

Polar Coordinates, Belnap Logic, and Probability Distributions on FOUR

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Abstract

This note rewrites the original four-valued section in semantic rather than field-theoretic terms. A complex number in polar form determines a two-sample experiment, and the four elementary outcomes of that experiment naturally form the four values of Belnap's logic: neither, true only, false only, and both. The main point is that the construction yields two compatible objects at once: a four-valued logic in the sense of Belnap/Dunn and a probability distribution on those four values. In this way, geometry in the unit disk, Venn-style decomposition of the sample space, and four-valued semantics fit into one consistent picture.

1. A complex number as a two-sample experiment

Let

$$z = re^{i\phi}, \quad r = |z|, \quad \phi \in [0, 2\pi).$$

For the probabilistic construction we use the effective radius

$$\rho := \begin{cases} r, & 0 \leq r \leq 1, \\ r^{-1}, & r > 1. \end{cases}$$

Thus $\rho \in [0, 1]$. We also set

$$s := \frac{\phi}{2\pi} \in [0, 1).$$

Now choose two independent random variables

$$u_1, u_2 \sim \text{Unif}[0, 1].$$

This gives a two-sample experiment with the threshold events

$$A := \{u_1 \leq \rho\}, \quad B := \{u_2 \leq s\}.$$

Hence there are exactly four elementary outcomes:

$$A \cap B, \quad A \cap B^c, \quad A^c \cap B, \quad A^c \cap B^c.$$

If $r \leq 1$, the point z itself determines the experiment. If $r > 1$, we first invert z at the unit circle and then read the experiment from the inverted point.

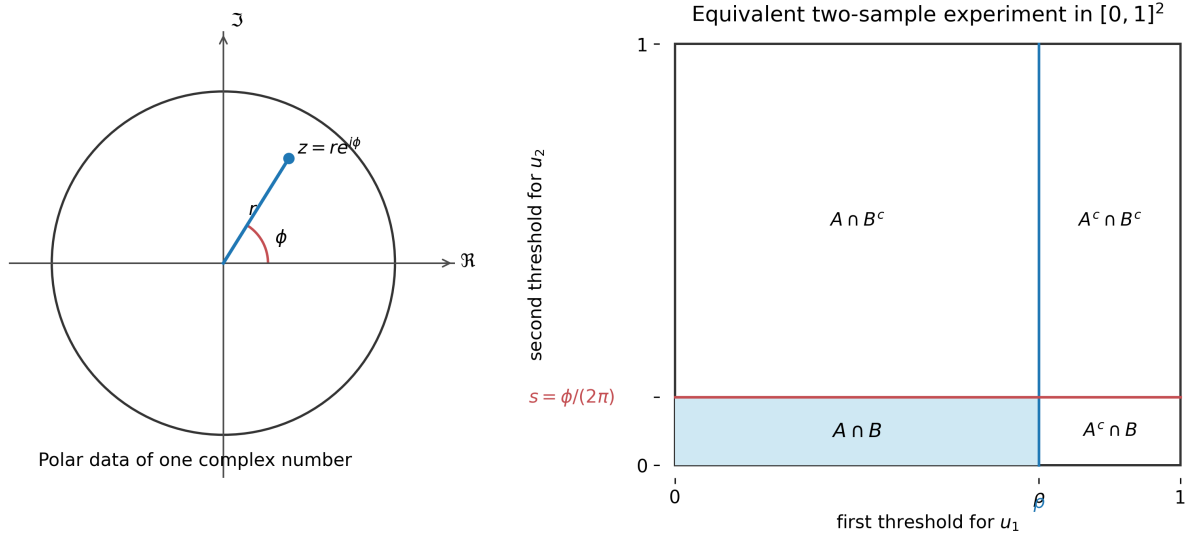


Figure 1: A complex number $z = re^{i\phi}$ determines two thresholds. The effective radius gives the first threshold ρ , and the normalized angle $s = \phi/(2\pi)$ gives the second. The shaded rectangle is the region $A \cap B$.

2. Venn decomposition and the four semantic states

The four regions of the Venn diagram can be read as four informational states of a proposition p :

Belnap value	region	interpretation
N	$A^c \cap B^c$	neither evidence for nor evidence against p
T	$A \cap B^c$	evidence for p , but not against p
F	$A^c \cap B$	evidence against p , but not for p
B	$A \cap B$	both evidence for and evidence against p

So the experiment does not merely produce four unlabeled atoms. It naturally produces the four epistemic values

$$\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\},$$

where N means “neither”, T means “true only”, F means “false only”, and B means “both”.

