Sequence of Graphs and the Riemann Hypothesis
From the MathOverflow Valuation Graph to the Pratt Cover Graph

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Contents

1 Introduction

This note is organized around two MathOverflow questions. The first asks about a graph on the natural numbers defined from ordinary prime-adic valuations. The second asks how one may reinterpret the prime number theorem and the Riemann hypothesis as cancellation statements for a sequence of graphs.

The goals of the note are the following.

(1) We write out the arithmetic and graph-theoretic content of those two MathOverflow questions in detail.

(2) We then restart the story with the *Pratt valuations* $m_p(n)$, define the correct Pratt graph as a cover graph, and ask whether this new sequence of graphs also satisfies the prime number theorem or the Riemann hypothesis *in the sense of the MathOverflow question.*

(3) We prove several basic facts about the Pratt cover graph, in particular that the vertex 1 has degree exactly 1.

One theme runs through the whole note. The ordinary valuation vector and the Pratt valuation vector are related by a lower unitriangular matrix

$$\phi(n) = A\, v(n).$$

So the Pratt story is not disconnected from the classical valuation story; rather, it is a recursively enriched coordinate system on the same multiplicative arithmetic.

2 The first MathOverflow graph: the valuation graph

2.1 Valuation vectors

Let $p_1 < p_2 < \cdots$ be the ordered sequence of prime numbers. For $n \in \mathbb{N}$, define the ordinary valuation vector

$$v(n) := (v_{p_1}(n), v_{p_2}(n), \dots) \in c_{00}(\mathbb{Z}),$$

where $c_{00}(\mathbb{Z})$ denotes the finitely supported integer sequences.

By unique factorization,

$$n = \prod_p p^{v_p(n)}.$$

Hence the map $n \mapsto v(n)$ records the multiplicative structure of n exactly.

2.2 Definition of the graph sequence

For each $N \geq 1$, define a graph $G_N = (V_N, E_N)$ by

$$V_N = \{1, 2, \dots, N\},$$

and

$$\{a, b\} \in E_N \quad \Longleftrightarrow \quad \frac{a}{b} \text{ or } \frac{b}{a} \text{ is prime.}$$

Equivalently, a and b are adjacent iff one obtains one of them from the other by multiplying or dividing by a single prime.

2.3 Distance formula

The key fact in the second MathOverflow question is an explicit formula for graph distance.

Theorem 2.1. *For all $a, b \in V_N$, the shortest-path distance in G_N is*

$$d_N(a,b) = \sum_p |v_p(a) - v_p(b)|.$$

Equivalently,

$$d_N(a,b) = \Omega\left(\frac{ab}{\gcd(a,b)^2}\right),$$

where

$$\Omega(m) := \sum_p v_p(m)$$

counts prime factors with multiplicity.

Proof. Write

$$a = \prod_p p^{\alpha_p}, \qquad b = \prod_p p^{\beta_p}.$$

Every edge changes exactly one prime exponent by $+1$ or -1. Therefore any path from a to b must change the p-th exponent at least $|\alpha_p - \beta_p|$ times. Summing over all primes gives the lower bound

$$d_N(a,b) \geq \sum_p |\alpha_p - \beta_p|.$$

For the reverse inequality, let

$$g := \gcd(a,b) = \prod_p p^{\min(\alpha_p, \beta_p)}.$$

Starting from a, divide successively by the prime factors of a/g, one prime at a time, until one reaches g. This takes

$$\Omega(a/g) = \sum_p (\alpha_p - \min(\alpha_p, \beta_p))$$

steps. Then multiply successively by the prime factors of b/g, one prime at a time, until one reaches b. This takes

$$\Omega(b/g) = \sum_p (\beta_p - \min(\alpha_p, \beta_p))$$

steps. So there is a path of length

$$\Omega(a/g) + \Omega(b/g) = \sum_p |\alpha_p - \beta_p|.$$

Hence equality holds.

Finally,

$$\frac{ab}{g^2} = \prod_p p^{|\alpha_p - \beta_p|},$$

so

$$\Omega\left(\frac{ab}{g^2}\right) = \sum_p |\alpha_p - \beta_p|.$$

This proves the second formula as well. □

2.4 Bipartiteness and the Liouville function

Define

$$\lambda(n) := (-1)^{\Omega(n)}.$$

Since each edge changes exactly one prime exponent by 1, every edge flips the parity of Ω. Therefore G_N is bipartite with bipartition determined by the sign of λ.

