theta_{ab}, \tag{24}$$

where $X^{\{ab\}}$ denotes the coordinate-geometric projection onto the codimension-2 edge.

For odd dimensions N, the Brianchon–Gram–Sommerville identity reduces the volume S_N to lower dimensions:

$$S_N(X)\frac{2}{O_N} = \left|\sum_{s\subset\{1,\dots,N\}} (-1)^{N-|s|}\frac{S_{N-|s|}(X^s)}{O_{N-|s|}}\right|, \tag{25}$$

using the conventions $S_1()/O_1 = 1/2$ and $S_0()/O_0 = 1$.

Iterating this leads to a single-parity reduction identity in which only even-dimensional boundary simplices appear:

$$\frac{S_{2m+1}(X)}{O_{2m+1}} = \left|\sum_{j=0}^{m}\sum_{\substack{s\subset\{1,\dots,2m+1\}\\|s|=2j+1}} t_{2j}\frac{S_{2m-2j}(X^s)}{O_{2m-2j}}\right|, \tag{26}$$

where the coefficients t_{2j} are defined by

$$\frac{1}{z}\tanh\left(\frac{z}{2}\right) = \sum_{j\geq 0} t_{2j}\frac{z^{2j}}{(2j+1)!}. \tag{27}$$

16 Application to the min-prime kernel

To calculate the spherical solid angle Ω_ε for dimensions $n \geq 3$, one may use Schläfli's formalism. The following steps summarize the procedure for the kernel $K(a, b) = a \wedge b$, in particular for $K(a, b) = \min(a, b)$ restricted to prime samples.

Step 1: Construct the normalized Gram matrix. Evaluate the kernel on $p_1, \ldots, p_n$ to obtain $G_{ij} = K(p_i, p_j)$. Normalizing gives

$$R_{ij} = \frac{G_{ij}}{\sqrt{G_{ii} G_{jj}}}. \tag{28}$$

This matrix plays the role of XX^T in Schläfli's geometry.

Step 2: Determine the dihedral angles. Let $C = R^{-1}$. Then for $i \neq j$,

$$\theta_{ij} = \pi - \arccos\left(\frac{C_{ij}}{\sqrt{C_{ii} C_{jj}}}\right). \tag{29}$$

Step 3: Evaluate the appropriate Schläfli function. For example:

- **Dimension** $n = 2$**:** $\Omega = \arccos(R_{12})$.
- **Dimension** $n = 3$**:**
$$\Omega = S_3 = \theta_{12} + \theta_{13} + \theta_{23} - \pi. \tag{30}$$
- **Dimension** $n = 4$**:** use the Murakami–Yano–Ushijima formula $F_4(\theta)$.
- **Dimension** $n = 5$**:**
$$\Omega = S_5 = \frac{2}{3} \sum_{a=1}^{5} F_4(\theta^{\text{without } a}) - \frac{\pi}{3} \sum_{a<b} \theta_{ab} + \frac{4\pi^2}{3}. \tag{31}$$

Step 4: Other sign patterns. For a sign pattern $\varepsilon \in \{\pm 1\}^n$, replace R by

$$R_{ij}^{(\varepsilon)} = \varepsilon_i \varepsilon_j R_{ij} \tag{32}$$

and repeat the previous steps.

16.1 Simplification for the minimum kernel: orthoschemes

If one chooses $K(a, b) = \min(a, b)$ on an increasing sample, the matrix $G_{ij} = \min(p_i, p_j)$ is the covariance matrix of a Brownian-type process. Its inverse is tridiagonal. Consequently,

$$C_{ij} = 0 \qquad \text{for all } |i - j| > 1. \tag{33}$$

Substituting this into the dihedral-angle formula yields

$$\theta_{ij} = \pi - \arccos(0) = \frac{\pi}{2} \quad \text{for all } |i - j| > 1. \tag{34}$$

Thus almost all bounding hyperplanes are orthogonal, and the simplex is an orthoscheme.

For orthoschemes, one does not need the full general formulas in every case. The simplex is determined by a path of $n - 1$ non-right dihedral angles. In odd curved dimensions, this often leads to formulas in terms of Lobachevsky-type functions.