3. The induced logic is Belnap’s FOUR

The natural semantic encoding of one outcome ω is the indicator pair

$$v_p(\omega) := (\mathbf{1}_A(\omega), \mathbf{1}_B(\omega)) \in \{0, 1\}^2.$$

This produces exactly four possibilities:

$$(0, 0), \quad (1, 0), \quad (0, 1), \quad (1, 1).$$

We identify them as

$$\mathbf{N} := (0, 0), \quad \mathbf{T} := (1, 0), \quad \mathbf{F} := (0, 1), \quad \mathbf{B} := (1, 1).$$

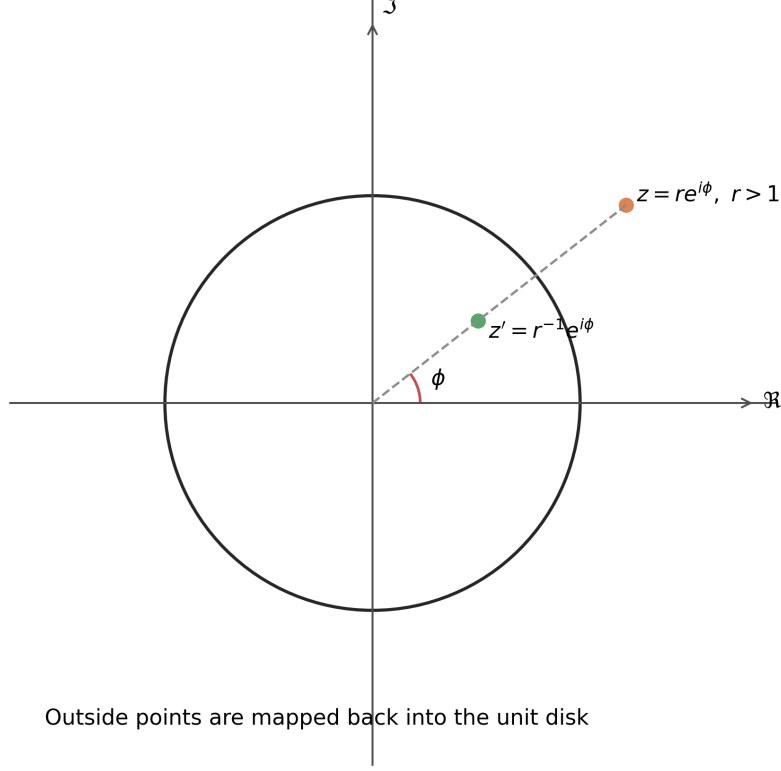


Figure 2: If $|z| > 1$, the point is mapped back to the unit disk by inversion on the same ray. The two-sample experiment is then read from $z' = r^{-1}e^{i\phi}$.

Theorem 1. *Let A be interpreted as evidence for a proposition p , and let B be interpreted as evidence against p . Then the four outcome types of the two-sample experiment form Belnap's four-valued semantics. More precisely, under the identification above, the operations*

$$\neg(x, y) := (y, x),$$

$$(x_1, y_1) \wedge (x_2, y_2) := (x_1 \wedge x_2, y_1 \vee y_2),$$

$$(x_1, y_1) \vee (x_2, y_2) := (x_1 \vee x_2, y_1 \wedge y_2)$$

are exactly the standard Belnap connectives on $\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}$.

Proof. The pair (x, y) records two independent pieces of information: whether there is positive support for p , and whether there is negative support against p . Negation swaps these two roles, so $\neg(x, y) = (y, x)$. For conjunction, the statement $p \wedge q$ has positive support only if both p and q have positive support, while negative support is already present if at least one conjunct has negative support. Hence

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2).$$

For disjunction, positive support is present if at least one disjunct has positive support, whereas negative support is present only when both disjuncts have negative support. Hence

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2).$$

These are precisely the usual Belnap/Dunn truth functions on the four values $\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}$. Therefore the induced four-valued logic is Belnap's FOUR. \square

In this semantics the designated values are $\{\mathbf{T}, \mathbf{B}\}$, since both carry positive support for the proposition.