Moreover, taking $a = 1$ in Theorem 2.1 gives

$$d_N(1, n) = \Omega(n).$$

Hence

$$\lambda(n) = (-1)^{\Omega(n)} = (-1)^{d_N(1,n)}.$$

This is the central observation in the second MathOverflow question.

3 The second MathOverflow question: graph-theoretic PNT and RH

The second MathOverflow question proposes the following general point of view.

Definition 3.1. *Let*

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_N \subseteq \cdots$$

be an increasing sequence of finite connected graphs, and suppose each graph contains a distinguished base vertex v_0. We say that this sequence satisfies:

(a) the prime number theorem in the sense of the MathOverflow question *if*

$$\lim_{N\to\infty} \frac{\sum_{v\in V(\Gamma_N)} (-1)^{d_{\Gamma_N}(v_0,v)}}{|V(\Gamma_N)|} = 0,$$

(b) the Riemann hypothesis in the sense of the MathOverflow question *if for every $\varepsilon > 0$,*

$$\lim_{N\to\infty} \frac{\sum_{v\in V(\Gamma_N)} (-1)^{d_{\Gamma_N}(v_0,v)}}{|V(\Gamma_N)|^{1/2+\varepsilon}} = 0.$$

For the valuation graph G_N with basepoint 1, these conditions become the classical Liouville summatory statements

$$\frac{1}{N} \sum_{n\le N} \lambda(n) \to 0$$

and

$$\sum_{n\le N} \lambda(n) = o_\varepsilon(N^{1/2+\varepsilon}).$$

So in this special case the graph-theoretic formulation is literally equivalent to the classical one.

3.1 The easy path-graph example

The MathOverflow post also gives a very simple example.

Definition 3.2. *Let P_N be the path graph on $\{1, 2, \ldots, N\}$, with edges*

$$\{x, y\} \in E(P_N) \iff |x - y| = 1.$$

Take the basepoint to be 1.

Proposition 3.3. *For the path graph P_N, one has*

$$\sum_{v \in V(P_N)} (-1)^{d_{P_N}(1,v)} = \begin{cases} 1, & N \text{ odd}, \\ 0, & N \text{ even}. \end{cases}$$

In particular, the sequence $(P_N, 1)$ satisfies both the prime number theorem and the Riemann hypothesis in the sense of the MathOverflow question.

Proof. Since P_N is a path, the distance from 1 to v is simply

$$d_{P_N}(1, v) = v - 1.$$

Therefore

$$\sum_{v \in V(P_N)} (-1)^{d_{P_N}(1,v)} = \sum_{v=1}^{N} (-1)^{v-1}.$$

This alternating sum is 1 if N is odd and 0 if N is even.

Hence

$$\left| \sum_{v \in V(P_N)} (-1)^{d_{P_N}(1,v)} \right| \leq 1.$$

Dividing by N proves the prime number theorem in the sense of the MathOverflow question, and dividing by $N^{1/2+\varepsilon}$ proves the Riemann hypothesis in that same sense. □

The path graph also has

$$\deg_{P_N}(1) = 1 \qquad (N \geq 2).$$

This fact will reappear for the Pratt graph.

4 Pratt valuations and the matrix A

4.1 Pratt forests and Pratt valuations

For a prime q, define the Pratt tree T_q recursively:

- T_2 consists of one single vertex labeled 2;
- if $q > 2$ is prime, then the root is labeled q, and for each prime divisor $r \mid q - 1$ one attaches $v_r(q - 1)$ children labeled r, each carrying a copy of T_r.

For a general positive integer

$$n = \prod_q q^{v_q(n)},$$

the Pratt forest of n is the disjoint union of $v_q(n)$ copies of T_q for all primes q.

