Ehrhart Polynomials and Pratt Exponents

Lattice theoretic and analytic investigations

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Abstract

This note revisits the Ehrhart-theoretic viewpoint of *Erhart polynomials and prime numbers* after replacing the ordinary valuation vector $(v_p(n))_p$ by the Pratt valuation vector $(m_p(n))_p$ from *Sequence of Graphs and the Riemann Hypothesis: From the MathOverflow Valuation Graph to the Pratt Cover Graph.*

We develop two complementary constructions. First, we transport the classical prime simplex through the finite Pratt matrix; since this matrix is unimodular, the corresponding Ehrhart polynomial is unchanged. Second, we introduce the genuine Pratt simplex defined by the cutoff

$$\Omega_{\mathrm{P}}(n) := \sum_p m_p(n) \le r,$$

and derive its associated lattice-point and generating-series theory.

A key new feature of the Pratt setting is the existence of a single global positive weight vector

$$w = \sum_{p \in \mathbb{P}} -\log\left(1 - \frac{1}{p}\right) e_p$$

such that

$$n = \exp\big(\langle \Phi_{\mathrm{P}}(n), w \rangle\big) \qquad (n \in \mathbb{N}).$$

In particular, $\Omega_{\mathrm{P}}(n) \le r$ implies $n \le 2^r$, so a Pratt-height cutoff automatically yields an ordinary size cutoff.

In a second part, we investigate the associated prime-counting function

$$\Pi_{\mathrm{P}}(r) := \#\{p \in \mathbb{P} : \Omega_{\mathrm{P}}(p) \le r\},$$

prove elementary lower bounds, introduce the one- and two-variable Pratt zeta functions, and formulate heuristic links with explicit-formula methods and a Pratt analogue of the Riemann-hypothesis philosophy.

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1 Introduction

The starting point of the paper *Erhart polynomials and prime numbers* is the classical exponent vector

$$v(n) = (v_p(n))_{p\in\mathbb{P}},$$

which records the prime factorization of an integer n. In finite prime dimension this turns multiplicative questions into lattice questions: if only primes $\leq N$ are allowed, then the vectors $v(n)$ form the lattice points of the positive orthant $\mathbb{Z}_{\geq 0}^{\pi(N)}$, and the condition

$$\Omega(n) = \sum_p v_p(n) \leq t$$

becomes the condition of lying in a dilated standard simplex. In this way, ordinary divisor theory acquires an Ehrhart-theoretic interpretation. At the same time, the parity of $\Omega(n)$ governs the Liouville function, so the same lattice picture also interacts with classical multiplicative functions.

The present paper asks what remains true if one replaces the ordinary valuation vector by the Pratt valuation vector

$$\Phi_{\mathrm{P}}(n) = (m_p(n))_{p\in\mathbb{P}},$$

introduced in *Sequence of Graphs and the Riemann Hypothesis: From the MathOverflow Valuation Graph to the Pratt Cover Graph.* Here $m_p(n)$ counts the multiplicity with which the prime p occurs in the Pratt forest of n. The key result from that paper is that Φ_{P} is still additive:

$$\Phi_{\mathrm{P}}(mn) = \Phi_{\mathrm{P}}(m) + \Phi_{\mathrm{P}}(n),$$

and moreover there exists a lower unitriangular matrix A such that

$$\Phi_{\mathrm{P}}(n) = A\,v(n).$$

This already suggests that an Ehrhart-theoretic framework should persist after passing from ordinary prime exponents to Pratt valuations.

However, the Pratt setting is not merely a change of coordinates. It has a structural feature absent from the ordinary valuation model: there exists a single global positive weight vector

$$w = \sum_{p\in\mathbb{P}} -\log\left(1-\frac{1}{p}\right) e_p$$

such that

$$\log n = \langle \Phi_{\mathrm{P}}(n), w\rangle \qquad \text{and hence} \qquad n = \exp(\langle \Phi_{\mathrm{P}}(n), w\rangle).$$

As a consequence, a bound on the total Pratt height

$$\Omega_{\mathrm{P}}(n) := \sum_p m_p(n)$$

immediately yields a numerical bound on n, namely

$$\Omega_{\mathrm{P}}(n) \le r \implies n \le 2^r.$$

This is a decisive difference from the ordinary setting: the condition $\Omega(n) \le r$ alone does not control the size of n, since every prime p satisfies $\Omega(p) = 1$. In contrast, the Pratt height behaves much more like a genuine radial parameter in the infinite prime lattice.

The aim of this note is therefore twofold. First, we develop the geometric and polyhedral theory obtained by replacing $v_p(n)$ with $m_p(n)$. Second, we investigate whether this added rigidity can be exploited analytically, especially in connection with counting primes.

More precisely, the paper has the following goals.

1. We restate the necessary material on Pratt forests, Pratt valuations, and the finite lower unitriangular Pratt matrix in a self-contained form.

2. We construct the corresponding lattice and polyhedral framework and distinguish carefully between two different objects: the transported classical simplex and the genuine Pratt simplex cut out by the total Pratt height.

3. We show that the transported simplex is Ehrhart-equivalent to the classical one, while the genuine Pratt cutoff leads naturally to a rational generating-series theory.

4. We study the associated counting function

$$\Pi_{\mathrm{P}}(r) := \#\{p \in \mathbb{P} : \Omega_{\mathrm{P}}(p) \le r\},$$

   prove elementary lower bounds, and explain how the one- and two-variable Pratt zeta functions arise as natural analytic companions of the geometry.

5. Finally, we formulate a heuristic bridge between the two-variable Pratt zeta function, explicit-formula methods, and a possible Pratt analogue of the Riemann-hypothesis philosophy.

The paper is therefore divided into two complementary parts. The first part is geometric: it develops the Ehrhart and lattice-point theory of Pratt coordinates. The second part is analytic: it studies the counting function $\Pi_{\mathrm{P}}(r)$, introduces the corresponding zeta functions, and separates carefully between what can be proved rigorously, what is supported numerically, and what remains conjectural.

2 Ordinary exponent vectors and the standard prime simplex

We briefly recall the part of the first source that we shall generalize.

Definition 2.1 (Ordinary exponent vector)**.** *For $n \in \mathbb{N}$ define*

$$v(n) := (v_p(n))_{p\in\mathcal{P}} \in \bigoplus_{p\in\mathcal{P}} \mathbb{Z}e_p,$$

where $v_p(n)$ is the exponent of p in the prime factorization of n. We also write

$$\Omega(n) := \sum_{p\in\mathcal{P}} v_p(n).$$

Fix $N \in \mathbb{N}$, and list the primes up to N as

$$p_1 < p_2 < \cdots < p_d, \qquad d = \pi(N).$$

The standard prime simplex is

$$Q_N := \operatorname{conv}\{0, e_{p_1}, \dots, e_{p_d}\} \subset \mathbb{R}^d.$$

Proposition 2.2 (Classical Ehrhart polynomial)**.** *For every $t \in \mathbb{N}_0$,*

$$L(Q_N, t) := \#(tQ_N \cap \mathbb{Z}^d) = \binom{d+t}{d}.$$

Proof. By definition,

$$tQ_N = \Big\{(x_1, \dots, x_d) \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq t\Big\}.$$

Therefore $tQ_N \cap \mathbb{Z}^d$ is exactly the set of all d-tuples of nonnegative integers satisfying

$$x_1 + \cdots + x_d \leq t.$$

To count these tuples, introduce an auxiliary nonnegative integer

$$x_{d+1} := t - (x_1 + \cdots + x_d).$$

Then the condition $x_1 + \cdots + x_d \leq t$ becomes the equation

$$x_1 + \cdots + x_d + x_{d+1} = t,$$

with all $d+1$ variables nonnegative integers. The standard stars-and-bars count for weak compositions of t into $d+1$ parts gives

$$\#\{(x_1, \dots, x_{d+1}) \in \mathbb{Z}^{d+1}_{\geq 0} : x_1 + \cdots + x_{d+1} = t\} = \binom{d+t}{d}.$$

The correspondence between $(x_1, \dots, x_d)$ and $(x_1, \dots, x_d, x_{d+1})$ is bijective, so the same number counts $tQ_N \cap \mathbb{Z}^d$. $\square$

Proposition 2.3 (Arithmetic interpretation)**.** *Let*

$$B_{N,t} := \{n \in \mathbb{N} : \textit{every prime divisor of } n \textit{ is } \leq N,\ \Omega(n) \leq t\}.$$

Then

$$|B_{N,t}| = \binom{d+t}{d}.$$

Proof. The map

$$(x_1, \dots, x_d) \longmapsto n := p_1^{x_1} \cdots p_d^{x_d}$$

is a bijection from $\mathbb{Z}_{\geq 0}^d$ onto the set of integers all of whose prime divisors are among $p_1, \dots, p_d$. Under this bijection,

$$\Omega(n) = x_1 + \cdots + x_d.$$

Hence the subset $B_{N,t}$ corresponds exactly to the lattice points of tQ_N. The previous proposition gives the cardinality. □

This is the model that we now transport to the Pratt setting.

3 Pratt trees, Pratt valuations, and the matrix A

We now recall the material from the second source paper that will be used throughout.

Definition 3.1 (Pratt tree of a prime)**.** *For a prime q, the Pratt tree T_q is defined recursively as follows.*

- *T_2 consists of one single vertex labeled 2.*
- *If $q > 2$ is prime, then the root is labeled q, and for each prime $r \mid (q-1)$ one attaches exactly $v_r(q-1)$ children labeled r, each child carrying a copy of T_r.*

Definition 3.2 (Pratt forest of an integer)**.** *If*

$$n = \prod_q q^{v_q(n)},$$

then the Pratt forest of n is the disjoint union of $v_q(n)$ copies of T_q for all primes q.

Definition 3.3 (Pratt exponents)**.** *For a prime p and an integer $n \geq 1$, let $m_p(n)$ be the number of vertices labeled p in the Pratt forest of n. The full Pratt valuation vector is*

$$\Phi_{\mathrm{P}}(n) := (m_p(n))_{p \in \mathcal{P}}.$$

We also define the total Pratt weight

$$\Omega_{\mathrm{P}}(n) := \sum_{p \in \mathcal{P}} m_p(n).$$

Proposition 3.4 (Additivity of Pratt valuations)**.** *For all positive integers x, y,*

$$\Phi_{\mathrm{P}}(xy) = \Phi_{\mathrm{P}}(x) + \Phi_{\mathrm{P}}(y).$$

Equivalently, for each prime p,

$$m_p(xy) = m_p(x) + m_p(y).$$

Consequently,

$$\Omega_{\mathrm{P}}(xy) = \Omega_{\mathrm{P}}(x) + \Omega_{\mathrm{P}}(y).$$

Proof. Write

$$x = \prod_q q^{\alpha_q}, \qquad y = \prod_q q^{\beta_q}.$$

Then

$$xy = \prod_q q^{\alpha_q + \beta_q}.$$

By definition, the Pratt forest of x is the disjoint union of α_q copies of T_q, and the Pratt forest of y is the disjoint union of β_q copies of T_q. Therefore the Pratt forest of xy is the disjoint union of $\alpha_q + \beta_q$ copies of T_q for each q. In other words, it is obtained by taking the disjoint union of the Pratt forest of x and the Pratt forest of y.

Now fix a prime p. The number of vertices labeled p in a disjoint union of finite labeled graphs is the sum of the numbers of vertices labeled p in the individual components. Hence

$$m_p(xy) = m_p(x) + m_p(y).$$

Since this holds for every prime p, the vector identity follows. Summing over p gives the statement for Ω_{P}. $\square$

Definition 3.5 (The Pratt matrix)**.** *Let $p_1 < p_2 < \dots$ be the ordered primes. Define the infinite matrix $A = (a_{ij})_{i,j \geq 1}$ by*

$$a_{ij} := m_{p_i}(p_j).$$

Thus the j-th column of A is the Pratt valuation vector of the prime p_j.

Proposition 3.6 (Relation between ordinary and Pratt coordinates)**.** *For every $n \in \mathbb{N}$,*

$$\Phi_{\mathrm{P}}(n) = Av(n).$$

Moreover, A is lower triangular with all diagonal entries equal to 1*.*

Proof. Write

$$n = \prod_{j \geq 1} p_j^{v_{p_j}(n)}.$$

Only finitely many exponents are nonzero. By additivity of Φ_{P},

$$\Phi_{\mathrm{P}}(n) = \sum_{j \geq 1} v_{p_j}(n)\, \Phi_{\mathrm{P}}(p_j).$$

By definition, $\Phi_{\mathrm{P}}(p_j)$ is the j-th column of A. Therefore the displayed formula is exactly the matrix identity

$$\Phi_{\mathrm{P}}(n) = Av(n).$$

We now prove triangularity. Fix j, and consider the Pratt tree T_{p_j}. The root is labeled p_j. Every child label comes from a prime divisor of $p_j - 1$, hence from a prime strictly smaller than p_j. Applying the same argument recursively to the descendants, one sees that no label larger than p_j can appear anywhere in the tree. Therefore

$$m_{p_i}(p_j) = 0 \qquad \text{for } i > j.$$

This means exactly that A is lower triangular.