17 Application to the min-prime kernel and orthoschemes

Let $M_n = (\min(p_i, p_j))_{1\le i,j\le n}$ be the prime–min Gram matrix. Write $\Delta_1 := p_1 = 2$ and $\Delta_i := p_i - p_{i-1}$ for $i \ge 2$. Then the inverse M_n^{-1} is the grounded Laplacian of a weighted path and is explicitly tridiagonal:

$$(M_n^{-1})_{ii} = \begin{cases} \frac{1}{\Delta_i} + \frac{1}{\Delta_{i+1}}, & i < n \\ \frac{1}{\Delta_n}, & i = n \end{cases} \tag{35}$$

and

$$(M_n^{-1})_{i,i+1} = (M_n^{-1})_{i+1,i} = -\frac{1}{\Delta_{i+1}}, \tag{36}$$

with all other entries equal to zero.

To compute the spherical volume Ω_ε for a sign pattern $\varepsilon \in \{\pm 1\}^n$, consider the normalized Gram matrix $R_{ij} = \frac{M_{ij}}{\sqrt{M_{ii}M_{jj}}}$. Let $C^{(\varepsilon)}$ be the inverse of $R^{(\varepsilon)}_{ij} = \varepsilon_i\varepsilon_j R_{ij}$. The scaling factors cancel, so the angles can be computed directly from M_n^{-1}:

$$\cos\left(\theta^{(\varepsilon)}_{ij}\right) = \frac{-C^{(\varepsilon)}_{ij}}{\sqrt{C^{(\varepsilon)}_{ii}C^{(\varepsilon)}_{jj}}} = \varepsilon_i\varepsilon_j \frac{-(M_n^{-1})_{ij}}{\sqrt{(M_n^{-1})_{ii}(M_n^{-1})_{jj}}}. \tag{37}$$

Because M_n^{-1} is tridiagonal,

$$\cos\left(\theta^{(\varepsilon)}_{ij}\right) = 0 \implies \theta^{(\varepsilon)}_{ij} = \frac{\pi}{2} \quad \text{for all } |i-j| > 1. \tag{38}$$

So every such simplex is a spherical orthoscheme.

Let $\alpha^{(\varepsilon)}_i := \theta^{(\varepsilon)}_{i,i+1}$ for $1 \le i \le n-1$. The base path angles α_i for the all-positive orthant satisfy

$$\cos(\alpha_i) = \frac{1/\Delta_{i+1}}{\sqrt{\left(\frac{1}{\Delta_i} + \frac{1}{\Delta_{i+1}}\right)\left(\frac{1}{\Delta_{i+1}} + \frac{1}{\Delta_{i+2}}\right)}} = \left[\left(1 + \frac{\Delta_{i+1}}{\Delta_i}\right)\left(1 + \frac{\Delta_{i+1}}{\Delta_{i+2}}\right)\right]^{-1/2} \tag{39}$$

for $1 \le i \le n-2$, while for $i = n-1$ one has

$$\cos(\alpha_{n-1}) = \frac{1/\Delta_n}{\sqrt{\left(\frac{1}{\Delta_{n-1}} + \frac{1}{\Delta_n}\right)\left(\frac{1}{\Delta_n}\right)}} = \left(1 + \frac{\Delta_n}{\Delta_{n-1}}\right)^{-1/2}. \tag{40}$$

For an arbitrary sign pattern,

$$\alpha^{(\varepsilon)}_i = \begin{cases} \alpha_i, & \text{if } \varepsilon_i = \varepsilon_{i+1}, \\ \pi - \alpha_i, & \text{if } \varepsilon_i \ne \varepsilon_{i+1}. \end{cases} \tag{41}$$

Hence

$$\Omega_\varepsilon = \mathrm{Orth}_{n-1}\left(\alpha^{(\varepsilon)}_1, \alpha^{(\varepsilon)}_2, \ldots, \alpha^{(\varepsilon)}_{n-1}\right). \tag{42}$$

For $n = 3$, this reduces to

$$\Omega_\varepsilon = \alpha^{(\varepsilon)}_1 + \alpha^{(\varepsilon)}_2 - \frac{\pi}{2}. \tag{43}$$

17.1 Explicit evaluation for the spherical tetrahedron ($n = 4$)

When the flat space dimension is $n = 4$, the intersection of the cone with the unit sphere forms a 3-dimensional spherical tetrahedron. As a spherical orthoscheme, it is determined by

$$A = \alpha_1^{(\varepsilon)}, \quad B = \alpha_2^{(\varepsilon)}, \quad C = \alpha_3^{(\varepsilon)}, \tag{44}$$

with the remaining three non-adjacent dihedral angles equal to $\pi/2$.