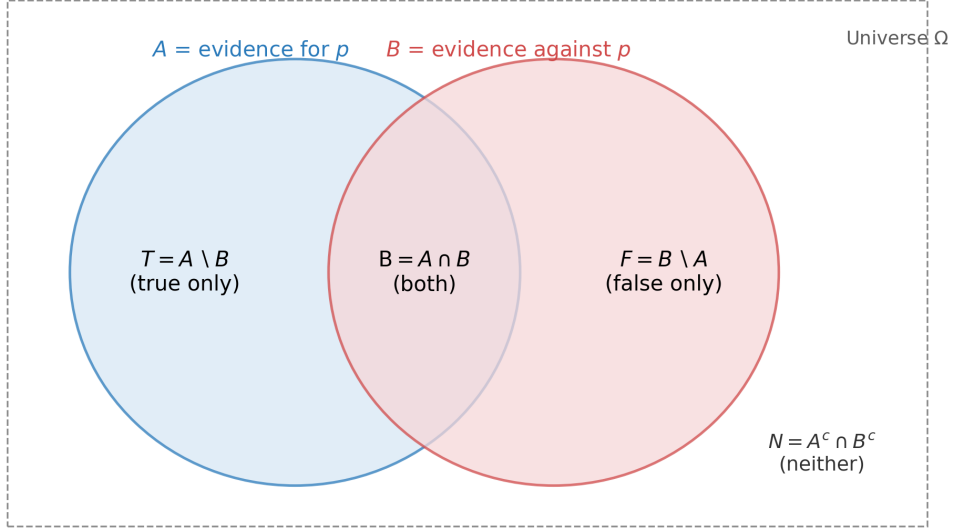


Figure 3: The four regions of the two-sample experiment match the four values of Belnap logic once A is read as evidence for p and B as evidence against p .

Truth tables

For completeness, here are the standard tables in the order N, T, F, B.

x	$\neg x$	\wedge	N	T	F	B
N	N	N	N	N	F	F
T	F	T	N	T	F	B
F	T	F	F	F	F	F
B	B	B	F	B	F	B

\vee	N	T	F	B
N	N	T	N	T
T	T	T	T	T
F	N	T	F	B
B	T	T	B	B

4. Probability distribution on Belnap values

The same experiment also carries a probability distribution on the four values. Since u_1 and u_2 are independent and uniform, we obtain

$$\begin{aligned} \mathbb{P}(\text{B}) &= \mathbb{P}(A \cap B) = \rho s, \\ \mathbb{P}(\text{T}) &= \mathbb{P}(A \cap B^c) = \rho(1 - s), \\ \mathbb{P}(\text{F}) &= \mathbb{P}(A^c \cap B) = (1 - \rho)s, \\ \mathbb{P}(\text{N}) &= \mathbb{P}(A^c \cap B^c) = (1 - \rho)(1 - s). \end{aligned}$$

Therefore the two-sample experiment defines a probability measure on $\{\text{N}, \text{T}, \text{F}, \text{B}\}$:

$$\mu_z : \{\text{N}, \text{T}, \text{F}, \text{B}\} \rightarrow [0, 1],$$

with

$$\mu_z(\text{N}) + \mu_z(\text{T}) + \mu_z(\text{F}) + \mu_z(\text{B}) = 1.$$

Proposition 1. Every complex number $z \in \mathbb{C}$ determines a probability distribution on Belnap's four values by the rule above, using inversion at the unit circle when $|z| > 1$.

Proof. The construction reduces every complex number to a point in or on the unit disk with effective radius $\rho \in [0, 1]$ and angle parameter $s \in [0, 1)$. The four probabilities listed above are nonnegative and sum to

$$\rho s + \rho(1 - s) + (1 - \rho)s + (1 - \rho)(1 - s) = 1.$$

Hence they define a probability distribution on the four Belnap values. □

So the paper should be read in two layers:

- the logical layer: the four outcome types form Belnap's FOUR;
- the probabilistic layer: the complex number z determines a measure on those four values.