Finally, the root of T_{p_j} contributes one occurrence of the label p_j. No other vertex in T_{p_j} can have label p_j, because all descendants are labeled by smaller primes. Hence

$$m_{p_j}(p_j) = 1.$$

Thus every diagonal entry of A is equal to 1. $\square$

Corollary 3.7 (Finite Pratt blocks are unimodular). *Fix $N \in \mathbb{N}$ and let $p_1 < \cdots < p_d$ be the primes $\leq N$. Let A_N be the $d \times d$ northwest block of A. Then $A_N \in \mathrm{GL}_d(\mathbb{Z})$ and $\det(A_N) = 1$.*

Proof. The matrix A_N is lower triangular, and each diagonal entry is 1. The determinant of any triangular matrix is the product of its diagonal entries. Hence

$$\det(A_N) = 1 \cdot 1 \cdots 1 = 1.$$

Therefore A_N is invertible over $\mathbb{Q}$. Because A_N is an integer matrix with determinant 1, its inverse is again an integer matrix. Equivalently, $A_N \in \mathrm{GL}_d(\mathbb{Z})$. □

4 The transported Pratt simplex and its unchanged Ehrhart polynomial

The first and easiest way to build a Pratt version of the ordinary simplex is to transport the ordinary exponent lattice through the finite matrix A_N. This produces a new simplex in Pratt coordinates, but because the change of variables is unimodular, the Ehrhart polynomial stays exactly the same.

Fix again $N \in \mathbb{N}$, let $p_1 < \cdots < p_d$ be the primes $\leq N$, and let A_N be as above. Write

$$u_j := A_N e_j = \Phi_{\mathrm{P}}(p_j) \in \mathbb{Z}^d \qquad (1 \leq j \leq d).$$

Define the transported Pratt simplex by

$$\widetilde{Q}_N := \mathrm{conv}\{0, u_1, \ldots, u_d\} \subset \mathbb{R}^d.$$

Theorem 4.1 (Unimodular transport). *For every $t \in \mathbb{N}_0$,*

$$L(\widetilde{Q}_N, t) = L(Q_N, t) = \binom{d+t}{d}.$$

Proof. Because $A_N \in \mathrm{GL}_d(\mathbb{Z})$, the map

$$T_N : \mathbb{R}^d \to \mathbb{R}^d, \qquad T_N(x) = A_N x,$$

is a bijective linear map with integer matrix and integer inverse. Therefore it induces a bijection of the integer lattice $\mathbb{Z}^d$ onto itself. Indeed, if $x \in \mathbb{Z}^d$, then $A_N x \in \mathbb{Z}^d$ because A_N has integer entries; conversely, if $y \in \mathbb{Z}^d$, then $x = A_N^{-1} y \in \mathbb{Z}^d$ because A_N^{-1} is again an integer matrix. Thus

$$T_N(\mathbb{Z}^d) = \mathbb{Z}^d.$$

Now T_N sends the vertices of Q_N to the vertices of $\widetilde{Q}_N$:

$$T_N(0) = 0, \qquad T_N(e_j) = A_N e_j = u_j.$$

Since linear maps commute with convex hulls,

$$T_N(Q_N) = \widetilde{Q}_N.$$

Because T_N is linear, it also commutes with dilations:

$$T_N(tQ_N) = tT_N(Q_N) = t\widetilde{Q}_N.$$

Therefore

$$T_N(tQ_N \cap \mathbb{Z}^d) = t\widetilde{Q}_N \cap \mathbb{Z}^d,$$

and this map is bijective because T_N is a lattice automorphism. Taking cardinalities yields

$$L(\widetilde{Q}_N, t) = L(Q_N, t).$$

The classical formula for $L(Q_N, t)$ then gives

$$L(\widetilde{Q}_N, t) = \binom{d+t}{d}.$$

$\square$

Remark 4.2. *This theorem is conceptually important but arithmetically modest. It says that if one merely rewrites the ordinary exponent lattice in Pratt coordinates using the unitriangular matrix A_N, then Ehrhart theory does not change. The real novelty begins only when one cuts by the genuine Pratt quantity Ω_{P}, not by the transported copy of Ω.*

5 The genuine Pratt-weight simplex

We now turn to the construction that really replaces $v_p(n)$ by $m_p(n)$.

5.1 Prime weights

Definition 5.1 (Pratt weight of a prime)**.** *For each prime p_j, define*

$$c_j := \Omega_{\mathrm{P}}(p_j) = \sum_i m_{p_i}(p_j).$$

Equivalently, c_j is the total number of vertices in the Pratt tree of p_j.

Proposition 5.2 (Linear formula for Ω_{P} in ordinary coordinates)**.** *For every integer*

$$n = \prod_{j=1}^{d} p_j^{x_j}$$

with all prime factors at most N, one has

$$\Omega_{\mathrm{P}}(n) = \sum_{j=1}^{d} c_j x_j.$$

Proof. By additivity of Ω_{P},

$$\Omega_{\mathrm{P}}(n) = \Omega_{\mathrm{P}}\left(\prod_{j=1}^{d} p_j^{x_j}\right) = \sum_{j=1}^{d} x_j\, \Omega_{\mathrm{P}}(p_j) = \sum_{j=1}^{d} c_j x_j.$$

No further argument is needed. $\square$

Definition 5.3 (The Pratt simplex)**.** *For fixed N, with primes $p_1, \ldots, p_d \le N$ and corresponding weights $c_1, \ldots, c_d$, define*

$$P_N := \Big\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d_{\ge 0} : c_1 x_1 + \cdots + c_d x_d \le 1\Big\}.$$

Equivalently,

$$P_N = \operatorname{conv}\Big\{0, \frac{1}{c_1} e_1, \ldots, \frac{1}{c_d} e_d\Big\}.$$

Since the coefficients c_j are positive integers, this is a rational simplex. The dilation tP_N is

$$tP_N = \Big\{x \in \mathbb{R}^d_{\geq 0} : c_1 x_1 + \cdots + c_d x_d \leq t\Big\}.$$

Theorem 5.4 (Arithmetic meaning of the Pratt simplex)**.** *For every $t \in \mathbb{N}_0$,*

$$tP_N \cap \mathbb{Z}^d$$

corresponds bijectively to the integers n all of whose prime factors are at most N and satisfying $\Omega_{\mathrm{P}}(n) \leq t$. More precisely, the map

$$(x_1, \dots, x_d) \longmapsto n = p_1^{x_1} \cdots p_d^{x_d}$$

is a bijection from $tP_N \cap \mathbb{Z}^d$ onto

$$C_{N,t} := \{n \in \mathbb{N} : \textit{every prime divisor of } n \textit{ is } \leq N,\ \Omega_{\mathrm{P}}(n) \leq t\}.$$

Hence

$$L(P_N, t) = |C_{N,t}|.$$

Proof. As in the ordinary case, unique factorization shows that the map

$$(x_1, \dots, x_d) \mapsto p_1^{x_1} \cdots p_d^{x_d}$$

is a bijection from $\mathbb{Z}^d_{\geq 0}$ onto the integers all of whose prime factors are among $p_1, \dots, p_d$. It remains to understand the inequality defining tP_N.

Let

$$n = p_1^{x_1} \cdots p_d^{x_d}.$$

By the preceding proposition,

$$\Omega_{\mathrm{P}}(n) = \sum_{j=1}^{d} c_j x_j.$$

Therefore

$$(x_1, \dots, x_d) \in tP_N \iff x_j \geq 0 \text{ for all } j \text{ and } \sum_{j=1}^{d} c_j x_j \leq t \iff \Omega_{\mathrm{P}}(n) \leq t.$$

Thus the lattice points of tP_N are exactly the exponent vectors of the integers in $C_{N,t}$. Taking cardinalities proves the statement. □

Corollary 5.5 (Boundary layers)**.** *For*

$$A_{N,t} := \{n \in \mathbb{N} : \textit{all prime factors of } n \textit{ are } \leq N,\ \Omega_{\mathrm{P}}(n) = t\},$$

one has

$$|A_{N,t}| = L(P_N, t) - L(P_N, t-1) \qquad (t \geq 1),$$

with $|A_{N,0}| = 1$.

Proof. By the theorem, $L(P_N, t)$ counts the integers with $\Omega_{\mathrm{P}}(n) \leq t$, whereas $L(P_N, t-1)$ counts those with $\Omega_{\mathrm{P}}(n) \leq t-1$. Subtracting the two counts leaves exactly the integers with $\Omega_{\mathrm{P}}(n) = t$. For $t = 0$, the only possibility is $n = 1$. □

5.2 Ehrhart series and layer generating functions

The rational simplex P_N does not usually have an ordinary polynomial Ehrhart function; in general one obtains a quasipolynomial. For our purposes the generating function is the cleanest object.

Theorem 5.6 (Exact generating function for Pratt layers)**.** *Define*

$$a_{N,t} := |A_{N,t}| = \#\{n \in \mathbb{N} : \textit{all prime factors of } n \textit{ are } \le N,\ \Omega_{\mathrm{P}}(n) = t\}.$$

Then the ordinary generating function is

$$\sum_{t \ge 0} a_{N,t} z^t = \prod_{j=1}^{d} \frac{1}{1 - z^{c_j}}.$$

Consequently,

$$\sum_{t \ge 0} L(P_N, t) z^t = \frac{1}{1 - z} \prod_{j=1}^{d} \frac{1}{1 - z^{c_j}}.$$

Proof. Every integer all of whose prime factors are at most N has a unique factorization

$$n = p_1^{x_1} \cdots p_d^{x_d} \qquad (x_j \in \mathbb{Z}_{\ge 0}).$$

Its total Pratt weight is

$$\Omega_{\mathrm{P}}(n) = c_1 x_1 + \cdots + c_d x_d.$$

Therefore the formal series that counts such integers by their Pratt weight is

$$\sum_{x_1, \ldots, x_d \ge 0} z^{c_1 x_1 + \cdots + c_d x_d}.$$

Because the exponent is a sum of terms depending on separate variables, the multiple series factors as a product:

$$\sum_{x_1, \ldots, x_d \ge 0} z^{c_1 x_1 + \cdots + c_d x_d} = \prod_{j=1}^{d} \left(\sum_{x_j \ge 0} z^{c_j x_j} \right).$$

Each inner sum is a geometric series,

$$\sum_{x_j \ge 0} z^{c_j x_j} = 1 + z^{c_j} + z^{2c_j} + \cdots = \frac{1}{1 - z^{c_j}}.$$

Hence

$$\sum_{x_1, \ldots, x_d \ge 0} z^{c_1 x_1 + \cdots + c_d x_d} = \prod_{j=1}^{d} \frac{1}{1 - z^{c_j}}.$$

By construction, the coefficient of z^t on the left counts exactly the exponent vectors, and hence exactly the integers, with total Pratt weight t. This proves the first formula.

For the cumulative counting function, note that

$$L(P_N, t) = \sum_{k=0}^{t} a_{N,k}.$$

Multiplying by z^t and summing over $t \ge 0$, we obtain

$$\sum_{t \ge 0} L(P_N, t) z^t = \sum_{t \ge 0} \sum_{k=0}^{t} a_{N,k} z^t.$$

Interchanging the order of summation,

$$\sum_{t\ge 0}\sum_{k=0}^{t} a_{N,k} z^t = \sum_{k\ge 0} a_{N,k} \sum_{t\ge k} z^t = \sum_{k\ge 0} a_{N,k} \frac{z^k}{1-z} = \frac{1}{1-z}\sum_{k\ge 0} a_{N,k} z^k.$$

Inserting the first formula gives the second. □

Remark 5.7. *The poles of this generating series occur at roots of unity whose orders divide one of the integers c_j. This is the standard reason the counting function of a rational simplex is a quasipolynomial rather than, in general, an ordinary polynomial.*

5.3 The Pratt-Liouville sign

The second source considers the sign $(-1)^{\Omega_{\mathrm{P}}(n)}$. Inside our simplex this sign is again encoded by a generating function.