Its exact volume $\Omega_\varepsilon = \mathrm{Orth}_3(A, B, C)$ can be evaluated analytically via a Schläfli-type formula. By analytic continuation from the hyperbolic case, one may write it as

$$\begin{aligned} \Omega_\varepsilon = \frac{1}{4}\Big[&\Lambda_{\mathrm{s}}(A+\delta) - \Lambda_{\mathrm{s}}(A-\delta) + \Lambda_{\mathrm{s}}(C+\delta) - \Lambda_{\mathrm{s}}(C-\delta) \\ &- \Lambda_{\mathrm{s}}\left(\frac{\pi}{2} - B + \delta\right) + \Lambda_{\mathrm{s}}\left(\frac{\pi}{2} - B - \delta\right) + 2\Lambda_{\mathrm{s}}\left(\frac{\pi}{2} - \delta\right)\Big], \end{aligned} \tag{45}$$

where

$$\tan\delta = \frac{\sqrt{\cos^2 B - \sin^2 A \sin^2 C}}{\cos A \cos C}. \tag{46}$$

Because the simplex is an orthoscheme, the general formula simplifies substantially.

17.2 Parity reductions and the $n = 5$ case

For $n = 5$ flat dimensions, the resulting spherical orthoscheme is 4-dimensional. By the generalized Brianchon–Gram–Sommerville identity, the 4-volume may be expressed in terms of lower-dimensional boundary pieces. Thus the $n = 5$ volume Ω_ε can be reduced to a combination of path-angle data and lower-dimensional spherical volumes.

17.3 Summary of the arithmetic-geometric pipeline

The prime–min kernel gives a direct pipeline from consecutive prime gaps to spherical orthoscheme data:

1. **Arithmetic input:** the consecutive prime gaps $\Delta_1, \Delta_2, \ldots, \Delta_n$.
2. **Spectral geometry:** compute the path angles α_i from the tridiagonal inverse matrix.
3. **Sign reflection:** replace α_i by $\pi - \alpha_i$ whenever adjacent signs differ.
4. **Schläfli evaluation:** evaluate the corresponding orthoscheme volume.

This is a structural reformulation of the volume computation; it should not be confused with a new arithmetic theorem.

Proposition 17.1 (Odd-dimensional evaluation by DAO decomposition)**.** *Let Ω_ε be the spherical orthoscheme associated with the prime–min kernel*

$$R_{ij} = \sqrt{\frac{p_{\min(i,j)}}{p_{\max(i,j)}}}, \qquad 2 \le p_1 < \cdots < p_n,$$

with essential dihedral angles

$$\alpha_i^{(\varepsilon)} = \pi - \arccos\big(-\varepsilon_i \varepsilon_{i+1} \cos\alpha_i\big), \qquad 1 \le i \le n-1,$$

where the base cosines $\cos\alpha_i$ are explicitly determined by the prime gaps.

If the flat dimension n is even (so that the curved dimension $n-1$ is odd), then Ω_ε admits a finite alternating decomposition into doubly-asymptotic orthoschemes:

$$\Omega_\varepsilon = \sum_{\nu=1}^{N(n)} s_\nu \ \mathrm{DAO}_{n-1}(\alpha_0^{(\nu)}, \ldots, \alpha_{n-1}^{(\nu)}), \qquad s_\nu \in \{\pm 1\}.$$

Each DAO block satisfies the continued-fraction relation

$$\cos^2\alpha_0 = 1 - \cfrac{\cos^2\alpha_1}{1 - \cfrac{\cos^2\alpha_2}{1 - \cdots - \cfrac{\cos^2\alpha_{n-4}}{1-\cos^2\alpha_{n-3}}}},$$

together with its cyclic permutations.

For $n = 4$ (curved dimension 3),

$$\mathrm{DAO}_3(\alpha) = \frac{1}{2}\Lambda_2(\alpha), \qquad \Lambda_2(\alpha) = -\frac{1}{2}\int_0^\alpha \ln(4\sin^2 u)\, du.$$

For $n = 6$ (curved dimension 5), each DAO_5 *term is given by the Kellerhals formula*

$$32\ \mathrm{DAO}_5 = \sum c_j\, K(\lambda, b_j; x_j), \qquad K(a, b; x) = \int_{\pi/2}^{x} \Lambda_2(\arctan(\tan v/a) - b)\, dv,$$

with

$$\lambda = \cot\alpha_0 \tan\alpha_3 = \tan\alpha_1 \cot\alpha_4 = \cot\alpha_2 \tan\alpha_5.$$

Hence, for even n, the spherical volume Ω_ε reduces to a finite combination of Lobachevsky-type terms.