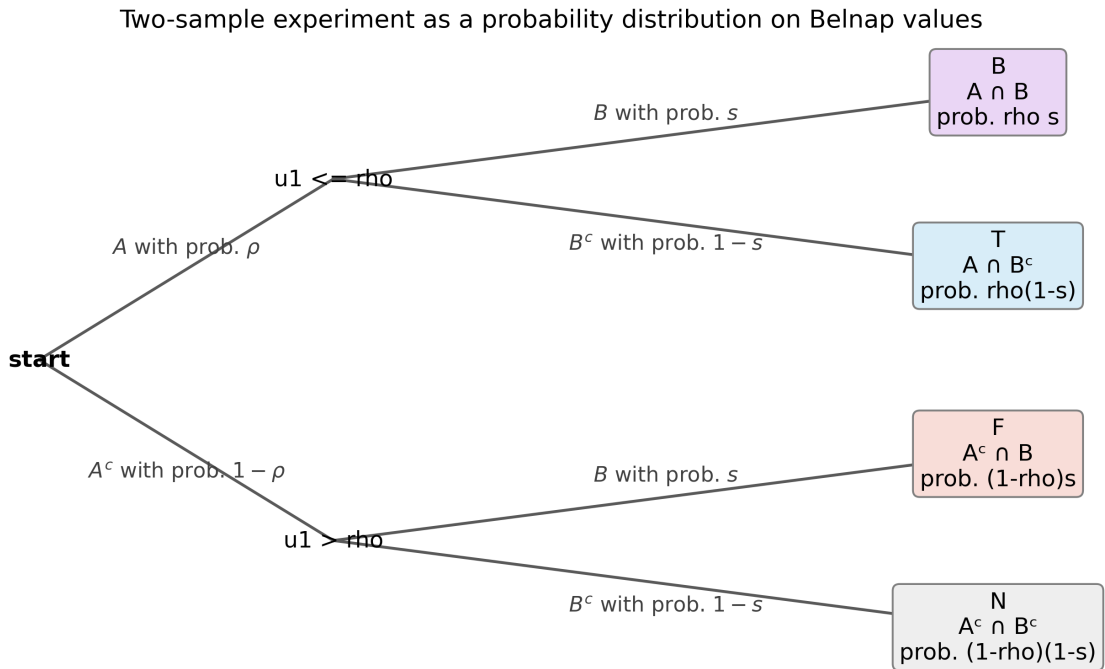


Figure 4: A visual tree of the two-sample experiment. The leaves are the Belnap values B, T, F, N, together with their probabilities.

5. Relation back to complex numbers

Conversely, if one starts with two thresholds

$$\rho \in [0, 1], \quad s \in [0, 1),$$

then one can reconstruct a point in polar form by setting

$$z = \rho e^{2\pi i s}.$$

Thus the unit disk packages the two parameters of the experiment into one complex number. The radial part controls the first threshold, and the angular part controls the second threshold. In this sense, geometry in polar coordinates provides a compact parametrization of probability distributions on $\{N, T, F, B\}$.

Remark 1. When $s = 0$, the second threshold disappears and the construction degenerates to an ordinary Bernoulli split with parameter ρ . The full four-valued structure becomes visible only when both coordinates are active.

6. Polynomial roots as families of Belnap distributions

Let

$$P(x) = a_n \prod_{k=1}^n (x - z_k)$$

be a complex polynomial with roots z_1, \dots, z_n . Each root $z_k = r_k e^{i\phi_k}$ induces its own effective radius ρ_k and angular threshold $s_k = \phi_k / (2\pi)$. Therefore each root determines its own distribution

$$\mu_{z_k} : \{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\} \rightarrow [0, 1].$$

So a polynomial naturally gives rise to a family of four-valued probabilistic experiments, one for each root. This point of view links algebra, through the roots of the polynomial, geometry, through polar coordinates and inversion, and logic plus probability, through Belnap values equipped with the measures μ_{z_k} .

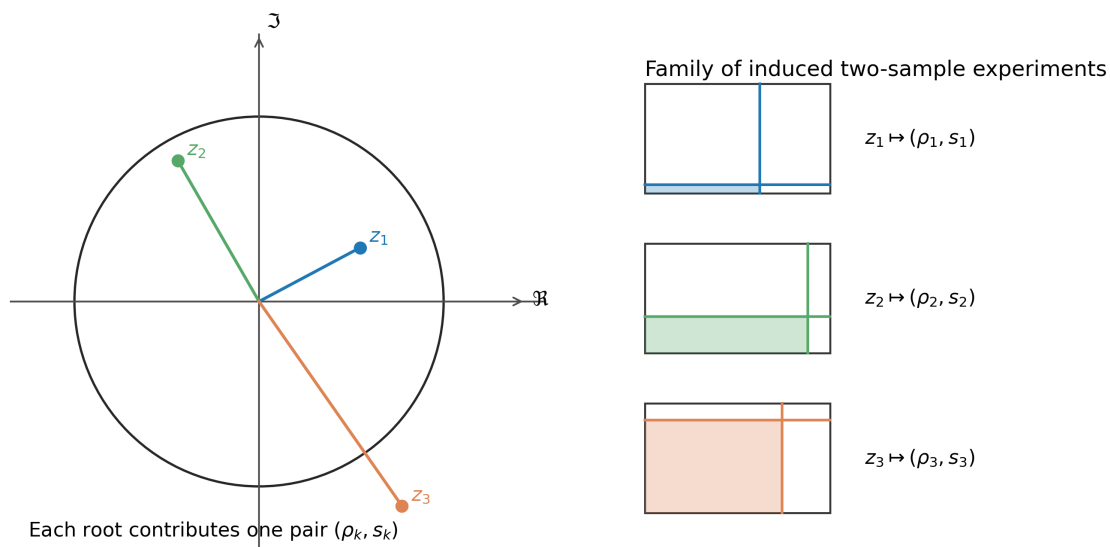


Figure 5: Each root of a polynomial contributes one pair of thresholds and therefore one probability distribution on the four Belnap values.

7. Conclusion

The main correction in this rewritten note is conceptual. The four-valued structure generated by the two-sample experiment is best understood as Belnap logic, not as a logic derived from the field structure of four elements. The experiment produces four semantic states $\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}$, and these states carry the standard Belnap connectives. At the same time, every complex number determines a probability distribution on those four states. This yields a coherent picture:

complex number in polar form \longrightarrow two-sample experiment \longrightarrow Belnap value plus probability.

In this formulation, the geometry is simple, the logic is standard, and the probabilistic interpretation is explicit.