Proposition 5.8 (Signed generating function)**.** *Define*

$$b_{N,t} := \sum_{\substack{n\ge 1\\ p\mid n\Rightarrow p\le N\\ \Omega_{\mathrm{P}}(n)=t}} (-1)^{\Omega_{\mathrm{P}}(n)}.$$

Then

$$b_{N,t} = (-1)^t a_{N,t},$$

and therefore

$$\sum_{t\ge 0} b_{N,t} z^t = \prod_{j=1}^{d} \frac{1}{1-(-1)^{c_j} z^{c_j}}.$$

Proof. On the set $A_{N,t}$, the total Pratt weight is identically equal to t. Hence every summand contributes the same sign $(-1)^t$, and so

$$b_{N,t} = (-1)^t |A_{N,t}| = (-1)^t a_{N,t}.$$

Summing over t gives

$$\sum_{t\ge 0} b_{N,t} z^t = \sum_{t\ge 0} a_{N,t} (-z)^t.$$

Now substitute $-z$ into the generating function of the previous theorem:

$$\sum_{t\ge 0} a_{N,t}(-z)^t = \prod_{j=1}^{d} \frac{1}{1-(-z)^{c_j}} = \prod_{j=1}^{d} \frac{1}{1-(-1)^{c_j} z^{c_j}}.$$

This is the claimed formula. □

6 A Pratt lattice and the role of the matrix A_N

The ordinary exponent vectors with support inside $\{p_1,\dots,p_d\}$ form the lattice $\mathbb{Z}^d$. Applying the unimodular matrix A_N produces another full lattice in $\mathbb{R}^d$, namely

$$\Lambda_N^{\mathrm{P}} := A_N \mathbb{Z}^d.$$

But because A_N is unimodular, this is not a new lattice in the abstract sense: it is again isomorphic to $\mathbb{Z}^d$. What is new is the interpretation of its distinguished vectors.

Proposition 6.1. *The set of Pratt vectors of integers supported on primes $\le N$ is exactly the affine semigroup generated by the columns of A_N:*

$$\{\Phi_{\mathrm{P}}(n) : p \mid n \Rightarrow p \le N\} = \Big\{\sum_{j=1}^{d} x_j u_j : x_j \in \mathbb{Z}_{\ge 0}\Big\}.$$

Proof. If all prime divisors of n are $\le N$, then

$$n = p_1^{x_1} \cdots p_d^{x_d} \qquad (x_j \in \mathbb{Z}_{\ge 0}).$$

By the matrix identity $\Phi_{\mathrm{P}}(n) = A_N x$, where $x = (x_1, \ldots, x_d)^T$. Since $u_j = A_N e_j$, this means

$$\Phi_{\mathrm{P}}(n) = \sum_{j=1}^{d} x_j u_j.$$

Thus every such Pratt vector lies in the displayed semigroup.

Conversely, given nonnegative integers $x_1, \ldots, x_d$, define

$$n := p_1^{x_1} \cdots p_d^{x_d}.$$

Then the same calculation shows that its Pratt vector is exactly $\sum_j x_j u_j$. So every element of the semigroup occurs. □

Remark 6.2. *This proposition is the precise sense in which the Pratt valuation theory is a lattice theory. The semigroup of admissible Pratt vectors is not the whole positive orthant in the standard basis, but it is the positive cone spanned by the column vectors u_j. Because those columns form a unimodular basis, this cone is lattice-equivalent to the standard orthant. Again, the genuinely new arithmetic enters only when we use the special linear functional Ω_{P} or, still more deeply, the global weight vector w of the next section.*

7 The global weight vector and reconstruction of n

We now prove the statement emphasized in the abstract. This is the key structural fact distinguishing the Pratt setting from the ordinary v_p-setting.

Definition 7.1 (Global Pratt weight vector)**.** *Define*

$$w := \sum_{p \in \mathcal{P}} w_p e_p, \qquad w_p := -\log\Big(1 - \frac{1}{p}\Big) = \log\Big(\frac{p}{p-1}\Big).$$

Each coefficient is positive. Since $w_p \sim 1/p$ as $p \to \infty$, the vector w is not in ℓ^1, but it is in ℓ^2 and it pairs perfectly well with every finitely supported Pratt vector $\Phi_{\mathrm{P}}(n)$.

Lemma 7.2 (Prime recursion for the weight pairing)**.** *For every prime q,*

$$\langle \Phi_{\mathrm{P}}(q), w \rangle = \log q.$$

Proof. We argue by induction on the prime q.

Base case. If $q = 2$, then the Pratt tree T_2 consists of a single vertex labeled 2. Hence

$$\Phi_{\mathrm{P}}(2) = e_2.$$

Therefore

$$\langle \Phi_{\mathrm{P}}(2), w \rangle = w_2 = -\log\Big(1 - \frac{1}{2}\Big) = \log 2.$$

So the formula holds for $q = 2$.

Induction step. Assume the formula holds for all primes smaller than a given prime $q > 2$. By the recursive definition of the Pratt tree, the tree T_q consists of a root labeled q, together with, for each prime divisor $r \mid (q-1)$, exactly $v_r(q-1)$ children labeled r, each carrying a copy of T_r. Consequently the Pratt valuation vector satisfies

$$\Phi_{\mathrm{P}}(q) = e_q + \sum_{r \mid (q-1)} v_r(q-1)\, \Phi_{\mathrm{P}}(r).$$

Pairing with w yields

$$\langle \Phi_{\mathrm{P}}(q), w \rangle = w_q + \sum_{r \mid (q-1)} v_r(q-1)\, \langle \Phi_{\mathrm{P}}(r), w \rangle.$$

By the induction hypothesis, $\langle \Phi_{\mathrm{P}}(r), w \rangle = \log r$ for all primes $r < q$, and every prime divisor of $q-1$ is indeed smaller than q. Thus

$$\langle \Phi_{\mathrm{P}}(q), w \rangle = w_q + \sum_{r \mid (q-1)} v_r(q-1) \log r.$$

The sum on the right is exactly $\log(q-1)$, because the prime factorization of $q-1$ is

$$q - 1 = \prod_{r \mid (q-1)} r^{v_r(q-1)}.$$

Hence

$$\sum_{r \mid (q-1)} v_r(q-1) \log r = \log(q-1).$$

Using the definition of w_q, we conclude

$$\langle \Phi_{\mathrm{P}}(q), w \rangle = \log\left(\frac{q}{q-1}\right) + \log(q-1) = \log q.$$

This completes the induction. □

Theorem 7.3 (Global reconstruction formula)**.** *For every positive integer n,*

$$\langle \Phi_{\mathrm{P}}(n), w \rangle = \log n.$$

Equivalently,

$$n = \exp\big(\langle \Phi_{\mathrm{P}}(n), w \rangle\big).$$

Proof. Write the prime factorization of n as

$$n = \prod_q q^{v_q(n)}.$$

By additivity of the Pratt valuation,

$$\Phi_{\mathrm{P}}(n) = \sum_q v_q(n)\, \Phi_{\mathrm{P}}(q).$$

Pairing with w,

$$\langle \Phi_{\mathrm{P}}(n), w \rangle = \sum_q v_q(n)\, \langle \Phi_{\mathrm{P}}(q), w \rangle.$$

By the previous lemma, each inner product equals $\log q$. Therefore

$$\langle \Phi_{\mathrm{P}}(n), w \rangle = \sum_q v_q(n) \log q.$$

But this is exactly the logarithm of the prime factorization of n:

$$\sum_q v_q(n)\log q = \log\left(\prod_q q^{v_q(n)}\right) = \log n.$$

Exponentiating both sides gives

$$n = \exp\big(\langle \Phi_{\mathrm{P}}(n), w\rangle\big).$$

□

Corollary 7.4 (The matrix identity $A^T w = \log p$)**.** *For every prime index j,*

$$\sum_i a_{ij} w_{p_i} = \log p_j.$$

Equivalently,

$$A^T w = (\log p_1, \log p_2, \dots)^T.$$

Proof. The j-th column of A is $\Phi_{\mathrm{P}}(p_j)$. Hence

$$\sum_i a_{ij} w_{p_i} = \langle \Phi_{\mathrm{P}}(p_j), w\rangle.$$

The previous lemma identifies this quantity with $\log p_j$. That is exactly the claimed matrix identity. □

8 The numerical meaning of the cutoff $\Omega_{\mathrm{P}}(n) \le r$

We now draw the central consequence of the global reconstruction formula.

Theorem 8.1 (A Pratt-weight cutoff is automatically a size cutoff)**.** *For every $n \in \mathbb{N}$,*

$$\log n \le (\log 2)\,\Omega_{\mathrm{P}}(n).$$

Equivalently,

$$n \le 2^{\Omega_{\mathrm{P}}(n)}.$$

In particular, if $\Omega_{\mathrm{P}}(n) \le r$, then

$$n \le 2^r.$$

Proof. We start from the reconstruction formula

$$\log n = \langle \Phi_{\mathrm{P}}(n), w\rangle = \sum_p m_p(n) w_p.$$

Each coefficient w_p is positive, and because the function

$$p \mapsto \log\left(\frac{p}{p-1}\right)$$

is strictly decreasing for $p \ge 2$, its largest value occurs at $p = 2$. Thus

$$0 < w_p \le w_2 = \log 2 \qquad \text{for every prime } p.$$

Substituting this bound term by term into the sum, we obtain

$$\log n = \sum_p m_p(n) w_p \le \sum_p m_p(n)\log 2 = (\log 2)\sum_p m_p(n) = (\log 2)\Omega_{\mathrm{P}}(n).$$

This proves the logarithmic inequality. Exponentiating gives

$$n \le e^{(\log 2)\Omega_{\mathrm{P}}(n)} = 2^{\Omega_{\mathrm{P}}(n)}.$$

If $\Omega_{\mathrm{P}}(n) \le r$, then the right-hand side is at most 2^r, so $n \le 2^r$. □

Corollary 8.2 (Finiteness without a prime-support cutoff)**.** *For every $r \in \mathbb{N}_0$, the set*

$$E_r := \{n \in \mathbb{N} : \Omega_{\mathrm{P}}(n) \le r\}$$

is finite. More precisely,

$$E_r \subseteq \{1, 2, \ldots, 2^r\}.$$

Hence

$$\#E_r = \#\{n \le 2^r : \Omega_{\mathrm{P}}(n) \le r\}.$$

Proof. If $n \in E_r$, then by definition $\Omega_{\mathrm{P}}(n) \le r$. The theorem therefore implies $n \le 2^r$. So every element of E_r lies in the finite set $\{1, 2, \ldots, 2^r\}$. This proves finiteness and the stated inclusion. The last equality is then immediate. □

Corollary 8.3 (Stabilization of the finite-dimensional Ehrhart count)**.** *For every $r \in \mathbb{N}_0$, if $N \ge 2^r$, then*

$$E_r = C_{N,r}.$$

Consequently,

$$\#E_r = L(P_N, r) \qquad (N \ge 2^r).$$

In particular,

$$\#E_r = L(P_{2^r}, r).$$

Proof. Take $N \ge 2^r$. If $n \in E_r$, then the previous corollary gives $n \le 2^r \le N$. Every prime divisor of n is therefore at most n, hence at most N. Thus n belongs to $C_{N,r}$, because it also satisfies $\Omega_{\mathrm{P}}(n) \le r$. So $E_r \subseteq C_{N,r}$.

Conversely, if $n \in C_{N,r}$, then by definition $\Omega_{\mathrm{P}}(n) \le r$, so $n \in E_r$. Thus $C_{N,r} \subseteq E_r$. The two sets are equal.

The formula for their cardinality follows from the arithmetic interpretation of $L(P_N, r)$. Setting $N = 2^r$ gives the final statement. □

Remark 8.4. *This stabilization phenomenon is the rigorous form of the intuition in the abstract. In the ordinary v_p-theory, the layer $\Omega(n) \le r$ is infinite unless one separately imposes a support cutoff on the primes. In the Pratt theory, no such auxiliary cutoff is needed: the linear inequality in Pratt coordinates already forces a finite interval in the ordinary size variable. That is a genuine qualitative advantage.*

9 Can these Ehrhart objects help count primes?

We now address the final question. The answer is: yes, they help in a meaningful one-sided way, but by themselves they do not yet give a direct formula for $\pi(x)$. We will make both points precise.

9.1 What can be proved directly

Definition 9.1. *For $r \in \mathbb{N}_0$, define*

$$\Pi_{\mathrm{P}}(r) := \#\{p \text{ prime} : \Omega_{\mathrm{P}}(p) \le r\}.$$

This is the number of primes lying in the global Pratt ball of radius r.