18 Numerical computations

18.1 Methodology: Gaussian orthant probabilities from the inverse Gram matrix

For the numerical evaluation of the spherical volumes Ω_ε, one may use the standard Gaussian orthant representation of spherical cone volumes.

Let

$$R_{ij} = \sqrt{\frac{p_{\min(i,j)}}{p_{\max(i,j)}}}, \qquad 1 \le i, j \le n,$$

be the normalized Gram matrix associated with the prime–min kernel, and let

$$C = R^{-1}.$$

For a sign pattern $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$, define the diagonal sign matrix

$$E_\varepsilon = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n).$$

Then the spherical cone Ω_ε is represented by the Gaussian orthant probability of the centered normal distribution with covariance matrix

$$\Sigma_\varepsilon = E_\varepsilon C E_\varepsilon.$$

Equivalently, if

$$X \sim \mathcal{N}(0, \Sigma_\varepsilon),$$

then

$$\Omega_\varepsilon = \mathrm{Surf}_n(1)\,\mathbb{P}(X_1 > 0, \ldots, X_n > 0), \qquad \mathrm{Surf}_n(1) = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{47}$$

This representation is numerically stable and avoids the branch-cut issues that can arise in direct polylogarithmic evaluation. For $n = 1$, the sphere S^0 consists of two points, and each sign cone contributes exactly one point. Hence

$$\Omega_{(+)}(1) = \Omega_{(-)}(1) = 1.$$

18.2 Results for the min-prime kernel

In the following table, we list the numerical evaluations for the kernel

$$K(p_i, p_j) = \min(p_i, p_j)$$

restricted to the first n primes, for the all-positive sign pattern

$$\varepsilon = (+, +, \ldots, +).$$

Thus $\Sigma_\varepsilon = C = R^{-1}$ in this case.

n	Primes $(p_1, \ldots, p_n)$	$\mathrm{Surf}_n(1)$	Orthant Prob. $\mathbb{P}(X > 0)$	Volume $\Omega_{(+\cdots+)}$
1	2	2.000000	0.5000000000	1.0000000000
2	$2, 3$	6.283185	0.0979544506	0.6154797087
3	$2, 3, 5$	12.566371	0.0179513677	0.2255921816
4	$2, 3, 5, 7$	19.739209	0.0024333789	0.0480332729
5	$2, \ldots, 11$	26.318945	0.0003537565	0.0093085108
6	$2, \ldots, 13$	31.006277	0.0000293537	0.0009101669
7	$2, \ldots, 17$	33.073362	0.0000024375	0.0000806051
8	$2, \ldots, 19$	32.469697	0.0000002204	0.0000071561
9	$2, \ldots, 23$	29.686580	0.0000000138	0.0000004087
10	$2, \ldots, 29$	25.501640	0.0000000011	0.0000000277

Table 1: Sphere surface areas, Gaussian orthant probabilities, and the resulting spherical cone volumes for the first n primes.

Table 2 shows the progression of the fundamental path angles α_i of the orthoscheme. For fixed i, the angle depends only on a local configuration of consecutive prime gaps and therefore stabilizes once the relevant primes are present.

19 Conclusion

The main content of this note is elementary but useful: a finite positive definite kernel sample gives a Gram matrix; a positive definite Gram matrix gives a basis; the associated sign cones partition space; and the induced spherical simplices partition the sphere. Consequently, one representative from each antipodal pair always has total volume

$$\frac{\pi^{n/2}}{\Gamma(n/2)}.$$

n	α_1	α_2	α_3	α_4	α_5	$\dots$
2	35.26°					
3	48.19°	54.74°				
4	48.19°	65.91°	45.00°			
5	48.19°	65.91°	54.74°	54.74°		
6	48.19°	65.91°	54.74°	70.53°	35.26°	
...	...	...	...	...	...	...
10	48.19°	65.91°	54.74°	70.53°	48.19°	$\dots,\ \alpha_9 = 50.77°$

Table 2: The angles α_i of the spherical orthoscheme in degrees.

The prime–min kernel provides a concrete family in which the normalized Gram matrices, orthoscheme data, and Gaussian orthant representations are all explicit. In low dimensions this yields closed formulas, and in higher dimensions it gives a stable computational framework.

The point is primarily organizational: it places a number of explicit identities into a common Gram-matrix and spherical-cone setting. Any stronger arithmetic interpretation would require additional input beyond the identities recorded here.

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