Proposition 9.2 (Pratt balls of primes lie in ordinary intervals)**.** *For every $r \in \mathbb{N}_0$,*

$$\Pi_{\mathrm{P}}(r) \le \pi(2^r).$$

More precisely,

$$\{p \text{ prime} : \Omega_{\mathrm{P}}(p) \le r\} \subseteq \{p \text{ prime} : p \le 2^r\}.$$

Proof. If p is prime and $\Omega_{\mathrm{P}}(p) \le r$, then the numerical cutoff theorem gives

$$p \le 2^r.$$

Thus every prime counted by $\Pi_{\mathrm{P}}(r)$ is also counted by $\pi(2^r)$. This is exactly the claimed inclusion, and taking cardinalities gives the inequality. □

Proposition 9.3 (Ehrhart counts control the number of integers in a size interval with bounded Pratt weight). *For every $r \in \mathbb{N}_0$,*

$$\#\{n \le 2^r : \Omega_{\mathrm{P}}(n) \le r\} = L(P_{2^r}, r).$$

Hence

$$\Pi_{\mathrm{P}}(r) \le L(P_{2^r}, r) - 1.$$

Proof. The first equality is just the stabilization corollary from the previous section. The set counted on the left contains 1 and all primes p with $\Omega_{\mathrm{P}}(p) \le r$. Therefore the number of such primes is at most the total number of elements of that set minus one. This gives

$$\Pi_{\mathrm{P}}(r) \le \#\{n \le 2^r : \Omega_{\mathrm{P}}(n) \le r\} - 1 = L(P_{2^r}, r) - 1.$$

□

The preceding proposition shows that Ehrhart theory does indeed produce finite combinatorial objects whose lattice-point counts dominate a certain prime-counting problem. In that precise sense, the answer to the question "can Ehrhart polynomials help count primes?" is yes.

9.2 Why this does not yet yield $\pi(x)$ directly

The obstacle is that Ehrhart theory counts *all* lattice points satisfying a linear inequality, whereas primality is not a linear condition in exponent space. The prime numbers correspond to the exponent vectors e_j in the ordinary coordinates, but after one passes to cumulative counting regions such as

$$\Omega_{\mathrm{P}}(n) \le r,$$

most lattice points represent composite numbers. Nothing in the bare Ehrhart count separates those primes from the composites.

We can make this obstruction explicit.

Proposition 9.4 (Ehrhart theory counts too much). *For every $r \ge 1$,*

$$L(P_{2^r}, r) = \#\{n \le 2^r : \Omega_{\mathrm{P}}(n) \le r\}$$

counts all integers in that Pratt ball, not only the primes. In particular, the contribution of composites is unavoidable in the raw Ehrhart count.

Proof. The equality is already proved. By definition, the counted set consists of all positive integers with $\Omega_{\mathrm{P}}(n) \le r$. If $a, b \ge 2$ and $\Omega_{\mathrm{P}}(a) + \Omega_{\mathrm{P}}(b) \le r$, then additivity gives

$$\Omega_{\mathrm{P}}(ab) = \Omega_{\mathrm{P}}(a) + \Omega_{\mathrm{P}}(b) \le r,$$

so the composite number ab is counted as well. Thus the counting function necessarily includes composites whenever such examples exist. For instance, if $r \ge 2$, then

$$\Omega_{\mathrm{P}}(4) = \Omega_{\mathrm{P}}(2^2) = 2,$$

so the composite number 4 is counted by $L(P_{2^r}, r)$ for every $r \ge 2$. Therefore the raw Ehrhart count cannot equal the prime-counting function. □

9.3 A more precise positive statement

Although the raw Ehrhart count does not isolate primes, the global weight vector still yields a genuine bridge between polyhedral counting and ordinary size. The bridge is the following theorem.

Theorem 9.5 (Pratt-Ehrhart regions are automatically arithmetically localized)**.** *For every $r \in \mathbb{N}_0$, the global Pratt ball*

$$E_r = \{n \in \mathbb{N} : \Omega_{\mathrm{P}}(n) \leq r\}$$

is exactly the set of lattice points of the finite-dimensional rational simplex rP_{2^r} under the usual factorization map. Hence every question about the distribution of primes inside E_r is a question about primes inside a finite Ehrhart region contained in the initial interval $[1, 2^r]$.

Proof. This is exactly the content of the stabilization corollary: for $N = 2^r$ we have

$$E_r = C_{2^r,r}.$$

By the arithmetic interpretation theorem, $C_{2^r,r}$ is counted by the lattice points of rP_{2^r}. Therefore the set E_r is precisely the integer set attached to that finite Ehrhart region. Any subset of E_r, in particular the subset of primes inside E_r, is therefore a subset of a finite polyhedral counting region inside $[1, 2^r]$. □

This theorem is the strongest unconditional statement available from the present formalism. It says that the Pratt-Ehrhart machine gives a finite polyhedral model for a subset of an ordinary initial interval. That is already impossible in the ordinary Ω-theory without imposing a separate support cutoff.

9.4 What would still be needed to count primes efficiently

To turn the preceding theorem into a direct estimate for $\pi(x)$, one would need additional information describing which points in the Pratt ball correspond to primes. A natural identity is

$$\Omega_{\mathrm{P}}(p) = 1 + \Omega_{\mathrm{P}}(p - 1) \qquad (p \text{ prime}),$$

which follows immediately from the recursive definition of the Pratt tree of a prime: the tree of p consists of one root together with the Pratt forest of $p - 1$. This identity shows that prime detection in the Pratt world is equivalent to understanding when an integer of the form $m + 1$ is prime, where m is constrained by the Pratt weight of m. That is an arithmetic problem of a very different kind than lattice-point enumeration.

So the clean conclusion is this.

> Ehrhart theory for Pratt exponents *does* help by converting the abstract coordinate cutoff $\Omega_{\mathrm{P}}(n) \leq r$ into a finite polyhedral counting problem inside the concrete interval $[1, 2^r]$. What it does *not* yet do is isolate primality among those lattice points. For that one needs additional arithmetic input beyond Ehrhart theory itself.

Part II. Analytical investigations

10 A first elementary lower bound for the Pratt prime counting function

In this part we study the prime counting function attached to the Pratt height,

$$\Pi_{\mathrm{P}}(r) := \#\{p \in \mathcal{P} : \Omega_{\mathrm{P}}(p) \leq r\}.$$

The first point is that, unlike the ordinary exponent vector, the Pratt vector carries a global linear reconstruction of the actual size of an integer. Recall that

$$\log n = \langle \Phi_{\mathrm{P}}(n), w \rangle, \qquad w := \sum_{p \in \mathcal{P}} -\log\left(1 - \frac{1}{p}\right) e_p,$$

with

$$0 < w_p = -\log\left(1 - \frac{1}{p}\right) \leq \log 2 \qquad (p \in \mathcal{P}).$$

Therefore

$$\log n = \sum_p m_p(n) w_p \leq (\log 2) \sum_p m_p(n) = (\log 2)\Omega_{\mathrm{P}}(n),$$

so that

$$n \leq 2^{\Omega_{\mathrm{P}}(n)}.$$

In particular, if $\Omega_{\mathrm{P}}(p) \leq r$, then automatically $p \leq 2^r$. Hence for every $n \geq 2$,

$$\pi(n) \geq \Pi_{\mathrm{P}}(\log_2 n).$$

This is the basic bridge from Pratt geometry to ordinary prime counting.

The interesting question is now to prove a nontrivial lower bound for $\Pi_{\mathrm{P}}(r)$.

Lemma 10.1 (Elementary upper bound for the Pratt height)**.** *For every integer $n \geq 1$, one has*

$$\Omega_{\mathrm{P}}(n) \leq n - 1.$$

Proof. We prove the statement by strong induction on n. The advantage of strong induction here is that in the composite case we shall need the estimate not only for $n - 1$, but for all proper factors of n.

For the initial step $n = 1$, one has $\Phi_{\mathrm{P}}(1) = 0$, hence also

$$\Omega_{\mathrm{P}}(1) = \sum_{p \in \mathcal{P}} m_p(1) = 0.$$

Therefore

$$\Omega_{\mathrm{P}}(1) = 0 = 1 - 1,$$

and the assertion is true in the initial case.

Now fix an integer $n \geq 2$ and assume as induction hypothesis that

$$\Omega_{\mathrm{P}}(m) \leq m - 1 \qquad \text{for every integer } 1 \leq m < n.$$

We must show that the same inequality holds for n. There are two cases.

Case 1: n is prime. In the Pratt theory one has the recursion

$$\Phi_{\mathrm{P}}(n) - \Phi_{\mathrm{P}}(n-1) = e_n.$$

This identity is an equality of prime-indexed coordinate vectors. If we now sum all coordinates on both sides, then the left-hand side becomes

$$\sum_p m_p(n) - \sum_p m_p(n-1) = \Omega_{\mathrm{P}}(n) - \Omega_{\mathrm{P}}(n-1),$$

while the right-hand side contributes exactly 1, because the standard basis vector e_n has a single nonzero coordinate, namely a 1 in the n-coordinate. Thus

$$\Omega_{\mathrm{P}}(n) - \Omega_{\mathrm{P}}(n-1) = 1,$$

that is,

$$\Omega_{\mathrm{P}}(n) = \Omega_{\mathrm{P}}(n-1) + 1.$$

Now $n-1 < n$, so the induction hypothesis applies to $n-1$. Hence

$$\Omega_{\mathrm{P}}(n-1) \leq (n-1) - 1 = n-2.$$

Substituting this into the previous identity gives

$$\Omega_{\mathrm{P}}(n) = \Omega_{\mathrm{P}}(n-1) + 1 \leq (n-2) + 1 = n-1.$$

So the desired inequality holds when n is prime.

Case 2: n is composite. Since n is composite, we may write

$$n = ab$$

with integers a, b satisfying

$$2 \leq a < n, \qquad 2 \leq b < n.$$

Because Φ_{P} is additive under multiplication, we have

$$\Phi_{\mathrm{P}}(n) = \Phi_{\mathrm{P}}(ab) = \Phi_{\mathrm{P}}(a) + \Phi_{\mathrm{P}}(b).$$

Summing coordinates again yields additivity of the total Pratt height:

$$\Omega_{\mathrm{P}}(n) = \Omega_{\mathrm{P}}(a) + \Omega_{\mathrm{P}}(b).$$

Now both a and b are strictly smaller than n, so the induction hypothesis applies to each of them:

$$\Omega_{\mathrm{P}}(a) \leq a-1, \qquad \Omega_{\mathrm{P}}(b) \leq b-1.$$

Adding these two inequalities gives

$$\Omega_{\mathrm{P}}(n) = \Omega_{\mathrm{P}}(a) + \Omega_{\mathrm{P}}(b) \leq (a-1) + (b-1) = a+b-2.$$

Thus it remains only to compare $a+b-2$ with $n-1 = ab-1$. This is elementary: since $a, b \geq 2$, we have

$$(a-1)(b-1) \geq 1.$$

Expanding the left-hand side gives

$$ab - a - b + 1 \geq 1,$$

which is equivalent to

$$ab - a - b \geq 0,$$

and therefore to

$$a + b \leq ab.$$

Subtracting 2 from both sides yields

$$a + b - 2 \leq ab - 2 < n - 1,$$

and in particular

$$a + b - 2 \leq ab - 1 = n - 1.$$

Combining this with the previous estimate for $\Omega_{\mathrm{P}}(n)$ we conclude that

$$\Omega_{\mathrm{P}}(n) \leq a + b - 2 \leq n - 1.$$

So the desired inequality also holds in the composite case.

Since both cases have been established, the induction is complete. $\square$

Theorem 10.2 (A first lower bound for $\Pi_{\mathrm{P}}(r)$)**.** *For every real $r \geq 1$,*

$$\Pi_{\mathrm{P}}(r) \geq \pi(r+1).$$

Consequently, for every $n \geq 2$,

$$\pi(n) \geq \Pi_{\mathrm{P}}(\log_2 n) \geq \pi(\log_2 n + 1).$$

Proof. We first prove the inequality

$$\Pi_{\mathrm{P}}(r) \geq \pi(r+1) \qquad (r \geq 1).$$

Recall that

$$\Pi_{\mathrm{P}}(r) = \#\{p \in \mathcal{P} : \Omega_{\mathrm{P}}(p) \leq r\}$$

counts those primes whose Pratt height is at most r. So in order to prove the desired lower bound, it is enough to show that every prime up to $r+1$ has Pratt height at most r.

Let p be any prime with $p \leq r+1$. By Lemma 10.1, applied to the integer $n = p$, we have

$$\Omega_{\mathrm{P}}(p) \leq p - 1.$$

Since $p \leq r+1$, subtracting 1 yields

$$p - 1 \leq r.$$

Combining the two inequalities, we obtain

$$\Omega_{\mathrm{P}}(p) \leq p - 1 \leq r.$$

Thus every prime $p \leq r+1$ belongs to the set

$$\{q \in \mathcal{P} : \Omega_{\mathrm{P}}(q) \leq r\}$$

that is counted by $\Pi_{\mathrm{P}}(r)$. Therefore there is an inclusion of sets

$$\{p \in \mathcal{P} : p \leq r+1\} \subseteq \{p \in \mathcal{P} : \Omega_{\mathrm{P}}(p) \leq r\}.$$

Taking cardinalities gives

$$\pi(r+1) = \#\{p \in \mathcal{P} : p \leq r+1\} \leq \#\{p \in \mathcal{P} : \Omega_{\mathrm{P}}(p) \leq r\} = \Pi_{\mathrm{P}}(r).$$

This proves the first assertion.

We now turn to the second chain of inequalities. Let $n \geq 2$ and put

$$r := \log_2 n.$$

By the global reconstruction formula,

$$\log m = \langle \Phi_{\mathrm{P}}(m), w \rangle$$

for every integer $m \geq 1$, and since every coordinate of w is at most $\log 2$, we have already shown above that

$$\log m \leq (\log 2)\Omega_{\mathrm{P}}(m).$$

Equivalently,

$$m \leq 2^{\Omega_{\mathrm{P}}(m)}.$$

Apply this to a prime p satisfying $\Omega_{\mathrm{P}}(p) \leq \log_2 n$. Then

$$p \leq 2^{\Omega_{\mathrm{P}}(p)} \leq 2^{\log_2 n} = n.$$

Hence every prime counted by $\Pi_{\mathrm{P}}(\log_2 n)$ is automatically at most n, so we have another inclusion of sets,

$$\{p \in \mathcal{P} : \Omega_{\mathrm{P}}(p) \leq \log_2 n\} \subseteq \{p \in \mathcal{P} : p \leq n\}.$$

Taking cardinalities yields

$$\Pi_{\mathrm{P}}(\log_2 n) \leq \pi(n).$$

Finally, applying the first part of the theorem with $r = \log_2 n$ gives

$$\Pi_{\mathrm{P}}(\log_2 n) \geq \pi(\log_2 n + 1).$$

Putting the two inequalities together, we obtain

$$\pi(n) \geq \Pi_{\mathrm{P}}(\log_2 n) \geq \pi(\log_2 n + 1),$$

as claimed. $\square$

Corollary 10.3 (A completely elementary analytic lower bound)**.** *For every $r \geq 1$,*

$$\Pi_{\mathrm{P}}(r) \geq \lfloor \log_2(r+1) \rfloor.$$

Equivalently, for every $n \geq 2$,

$$\pi(n) \geq \Pi_{\mathrm{P}}(\log_2 n) \geq \lfloor \log_2(\log_2 n + 1) \rfloor .$$

Proof. We begin with a standard consequence of Bertrand's postulate. Let $x \geq 2$, and let

$$M := \lfloor \log_2 x \rfloor.$$

Then M is a nonnegative integer and by definition of the floor function we have

$$M \leq \log_2 x < M + 1.$$

Exponentiating base 2 gives

$$2^M \leq x < 2^{M+1}.$$

Now apply Bertrand's theorem successively to the integers

$$2, 2^2, 2^3, \ldots, 2^M.$$

For each integer m with $0 \leq m \leq M - 1$, Bertrand's theorem yields at least one prime in the interval

$$(2^m, 2^{m+1}).$$

These intervals are pairwise disjoint, so they contribute at least one distinct prime each. Therefore there are at least M primes not exceeding 2^M, and hence certainly not exceeding x. Thus

$$\pi(x) \geq M = \lfloor \log_2 x \rfloor.$$

This proves the elementary estimate

$$\pi(x) \geq \lfloor \log_2 x \rfloor \qquad (x \geq 2).$$

We now apply Theorem 10.2, which states that

$$\Pi_{\mathrm{P}}(r) \geq \pi(r+1) \qquad (r \geq 1).$$

Substituting $x = r + 1$ into the inequality just proved, we obtain

$$\pi(r+1) \geq \lfloor \log_2(r+1) \rfloor.$$

Combining the two inequalities gives

$$\Pi_{\mathrm{P}}(r) \geq \pi(r+1) \geq \lfloor \log_2(r+1) \rfloor,$$

which is the first claim.

Finally, replace r by $\log_2 n$. Then

$$\Pi_{\mathrm{P}}(\log_2 n) \geq \lfloor \log_2(\log_2 n + 1) \rfloor .$$

Together with Theorem 10.2, which also gives

$$\pi(n) \geq \Pi_{\mathrm{P}}(\log_2 n),$$

this yields

$$\pi(n) \geq \Pi_{\mathrm{P}}(\log_2 n) \geq \lfloor \log_2(\log_2 n + 1) \rfloor .$$

This proves the corollary. □

Corollary 10.4 (Chebyshev-strength lower bound)**.** *Assume one uses Chebyshev's classical estimate $\pi(x) \gg x/\log x$. Then*

$$\Pi_{\mathrm{P}}(r) \gg \frac{r}{\log r} \qquad (r \geq 2),$$

and therefore

$$\Pi_{\mathrm{P}}(\log_2 n) \gg \frac{\log n}{\log \log n} \qquad (n \geq 4).$$

Proof. By Theorem 10.2, we already know that

$$\Pi_{\mathrm{P}}(r) \geq \pi(r+1) \qquad (r \geq 1).$$

Now invoke Chebyshev's classical lower bound for the ordinary prime counting function: there exists a positive absolute constant c_0 such that

$$\pi(x) \geq c_0 \frac{x}{\log x} \qquad (x \geq 2).$$

Apply this with $x = r + 1$. For $r \geq 2$ one obtains

$$\Pi_{\mathrm{P}}(r) \geq \pi(r+1) \geq c_0 \frac{r+1}{\log(r+1)}.$$

Since

$$\frac{r+1}{\log(r+1)} \asymp \frac{r}{\log r} \qquad (r \to \infty),$$

we may absorb the harmless change from $r+1$ to r into the implied constant and conclude that

$$\Pi_{\mathrm{P}}(r) \gg \frac{r}{\log r} \qquad (r \geq 2).$$

This is the first assertion.

For the second assertion, substitute $r = \log_2 n$. Then

$$\Pi_{\mathrm{P}}(\log_2 n) \gg \frac{\log_2 n}{\log(\log_2 n)}.$$

Now

$$\log_2 n = \frac{\log n}{\log 2},$$

and

$$\log(\log_2 n) = \log\left(\frac{\log n}{\log 2}\right) = \log\log n - \log\log 2.$$

Thus $\log(\log_2 n)$ differs from $\log\log n$ only by an additive constant, and so the two are comparable for $n \geq 4$. Consequently,

$$\Pi_{\mathrm{P}}(\log_2 n) \gg \frac{\log n}{\log\log n} \qquad (n \geq 4).$$

This proves the corollary. $\square$

The conceptual point is worth emphasizing. For the ordinary exponent vector $v(n)$, the condition $\Omega(n) \leq r$ does not control the numerical size of n: every prime p has $\Omega(p) = 1$, regardless of the size of p. In contrast, the Pratt identity

$$\log n = \langle \Phi_{\mathrm{P}}(n), w \rangle$$

with a *single global* positive weight vector w implies that the Pratt ball

$$\{\Phi_{\mathrm{P}}(n) : \Omega_{\mathrm{P}}(n) \leq r\}$$

automatically lies inside the initial interval $[1, 2^r]$. This is the precise sense in which the Pratt geometry is much more rigid than the ordinary valuation geometry.

11 Structural properties of $\Pi_{\mathrm{P}}(r)$

Let us write

$$h(n) := \Omega_{\mathrm{P}}(n) = \sum_{q \in \mathcal{P}} m_q(n).$$

Then

$$\Pi_{\mathrm{P}}(r) = \#\{p \in \mathcal{P} : h(p) \leq r\}.$$

The basic relations are

$$h(ab) = h(a) + h(b), \qquad h(p) = 1 + h(p-1) \qquad (p \in \mathcal{P}).$$

The second identity is simply the coordinate sum of the Pratt recursion $\Phi_{\mathrm{P}}(p) - \Phi_{\mathrm{P}}(p-1) = e_p$.

Proposition 11.1 (Elementary structural properties)**.** *The function $\Pi_{\mathrm{P}}(r)$, viewed on the nonnegative integers, satisfies the following properties.*

(a) *It is monotone nondecreasing.*

(b) *It is a step function with jumps*

$$a_r := \Pi_{\mathrm{P}}(r) - \Pi_{\mathrm{P}}(r-1) = \#\{p \in \mathcal{P} : h(p) = r\}.$$

(c) *Every jump is finite.*

(d) *One has the exact bounds*

$$\pi(r+1) \le \Pi_{\mathrm{P}}(r) \le \pi(2^r) \qquad (r \ge 1).$$

Proof. We prove the four assertions one after another.

Proof of (a). If a prime p satisfies $h(p) \le r$, then automatically $h(p) \le r+1$ as well, simply because $r \le r+1$. Hence we have an inclusion of sets

$$\{p \in \mathcal{P} : h(p) \le r\} \subseteq \{p \in \mathcal{P} : h(p) \le r+1\}.$$

Taking cardinalities yields

$$\Pi_{\mathrm{P}}(r) \le \Pi_{\mathrm{P}}(r+1).$$

Thus Π_{P} is monotone nondecreasing.

Proof of (b). For integer heights, the value $\Pi_{\mathrm{P}}(r)$ counts all primes whose Pratt height is at most r, while $\Pi_{\mathrm{P}}(r-1)$ counts all primes whose Pratt height is at most $r-1$. The primes counted by the first quantity but not by the second are exactly the primes with height equal to r. Set-theoretically,

$$\{p : h(p) \le r\} \setminus \{p : h(p) \le r-1\} = \{p : h(p) = r\}.$$

Therefore

$$\Pi_{\mathrm{P}}(r) - \Pi_{\mathrm{P}}(r-1) = \#\{p \in \mathcal{P} : h(p) = r\} =: a_r.$$

This is precisely the stated jump formula.

Proof of (c). Fix r. We must show that only finitely many primes have height at most r. Let p be such a prime. Then, by the global reconstruction formula,

$$\log p = \langle \Phi_{\mathrm{P}}(p), w \rangle.$$

Every coordinate of the weight vector w is positive and bounded above by $\log 2$, so

$$\log p = \sum_q m_q(p) w_q \le (\log 2) \sum_q m_q(p) = (\log 2) h(p).$$

Since $h(p) \le r$, this implies

$$\log p \le r \log 2.$$

Exponentiating gives

$$p \le 2^r.$$

Thus every prime with height at most r lies in the finite interval $[1, 2^r]$. There are only finitely many primes in any bounded interval, hence the jump set is finite. In particular, $a_r < \infty$.

Proof of (d). The lower bound

$$\pi(r+1) \le \Pi_{\mathrm{P}}(r)$$

was proved in Theorem 10.2. So only the upper bound remains. Suppose that a prime p is counted by $\Pi_{\mathrm{P}}(r)$, i.e. that $h(p) \le r$. By the argument just used in part (c), we then have $p \le 2^r$. Therefore

$$\{p \in \mathcal{P} : h(p) \le r\} \subseteq \{p \in \mathcal{P} : p \le 2^r\}.$$

Taking cardinalities yields

$$\Pi_{\mathrm{P}}(r) = \#\{p \in \mathcal{P} : h(p) \le r\} \le \#\{p \in \mathcal{P} : p \le 2^r\} = \pi(2^r).$$

This completes the proof of all four parts. $\square$

Proposition 11.2 (Exact reformulation of the jump sizes)**.** *For every integer $r \ge 1$,*

$$a_r = \#\{n \ge 1 : h(n) = r-1,\ n+1 \textit{ is prime}\}.$$

Proof. Start from the definition of the jump size:

$$a_r = \Pi_{\mathrm{P}}(r) - \Pi_{\mathrm{P}}(r-1) = \#\{p \in \mathcal{P} : h(p) = r\}.$$

So a_r counts precisely those primes whose Pratt height is exactly r.

Now let p be a prime. The Pratt recursion for prime arguments says that

$$\Phi_{\mathrm{P}}(p) - \Phi_{\mathrm{P}}(p-1) = e_p.$$

Summing all coordinates gives

$$h(p) - h(p-1) = 1,$$

that is,

$$h(p) = 1 + h(p-1).$$

Therefore the condition

$$h(p) = r$$

is equivalent to the condition

$$h(p-1) = r-1.$$

If we now define

$$n := p-1,$$

then the prime p is uniquely recovered as $n+1$, and the above condition becomes

$$h(n) = r-1, \qquad n+1 \text{ is prime}.$$

Thus every prime counted by a_r determines an integer n belonging to the set

$$\{n \ge 1 : h(n) = r-1,\ n+1 \text{ is prime}\}.$$

Conversely, suppose that an integer $n \ge 1$ satisfies

$$h(n) = r-1 \qquad \text{and} \qquad n+1 \text{ is prime}.$$

Set $p := n+1$. Since p is prime, the same recursion applies and gives

$$h(p) = 1 + h(p-1) = 1 + h(n) = 1 + (r-1) = r.$$

Hence p is counted by a_r.

We have therefore constructed inverse correspondences

$$p \longmapsto p-1, \qquad n \longmapsto n+1,$$

between the two sets under consideration. So they are in bijection, and consequently have the same cardinality:

$$a_r = \#\{n \ge 1 : h(n) = r-1,\ n+1 \text{ is prime}\}.$$

This proves the proposition. $\square$

Proposition 11.2 isolates the exact point where prime number theory enters. The lattice-theoretic side counts integers on a fixed Pratt layer $h(n) = r-1$, while primality asks which of the translated points $n+1$ are prime. Thus $\Pi_{\mathrm{P}}(r)$ is not a pure Ehrhart count; it is an Ehrhart-type layer count followed by a primality filter on a translated boundary.

12 Numerical evidence and a first heuristic law

We now record numerical evidence obtained by direct computation of the recursion

$$h(1) = 0, \qquad h(p) = 1 + h(p-1) \quad (p \text{ prime}), \qquad h(mn) = h(m) + h(n).$$

Because $h(n) \le r \Rightarrow n \le 2^r$, one can compute all values of $\Pi_{\mathrm{P}}(r)$ up to a given height r by checking integers only up to 2^r. A computation up to 2^{22} gives the exact values

r	5	10	15	20	21	22
$\Pi_{\mathrm{P}}(r)$	7	41	356	3139	5027	7722

and more explicitly

r	10	11	12	13	14	15	16	17	18	19	20	21	22
$\Pi_{\mathrm{P}}(r)$	41	70	94	153	221	356	524	858	1282	2094	3139	5027	7722

These data strongly suggest that, for typical primes, the Pratt height grows roughly linearly with $\log p$. A linear regression of $h(p)$ against $\log p$ for primes $p \le 2 \cdot 10^6$ gives a slope close to 2.02. This suggests the heuristic relation

$$h(p) \approx 2 \log p,$$

which may be rewritten as

$$p \approx e^{r/2} \qquad \text{when } h(p) = r.$$

If one then combines this with the usual prime-density heuristic $1/\log x$, one is led to the conjectural growth law

$$\Pi_{\mathrm{P}}(r) \asymp \frac{e^{r/2}}{r}.$$

The quantity

$$\Pi_{\mathrm{P}}(r) \frac{r}{e^{r/2}}$$

is numerically rather stable for $10 \le r \le 22$, fluctuating around values between approximately 2.8 and 3.0. This is not a proof, but it is a very clear numerical signature.

Remark 12.1 (What is rigorous and what is not)**.** *The numerical data and the heuristic law above should be read with caution. What is proved rigorously in this note is the double inequality*

$$\pi(r+1) \le \Pi_{\mathrm{P}}(r) \le \pi(2^r).$$

What is not *proved is the asymptotic formula* $\Pi_{\mathrm{P}}(r) \asymp e^{r/2}/r$*. To prove such a statement one would need genuine information on how the Pratt layers*

$$\{n : h(n) = r\}$$

are distributed modulo small primes, because Proposition 11.2 asks how often $n+1$ *is prime on those layers. That is a real prime-distribution problem, not a purely Ehrhart-theoretic one.*

13 The Pratt zeta function in one variable

We now introduce the generating series that packages the entire height distribution.

Definition 13.1 (Pratt zeta and prime Pratt series)**.** *Define*

$$\zeta_{\mathrm{P}}(y) := \sum_{n\geq 1} y^{\Omega_{\mathrm{P}}(n)} = \sum_{r\geq 0} A_r y^r, \qquad A_r := \#\{n \geq 1 : \Omega_{\mathrm{P}}(n) = r\},$$

and

$$P_{\mathrm{P}}(y) := \sum_{p\in\mathcal{P}} y^{\Omega_{\mathrm{P}}(p)} = \sum_{r\geq 1} b_r y^r, \qquad b_r := \#\{p \in \mathcal{P} : \Omega_{\mathrm{P}}(p) = r\}.$$

Thus

$$\Pi_{\mathrm{P}}(r) = \sum_{j\leq r} b_j.$$

Proposition 13.2 (Euler product)**.** *As a formal power series one has*

$$\zeta_{\mathrm{P}}(y) = \prod_{p\in\mathcal{P}} \frac{1}{1 - y^{\Omega_{\mathrm{P}}(p)}}.$$

Proof. We argue first at the level of formal power series. Because Ω_{P} is additive under multiplication, we have for every prime p and every integer $k \geq 0$,

$$\Omega_{\mathrm{P}}(p^k) = \underbrace{\Omega_{\mathrm{P}}(p) + \cdots + \Omega_{\mathrm{P}}(p)}_{k \text{ times}} = k\,\Omega_{\mathrm{P}}(p).$$

Therefore the contribution of the prime powers of p to the height-generating series is the geometric series

$$\sum_{k\geq 0} y^{\Omega_{\mathrm{P}}(p^k)} = \sum_{k\geq 0} y^{k\Omega_{\mathrm{P}}(p)} = \frac{1}{1 - y^{\Omega_{\mathrm{P}}(p)}}.$$

Now consider a general integer $n \geq 1$ with prime factorization

$$n = \prod_{p} p^{v_p(n)}.$$

Additivity of Ω_{P} gives

$$\Omega_{\mathrm{P}}(n) = \sum_{p} v_p(n)\Omega_{\mathrm{P}}(p),$$

so that

$$y^{\Omega_{\mathrm{P}}(n)} = \prod_{p} y^{v_p(n)\Omega_{\mathrm{P}}(p)}.$$

Thus the coefficient attached to n is obtained by choosing, for each prime p, one term from the geometric series

$$\sum_{k\geq 0} y^{k\Omega_{\mathrm{P}}(p)}.$$

Because prime factorization is unique, this correspondence between integers n and choices of exponents $(v_p(n))_p$ is one-to-one. Hence multiplying all these geometric series together produces exactly one copy of $y^{\Omega_{\mathrm{P}}(n)}$ for each integer $n \geq 1$, and no other monomials. Therefore

$$\prod_{p\in\mathcal{P}} \frac{1}{1 - y^{\Omega_{\mathrm{P}}(p)}} = \prod_{p\in\mathcal{P}} \left(\sum_{k\geq 0} y^{k\Omega_{\mathrm{P}}(p)}\right) = \sum_{n\geq 1} y^{\Omega_{\mathrm{P}}(n)} = \zeta_{\mathrm{P}}(y).$$

This is the desired Euler product identity. $\square$

Proposition 13.3 (Logarithmic formula and M"obius inversion)**.** *One has the formal identities*

$$\log \zeta_{\mathrm{P}}(y) = \sum_{k\geq 1} \frac{1}{k} P_{\mathrm{P}}(y^k)$$

and

$$P_{\mathrm{P}}(y) = \sum_{m\geq 1} \frac{\mu(m)}{m} \log \zeta_{\mathrm{P}}(y^m).$$

Equivalently, if

$$\log \zeta_{\mathrm{P}}(y) = \sum_{r\geq 1} \ell_r y^r,$$

then

$$b_r = \sum_{d|r} \frac{\mu(d)}{d}\, \ell_{r/d}.$$

Proof. We begin with the Euler product from Proposition 13.2:

$$\zeta_{\mathrm{P}}(y) = \prod_{p\in\mathcal{P}} \frac{1}{1 - y^{\Omega_{\mathrm{P}}(p)}}.$$

Taking formal logarithms gives

$$\log \zeta_{\mathrm{P}}(y) = \sum_{p\in\mathcal{P}} \log\left(\frac{1}{1 - y^{\Omega_{\mathrm{P}}(p)}}\right) = -\sum_{p\in\mathcal{P}} \log(1 - y^{\Omega_{\mathrm{P}}(p)}).$$

Now use the formal identity

$$-\log(1-u) = \sum_{k\geq 1} \frac{u^k}{k}.$$

Substituting $u = y^{\Omega_{\mathrm{P}}(p)}$ for each prime p, we obtain

$$-\log(1 - y^{\Omega_{\mathrm{P}}(p)}) = \sum_{k\geq 1} \frac{y^{k\Omega_{\mathrm{P}}(p)}}{k}.$$

Summing over all primes gives

$$\log \zeta_{\mathrm{P}}(y) = \sum_{p} \sum_{k\geq 1} \frac{y^{k\Omega_{\mathrm{P}}(p)}}{k}.$$

Interchanging the order of summation at the formal level, we get

$$\log \zeta_{\mathrm{P}}(y) = \sum_{k\geq 1} \frac{1}{k} \sum_{p} y^{k\Omega_{\mathrm{P}}(p)}.$$

But the inner sum is exactly $P_{\mathrm{P}}(y^k)$, because

$$P_{\mathrm{P}}(y^k) = \sum_{p} (y^k)^{\Omega_{\mathrm{P}}(p)} = \sum_{p} y^{k\Omega_{\mathrm{P}}(p)}.$$

Therefore

$$\log \zeta_{\mathrm{P}}(y) = \sum_{k\geq 1} \frac{1}{k} P_{\mathrm{P}}(y^k),$$

which is the first formula.

We now extract coefficients. Write

$$P_{\mathrm{P}}(y) = \sum_{m\geq 1} b_m y^m, \qquad \log \zeta_{\mathrm{P}}(y) = \sum_{r\geq 1} \ell_r y^r.$$

Substituting the first series into the logarithmic identity yields

$$\sum_{r\geq 1} \ell_r y^r = \sum_{k\geq 1} \frac{1}{k} \sum_{m\geq 1} b_m y^{km}.$$

Fix $r \geq 1$. The coefficient of y^r on the right-hand side comes from precisely those pairs (k, m) with $km = r$, equivalently from the divisors $m \mid r$. For such a divisor m, one has $k = r/m$, and the corresponding contribution is

$$\frac{b_m}{k} = \frac{b_m}{r/m} = \frac{m}{r} b_m.$$

Summing over all divisors $m \mid r$, we get

$$\ell_r = \sum_{m\mid r} \frac{b_m}{r/m} = \frac{1}{r} \sum_{m\mid r} m b_m.$$

Equivalently,

$$r\ell_r = \sum_{m\mid r} m b_m.$$

Now define arithmetic functions

$$F(r) := r\ell_r, \qquad G(r) := r b_r.$$

Then the identity just proved is

$$F(r) = \sum_{d\mid r} G(d).$$

By classical M"obius inversion,

$$G(r) = \sum_{d\mid r} \mu(d) F(r/d).$$

Substituting back the definitions of F and G, we obtain

$$r b_r = \sum_{d\mid r} \mu(d) \frac{r}{d} \ell_{r/d}.$$

Dividing by r gives

$$b_r = \sum_{d\mid r} \frac{\mu(d)}{d} \ell_{r/d},$$

as claimed. Finally, since $\ell_{r/d} = [y^{r/d}] \log \zeta_{\mathrm{P}}(y)$, summing these coefficient identities back into a generating series yields

$$P_{\mathrm{P}}(y) = \sum_{m\geq 1} \frac{\mu(m)}{m} \log \zeta_{\mathrm{P}}(y^m).$$

This completes the proof. □

Corollary 13.4 (Explicit formula for the cumulative function)**.** *For every integer $r \geq 1$,*

$$\Pi_{\mathrm{P}}(r) = \sum_{j\leq r} \sum_{d\mid j} \frac{\mu(d)}{d} [y^{j/d}] \log \zeta_{\mathrm{P}}(y).$$

Equivalently, whenever $0 < \rho < 1$ lies in the disc of analyticity,

$$\Pi_{\mathrm{P}}(r) = \frac{1}{2\pi i} \int_{|y|=\rho} \frac{1}{(1-y)y^{r+1}} \left(\sum_{m\geq 1} \frac{\mu(m)}{m} \log \zeta_{\mathrm{P}}(y^m) \right) dy.$$

Proof. By Definition 13.1, the cumulative counting function is obtained by summing the jump sizes:

$$\Pi_{\mathrm{P}}(r) = \sum_{j \le r} b_j.$$

Proposition 13.3 gives, for each $j \ge 1$, the exact formula

$$b_j = \sum_{d|j} \frac{\mu(d)}{d} \, [y^{j/d}] \log \zeta_{\mathrm{P}}(y).$$

Substituting this into the expression for $\Pi_{\mathrm{P}}(r)$ and summing over $j \le r$ yields

$$\Pi_{\mathrm{P}}(r) = \sum_{j \le r} \sum_{d|j} \frac{\mu(d)}{d} \, [y^{j/d}] \log \zeta_{\mathrm{P}}(y),$$

which is the first formula.

For the contour integral, consider the ordinary generating series of the cumulative function:

$$\sum_{r \ge 0} \Pi_{\mathrm{P}}(r) y^r.$$

Since $\Pi_{\mathrm{P}}(r) = \sum_{j \le r} b_j$, we compute

$$\sum_{r \ge 0} \Pi_{\mathrm{P}}(r) y^r = \sum_{r \ge 0} \left(\sum_{j \le r} b_j \right) y^r.$$

Interchanging the order of summation gives

$$\sum_{r \ge 0} \Pi_{\mathrm{P}}(r) y^r = \sum_{j \ge 1} b_j \sum_{r \ge j} y^r.$$

The inner sum is a geometric tail:

$$\sum_{r \ge j} y^r = \frac{y^j}{1-y}.$$

Therefore

$$\sum_{r \ge 0} \Pi_{\mathrm{P}}(r) y^r = \sum_{j \ge 1} b_j \frac{y^j}{1-y} = \frac{1}{1-y} \sum_{j \ge 1} b_j y^j = \frac{P_{\mathrm{P}}(y)}{1-y}.$$

Now apply Cauchy's coefficient formula to this generating series. For any $0 < \rho < 1$ inside the disk of analyticity,

$$\Pi_{\mathrm{P}}(r) = [y^r] \frac{P_{\mathrm{P}}(y)}{1-y} = \frac{1}{2\pi i} \int_{|y|=\rho} \frac{P_{\mathrm{P}}(y)}{(1-y) y^{r+1}} \, dy.$$

Finally substitute the M"obius-inversion formula

$$P_{\mathrm{P}}(y) = \sum_{m \ge 1} \frac{\mu(m)}{m} \log \zeta_{\mathrm{P}}(y^m)$$

from Proposition 13.3. This yields

$$\Pi_{\mathrm{P}}(r) = \frac{1}{2\pi i} \int_{|y|=\rho} \frac{1}{(1-y) y^{r+1}} \left(\sum_{m \ge 1} \frac{\mu(m)}{m} \log \zeta_{\mathrm{P}}(y^m) \right) dy.$$

This is the desired integral formula. □

The corollary should be compared with the role of $\log \zeta(s)$ in the classical extraction of prime sums from the Riemann zeta function. The formal structure is very close: the logarithm of the global Euler product encodes the prime layers, and M"obius inversion removes the contribution of higher prime powers. What is missing, at present, is the analytic continuation machinery that would allow a Riemann-style contour shift and a sum over zeros.

14 The bivariate Pratt zeta: rigorous framework

The one-variable series counts by height only. To put height and numerical size into a single object, one is naturally led to a two-variable zeta function.

Definition 14.1 (Bivariate Pratt zeta). *For complex parameters y and s, define*

$$\zeta_{\mathrm{P}}(y,s) := \sum_{n\geq 1} \frac{y^{\Omega_{\mathrm{P}}(n)}}{n^s}.$$

Whenever the series converges absolutely, define also

$$\Lambda_{\mathrm{P},y}(p^k) := (\log p)\, y^{k\Omega_{\mathrm{P}}(p)} \qquad (p \in \mathcal{P},\ k \geq 1),$$

with $\Lambda_{\mathrm{P},y}(n) = 0$ if n is not a prime power, and let

$$\Psi_{\mathrm{P},y}(x) := \sum_{n\leq x} \Lambda_{\mathrm{P},y}(n).$$

Proposition 14.2 (Absolute convergence and Euler product). *If $|y| \leq 1$ and $\Re s > 1$, then the series defining $\zeta_{\mathrm{P}}(y,s)$ converges absolutely and*

$$\zeta_{\mathrm{P}}(y,s) = \prod_{p\in\mathcal{P}} \frac{1}{1 - y^{\Omega_{\mathrm{P}}(p)} p^{-s}}.$$

Moreover,

$$-\partial_s \log \zeta_{\mathrm{P}}(y,s) = \sum_{n\geq 1} \frac{\Lambda_{\mathrm{P},y}(n)}{n^s}$$

throughout that half-plane.

Proof. We divide the proof into three steps: absolute convergence, Euler product, and logarithmic derivative.

Step 1: absolute convergence. Assume that $|y| \leq 1$ and $\Re s > 1$. For every integer $n \geq 1$ we have $\Omega_{\mathrm{P}}(n) \geq 0$, hence

$$|y|^{\Omega_{\mathrm{P}}(n)} \leq 1.$$

Therefore

$$\left| \frac{y^{\Omega_{\mathrm{P}}(n)}}{n^s} \right| = \frac{|y|^{\Omega_{\mathrm{P}}(n)}}{n^{\Re s}} \leq \frac{1}{n^{\Re s}}.$$

Summing over all $n \geq 1$ gives the majorization

$$\sum_{n\geq 1} \left| \frac{y^{\Omega_{\mathrm{P}}(n)}}{n^s} \right| \leq \sum_{n\geq 1} \frac{1}{n^{\Re s}}.$$

The series on the right is the ordinary p-series, which converges because $\Re s > 1$. Hence the defining series for $\zeta_{\mathrm{P}}(y,s)$ converges absolutely in this region.

Step 2: Euler product. Define the arithmetic weight

$$f_{y,s}(n) := \frac{y^{\Omega_{\mathrm{P}}(n)}}{n^s}.$$

Because Ω_{P} is additive and $n \mapsto n^{-s}$ is multiplicative, this weight is completely multiplicative:

$$f_{y,s}(mn) = \frac{y^{\Omega_{\mathrm{P}}(mn)}}{(mn)^s} = \frac{y^{\Omega_{\mathrm{P}}(m)+\Omega_{\mathrm{P}}(n)}}{m^s n^s} = f_{y,s}(m) f_{y,s}(n).$$

For a fixed prime p, the local series is therefore

$$\sum_{k\geq 0} f_{y,s}(p^k) = \sum_{k\geq 0} \frac{y^{\Omega_{\mathrm{P}}(p^k)}}{p^{ks}}.$$

Since $\Omega_{\mathrm{P}}(p^k) = k\Omega_{\mathrm{P}}(p)$, this becomes

$$\sum_{k\geq 0} \frac{y^{k\Omega_{\mathrm{P}}(p)}}{p^{ks}} = \sum_{k\geq 0} \left(y^{\Omega_{\mathrm{P}}(p)} p^{-s}\right)^k = \frac{1}{1 - y^{\Omega_{\mathrm{P}}(p)} p^{-s}}.$$

Now absolute convergence just proved allows the standard Euler-product argument, so that

$$\zeta_{\mathrm{P}}(y,s) = \sum_{n\geq 1} f_{y,s}(n) = \prod_{p\in\mathcal{P}} \left(\sum_{k\geq 0} f_{y,s}(p^k)\right) = \prod_{p\in\mathcal{P}} \frac{1}{1 - y^{\Omega_{\mathrm{P}}(p)} p^{-s}}.$$

Step 3: logarithmic derivative. Take logarithms of the Euler product:

$$\log \zeta_{\mathrm{P}}(y,s) = -\sum_p \log\left(1 - y^{\Omega_{\mathrm{P}}(p)} p^{-s}\right).$$

Because we are in a region of absolute convergence, the resulting prime-power expansion may be differentiated term by term with respect to s. Using

$$-\frac{d}{ds} \log(1 - zp^{-s}) = \sum_{k\geq 1} (\log p) z^k p^{-ks},$$

with $z = y^{\Omega_{\mathrm{P}}(p)}$, we obtain

$$-\partial_s \log \zeta_{\mathrm{P}}(y,s) = \sum_p \sum_{k\geq 1} (\log p)\, y^{k\Omega_{\mathrm{P}}(p)} p^{-ks}.$$

But by definition, the coefficient attached to p^k is exactly

$$\Lambda_{\mathrm{P},y}(p^k) = (\log p)\, y^{k\Omega_{\mathrm{P}}(p)},$$

while the coefficients at non-prime powers are zero. Thus the double sum may be rewritten as the Dirichlet series

$$-\partial_s \log \zeta_{\mathrm{P}}(y,s) = \sum_{n\geq 1} \frac{\Lambda_{\mathrm{P},y}(n)}{n^s}.$$

This proves the proposition. □

Proposition 14.3 (A second rigorous convergence region)**.** *For $|y| < 1$, the series defining $\zeta_{\mathrm{P}}(y,s)$ converges absolutely in the larger half-plane*

$$\Re s > 1 + \log_2 |y|.$$

Proof. The proof is a direct comparison with an ordinary Dirichlet series. Start from the global reconstruction inequality already established earlier:

$$\log n = \langle \Phi_{\mathrm{P}}(n), w \rangle \leq (\log 2)\Omega_{\mathrm{P}}(n).$$

Divide both sides by $\log 2$. Since $\log 2 > 0$, the direction of the inequality is preserved, and we get

$$\Omega_{\mathrm{P}}(n) \geq \frac{\log n}{\log 2} = \log_2 n.$$

This lower bound on the Pratt height is the key input.

Now assume $|y| < 1$. Because the function $t \mapsto |y|^t$ is decreasing for positive t, the inequality $\Omega_{\mathrm{P}}(n) \geq \log_2 n$ implies

$$|y|^{\Omega_{\mathrm{P}}(n)} \leq |y|^{\log_2 n}.$$

Using the identity

$$a^{\log_b n} = n^{\log_b a} \qquad (a > 0, b > 0, b \neq 1),$$

with $a = |y|$ and $b = 2$, we rewrite the right-hand side as

$$|y|^{\log_2 n} = n^{\log_2 |y|}.$$

Hence each term of the bivariate Pratt zeta satisfies

$$\left| \frac{y^{\Omega_{\mathrm{P}}(n)}}{n^s} \right| = \frac{|y|^{\Omega_{\mathrm{P}}(n)}}{n^{\Re s}} \leq \frac{n^{\log_2 |y|}}{n^{\Re s}} = n^{-(\Re s - \log_2 |y|)}.$$

Therefore the series

$$\sum_{n \geq 1} \left| \frac{y^{\Omega_{\mathrm{P}}(n)}}{n^s} \right|$$

is dominated termwise by the p-series

$$\sum_{n \geq 1} n^{-(\Re s - \log_2 |y|)}.$$

This converges precisely when its exponent is strictly greater than 1, namely when

$$\Re s - \log_2 |y| > 1.$$

Equivalently,

$$\Re s > 1 + \log_2 |y|.$$

Thus the defining series for $\zeta_{\mathrm{P}}(y, s)$ converges absolutely throughout that half-plane. $\square$

Corollary 14.4 (Perron-type explicit formula)**.** *Fix $|y| \leq 1$ and $c > 1$. Then formally, and analytically whenever the usual Perron hypotheses are justified,*

$$\Psi_{\mathrm{P},y}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (-\partial_s \log \zeta_{\mathrm{P}}(y, s)) \frac{x^s}{s} \, ds.$$

Proof. By Proposition 14.2, in the half-plane $\Re s > 1$ one has the Dirichlet series expansion

$$-\partial_s \log \zeta_{\mathrm{P}}(y, s) = \sum_{n \geq 1} \frac{\Lambda_{\mathrm{P},y}(n)}{n^s}.$$

The coefficients $\Lambda_{\mathrm{P},y}(n)$ are supported on prime powers, but for the Perron argument that support property is not essential; what matters is that we have a Dirichlet series with coefficients $\Lambda_{\mathrm{P},y}(n)$.

Now recall the classical Perron inversion principle: if

$$F(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

converges absolutely in some half-plane $\Re s > c_0$, then for every $c > c_0$ one has formally, and rigorously under the usual growth hypotheses,

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} \, ds.$$

We apply this with

$$a_n := \Lambda_{\mathrm{P},y}(n), \qquad F(s) := -\partial_s \log \zeta_{\mathrm{P}}(y,s).$$

Since $c > 1$, we are in the region of absolute convergence provided by Proposition 14.2. Therefore Perron's formula yields

$$\sum_{n \le x} \Lambda_{\mathrm{P},y}(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\partial_s \log \zeta_{\mathrm{P}}(y,s)\right) \frac{x^s}{s}\, ds.$$

By Definition 14.1, the sum on the left-hand side is exactly $\Psi_{\mathrm{P},y}(x)$. Hence

$$\Psi_{\mathrm{P},y}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\partial_s \log \zeta_{\mathrm{P}}(y,s)\right) \frac{x^s}{s}\, ds.$$

This is the claimed Perron-type explicit formula. □

At this point the analogy with the classical theory is already visible. For the ordinary zeta function one continues $\zeta(s)$ analytically, studies its zeros, and then shifts the Perron contour to derive an explicit formula for the Chebyshev function. For $\zeta_{\mathrm{P}}(y,s)$ we presently have only the Euler product and its region of absolute convergence. We do *not* yet have an analytic continuation, a functional equation, or a proved description of the zero set.

15 The general bivariate heuristic and the candidate singular line

We now pass from rigorous statements to a heuristic picture. The guiding numerical observation is that for typical primes one seems to have

$$\Omega_{\mathrm{P}}(p) \approx 2 \log p.$$

Assume this heuristic relation for the moment. Then

$$y^{\Omega_{\mathrm{P}}(p)} \approx y^{2\log p} = e^{(2\log y)\log p} = p^{2\log y}.$$

Substituting this into the Euler product of Proposition 14.2 gives the formal model

$$\zeta_{\mathrm{P}}(y,s) \approx \prod_p \frac{1}{1 - p^{-(s-2\log y)}} = \zeta(s - 2\log y).$$

This comparison suggests a very simple geometric rule in the (y,s)-plane:

- the classical pole at $s = 1$ should move to the candidate pole line

$$\Re s = 1 + 2\log|y|;$$

- the classical critical line $\Re s = 1/2$ should move to the candidate zero line

$$\Re s = \frac{1}{2} + 2\log|y|.$$

We stress that this is a *heuristic model*, not a theorem. Nevertheless it organizes the numerical data very naturally.

16 Specialization I: the classical Riemann setting

Let us first specialize the bivariate heuristic to the classical point $y = 1$. Then

$$\zeta_{\mathrm{P}}(1, s) = \sum_{n \geq 1} n^{-s} = \zeta(s),$$

because the weight $1^{\Omega_{\mathrm{P}}(n)}$ is identically equal to 1. Therefore the candidate pole line becomes

$$\Re s = 1,$$

and the candidate critical line becomes

$$\Re s = \frac{1}{2},$$

which is exactly the classical Riemann picture. In particular, the whole bivariate formalism is compatible with the ordinary zeta function at $y = 1$. Thus the proposed bivariate heuristic is not an arbitrary analogy; it really extends the classical theory at one distinguished boundary value.

17 Specialization II: the pure Pratt setting and the radius formula

The one-variable Pratt zeta is the slice

$$\zeta_{\mathrm{P}}(y) = \zeta_{\mathrm{P}}(y, 0).$$

If one applies the heuristic pole-line formula

$$\Re s = 1 + 2 \log |y|,$$

to the slice $s = 0$, one arrives at the condition

$$0 = 1 + 2 \log |y|.$$

Solving for $|y|$ gives the candidate dominant singularity

$$\boxed{R_{\mathrm{heur}} = e^{-1/2} \approx 0.60653.}$$

This agrees strikingly well with the direct numerical estimation of the radius of convergence of $\zeta_{\mathrm{P}}(y)$ obtained from the coefficients A_r. A computation of the coefficients up to height 22 gives a regression

$$\log A_r \approx 0.5024\, r - 0.3956,$$

which corresponds to

$$R \approx e^{-0.5024} \approx 0.605.$$

The numerical agreement with $e^{-1/2}$ is strong evidence that the shifted-zeta heuristic captures a real structural phenomenon. However, it is still only evidence. No proof is currently known.

18 A Pratt-Riemann hypothesis for zeros

We are now in a position to formulate the natural zero-distribution conjecture suggested by the previous section.

Question 18.1 (Heuristic Pratt-Riemann hypothesis)**.** *Fix $0 < |y| \le 1$. Does the nontrivial zero set of $\zeta_{\mathrm{P}}(y,s)$, after analytic continuation in s, lie on the vertical line*

$$\Re s = \frac{1}{2} + 2\log|y|?$$

This should be read as the direct bivariate analogue of RH. Indeed:

- at $y = 1$, the conjectural line is the classical critical line $\Re s = 1/2$;
- on the slice $s = 0$, the candidate pole location predicts the radius formula $R = e^{-1/2}$.

So the two apparently different phenomena — classical RH on the one hand, and the Pratt radius formula on the other — become aspects of a single bivariate picture.

It is important to separate three levels of certainty.

1. **Proved.** The Euler product for $\zeta_{\mathrm{P}}(y,s)$ and the regions of absolute convergence in Propositions 14.2 and 14.3.
2. **Numerically supported.** The approximate law $\Omega_{\mathrm{P}}(p) \approx 2\log p$ and the radius estimate $R \approx 0.605$.
3. **Conjectural.** Analytic continuation, a functional equation, an explicit formula by contour shifting, and the zero-line statement in Question 18.1.

At present only the first level is rigorous. The second is experimental. The third is a genuine research program.

19 Can one deduce superadditivity from the explicit formulas?

One might ask whether the explicit formulas for $P_{\mathrm{P}}(y)$ and $\Pi_{\mathrm{P}}(r)$ force some structural inequality such as superadditivity. The answer is: not directly. The reason is that the extraction of prime layers from the global height zeta function uses M"obius inversion,

$$P_{\mathrm{P}}(y) = \sum_{m\ge1} \frac{\mu(m)}{m} \log \zeta_{\mathrm{P}}(y^m),$$

which contains strong sign changes. Therefore positivity properties are not transparent at the level of the explicit formula. In particular, a statement such as

$$\Pi_{\mathrm{P}}(r+s) \ge \Pi_{\mathrm{P}}(r) + \Pi_{\mathrm{P}}(s)$$

does not formally follow from the Euler product or from Corollary 13.4.

What *would* imply eventual superadditivity is a sufficiently precise singularity analysis. For instance, if one could prove an asymptotic formula of the shape

$$\Pi_{\mathrm{P}}(r) \sim C\frac{R^{-r}}{r^{\gamma}} \qquad (R<1),$$

then the exponential factor R^{-r} would imply

$$\Pi_{\mathrm{P}}(r+s) \gg \Pi_{\mathrm{P}}(r) + \Pi_{\mathrm{P}}(s)$$

for all sufficiently large r, s. Thus the route would be

analytic continuation and singularity analysis $\Longrightarrow$ asymptotic formula $\Longrightarrow$ eventual superadditivity.

But this route is not yet available rigorously.

20 Final separation: theorem, numerical law, conjecture

To close this analytical part, let us record the logical status of the main statements.

Rigorously proved in this note. The global weight formula $\log n = \langle \Phi_{\mathrm{P}}(n), w \rangle$, the inequality $n \le 2^{\Omega_{\mathrm{P}}(n)}$, the elementary bound $\Omega_{\mathrm{P}}(n) \le n - 1$, the lower bounds

$$\Pi_{\mathrm{P}}(r) \ge \pi(r+1), \qquad \Pi_{\mathrm{P}}(r) \ge \lfloor \log_2(r+1) \rfloor,$$

and the one- and two-variable Euler products and explicit coefficient formulas for the Pratt zeta functions.

Strongly suggested by computation. The approximate law $\Omega_{\mathrm{P}}(p) \approx 2 \log p$, the heuristic asymptotic

$$\Pi_{\mathrm{P}}(r) \asymp \frac{e^{r/2}}{r},$$

and the one-variable radius estimate

$$R \approx 0.605.$$

Conjectural research direction. The shifted-zeta heuristic

$$\zeta_{\mathrm{P}}(y, s) \approx \zeta(s - 2 \log y),$$

the exact radius formula

$$R = e^{-1/2},$$

and the bivariate Pratt-Riemann hypothesis that the nontrivial zeros lie on the line

$$\Re s = \frac{1}{2} + 2 \log |y|.$$

The usefulness of the bivariate point of view is that it places the classical Riemann picture and the purely Pratt picture into one common framework. At $y = 1$ one recovers the ordinary zeta function and its critical line. At $s = 0$ one sees the radius problem for the Pratt height series. The conjectural identity $R = e^{-1/2}$ is exactly the intersection of those two viewpoints.

21 Summary of the new picture

We end by collecting the main points.

1. The ordinary exponent theory uses the standard simplex

$$Q_N = \mathrm{conv}\{0, e_{p_1}, \dots, e_{p_d}\},$$

whose Ehrhart polynomial is $\binom{d+t}{d}$.

2. The Pratt vector satisfies

$$\Phi_{\mathrm{P}}(n) = Av(n),$$

where A is lower unitriangular. Therefore the transported simplex

$$\widetilde{Q}_N = \mathrm{conv}\{0, \Phi_{\mathrm{P}}(p_1), \dots, \Phi_{\mathrm{P}}(p_d)\}$$

has the same Ehrhart polynomial as Q_N.

3. The genuine Pratt cutoff uses the total Pratt weight

$$\Omega_{\mathrm{P}}(n) = \sum_p m_p(n) = \sum_{j=1}^{d} c_j x_j$$

for $n = p_1^{x_1} \cdots p_d^{x_d}$. This leads to the rational simplex

$$P_N = \{x \geq 0 : c_1 x_1 + \cdots + c_d x_d \leq 1\},$$

whose lattice points count integers with bounded Pratt weight.

4. The layer generating series is

$$\sum_{t \geq 0} a_{N,t} z^t = \prod_{j=1}^{d} \frac{1}{1 - z^{c_j}}.$$

Thus the Pratt-Ehrhart theory is explicit even though it is usually quasipolynomial rather than polynomial.

5. There is a global weight vector

$$w = \sum_p -\log\left(1 - \frac{1}{p}\right) e_p$$

with the exact reconstruction formula

$$\log n = \langle \Phi_{\mathrm{P}}(n), w \rangle, \qquad n = \exp(\langle \Phi_{\mathrm{P}}(n), w \rangle).$$

This is the key new rigidity absent from the ordinary valuation picture.

6. Because $w_p \leq \log 2$ for all primes p, one gets

$$n \leq 2^{\Omega_{\mathrm{P}}(n)}.$$

Hence the Pratt ball $\Omega_{\mathrm{P}}(n) \leq r$ is automatically contained in the interval $[1, 2^r]$. Therefore the global set $\{n : \Omega_{\mathrm{P}}(n) \leq r\}$ is finite and is counted by a finite-dimensional Ehrhart region.

7. This proves that Pratt-Ehrhart theory really does connect a coordinate cutoff to an ordinary size cutoff. That is the precise sense in which it can help count primes. However, the raw Ehrhart count still counts composites as well as primes, so additional arithmetic input is needed to recover $\pi(x)$ itself.

22 Conclusion

We have shown that the passage from the ordinary exponent vector $v(n)$ to the Pratt valuation vector $\Phi_{\mathrm{P}}(n)$ preserves enough linear structure to support a meaningful Ehrhart-theoretic framework, while at the same time introducing a rigidity that is absent in the classical setting.

On the geometric side, two distinct constructions emerge. The first is the transported simplex obtained from the ordinary exponent simplex through the finite Pratt matrix A. Since A is lower unitriangular, this construction is unimodularly equivalent to the classical one and therefore has the same Ehrhart polynomial. The second is the genuine Pratt-height simplex defined by the cutoff $\Omega_{\mathrm{P}}(n) \leq r$. This object is more intrinsic to the Pratt theory and leads naturally to a rational generating-series description of its lattice layers.

The main structural novelty is the global reconstruction formula

$$\log n = \langle \Phi_{\mathrm{P}}(n), w \rangle, \qquad w_p = -\log\left(1 - \frac{1}{p}\right),$$

which implies

$$\Omega_{\mathrm{P}}(n) \le r \implies n \le 2^r.$$

Thus a linear cutoff in Pratt coordinates automatically gives an ordinary size cutoff. This creates a direct bridge between lattice geometry in Pratt space and finite counting problems for integers and primes. In particular, it yields the elementary but conceptually important lower bound

$$\Pi_{\mathrm{P}}(r) \ge \pi(r+1),$$

and hence nontrivial lower bounds for $\Pi_{\mathrm{P}}(\log_2 x)$ depending only on x.

On the analytic side, we introduced the one-variable and two-variable Pratt zeta functions,

$$\zeta_{\mathrm{P}}(y) = \sum_{n \ge 1} y^{\Omega_{\mathrm{P}}(n)}, \qquad \zeta_{\mathrm{P}}(y, s) = \sum_{n \ge 1} \frac{y^{\Omega_{\mathrm{P}}(n)}}{n^s},$$

together with their Euler products, logarithmic expansions, and Möbius inversion formulae. These identities provide a formal analogue of the classical passage from $\zeta(s)$ to prime-counting information. At present, however, the decisive analytic ingredients of the classical theory — analytic continuation, functional equation, and a controlled zero set — are not yet available in the Pratt setting. For that reason, the analogy with explicit formulae and with the Riemann hypothesis remains heuristic rather than rigorous.

The numerical evidence nevertheless points in a striking direction. The data suggest that, for typical primes,

$$\Omega_{\mathrm{P}}(p) \approx 2 \log p,$$

equivalently

$$\Pi_{\mathrm{P}}(r) \asymp \frac{e^{r/2}}{r},$$

and that the dominant singularity R of the one-variable Pratt zeta may satisfy

$$R \approx e^{-1/2}.$$

In the two-variable picture this corresponds heuristically to the shifted model

$$\zeta_{\mathrm{P}}(y, s) \approx \zeta(s - 2\log y),$$

which in turn suggests a Pratt analogue of the critical line.

Several directions remain open. One would like to understand the growth of the layer counts

$$\#\{n : \Omega_{\mathrm{P}}(n) = r\},$$

to study the singularities of $\zeta_{\mathrm{P}}(y)$ and $\zeta_{\mathrm{P}}(y, s)$ by analytic-combinatorial methods, and to clarify whether any part of the observed exponential behavior of $\Pi_{\mathrm{P}}(r)$ can be proved without appealing to deep prime-distribution theorems. More ambitiously, one may ask whether the Pratt zeta functions admit an analytic theory strong enough to support a genuine explicit formula and a meaningful zero-distribution conjecture.

In this sense, the present paper should be viewed as a first step. It shows that Pratt valuations are not only compatible with the Ehrhart viewpoint, but in fact lead to a richer interaction between lattice geometry, multiplicative structure, and analytic prime-counting questions than the ordinary valuation model.