Definition 4.1. *For a prime p, let $m_p(n)$ be the number of vertices labeled p in the Pratt forest of n. The global Pratt valuation vector is*

$$\Phi_{\mathrm{P}}(n) := (m_p(n))_{p\in\mathbb{P}}.$$

Proposition 4.2. *The Pratt valuation map is additive under multiplication:*

$$\Phi_{\mathrm{P}}(xy) = \Phi_{\mathrm{P}}(x) + \Phi_{\mathrm{P}}(y), \qquad m_p(xy) = m_p(x) + m_p(y).$$

Proof. The Pratt forest of xy is obtained by taking the disjoint union of the Pratt forests of x and y, with multiplicities given by the exponents in the prime factorization of xy. Therefore the total number of vertices labeled p is the sum of the corresponding numbers for x and y. □

4.2 The matrix A

Let $p_1 < p_2 < \cdots$ be the ordered primes. Define the infinite matrix

$$A = (a_{ij})_{i,j\geq 1}, \qquad a_{ij} := m_{p_i}(p_j).$$

For each n, let

$$v(n) := (v_{p_1}(n), v_{p_2}(n), \dots)$$

be the ordinary valuation vector, and let

$$\phi(n) := (m_{p_1}(n), m_{p_2}(n), \dots)$$

be the Pratt valuation vector in the same coordinates.

Proposition 4.3. *For every $n \in \mathbb{N}$, one has*

$$\phi(n) = A\, v(n).$$

Moreover, A is lower triangular with diagonal entries equal to 1.

Proof. Write

$$n = \prod_j p_j^{v_{p_j}(n)}.$$

By Proposition 4.2,

$$\phi(n) = \sum_j v_{p_j}(n)\, \phi(p_j).$$

By definition, the vector $\phi(p_j)$ is the j-th column of A. Therefore the displayed identity is exactly the matrix equation $\phi(n) = Av(n)$.

Now fix a prime p_j. In the Pratt tree of p_j, no prime larger than p_j can occur, because every child comes from a prime divisor of $p_j - 1$, hence is strictly smaller than p_j, and the same remains true recursively. Thus $m_{p_i}(p_j) = 0$ for $i > j$, so A is lower triangular.

Finally, the root of T_{p_j} is labeled p_j, so $m_{p_j}(p_j) = 1$. Hence every diagonal entry is 1. □

4.3 The Pratt order

Definition 4.4. *For positive integers a, b, define*

$$a \leq_{\mathrm{P}} b \quad\Longleftrightarrow\quad m_p(a) \leq m_p(b) \text{ for every prime } p.$$

This is the coordinatewise order on the Pratt valuation vectors.

5 The Pratt cover graph

5.1 Definition

The correct graph attached to the Pratt order is the cover graph, not the full comparability graph.

Definition 5.1. *For each $N \geq 1$, define the finite graph*

$$\Gamma_N^{\mathrm{P}} = (V_N, E_N), \qquad V_N = \{1, 2, \ldots, N\},$$

by declaring that $\{a, b\} \in E_N$ if and only if one of the following equivalent conditions holds:

(i) $a <_{\mathrm{P}} b$ and there is no $c \in \{1, \ldots, N\}$ with

$$a <_{\mathrm{P}} c <_{\mathrm{P}} b;$$

(ii) $b <_{\mathrm{P}} a$ and there is no $c \in \{1, \ldots, N\}$ with

$$b <_{\mathrm{P}} c <_{\mathrm{P}} a.$$

In other words, Γ_N^{P} is the undirected Hasse graph of the finite poset $(\{1, \ldots, N\}, \leq_{\mathrm{P}})$.

5.2 A first structural fact: every nontrivial Pratt forest contains a 2

Lemma 5.2. *For every integer $n > 1$, one has*

$$m_2(n) \geq 1.$$

Proof. We first prove the claim for primes. If $q = 2$, then $m_2(2) = 1$ by definition. If $q > 2$ is prime, then $q - 1$ is even, so the prime 2 divides $q - 1$. Therefore the root of T_q has at least one child labeled 2, carrying a copy of T_2. Hence $m_2(q) \geq 1$.

Now let

$$n = \prod_q q^{v_q(n)}$$

with $n > 1$. Then some prime q satisfies $v_q(n) \geq 1$. The Pratt forest of n contains at least one copy of T_q, and we just proved that every such tree contains a vertex labeled 2. Hence $m_2(n) \geq 1$. □

5.3 The unique neighbor of 1

Proposition 5.3. *For every $N \geq 2$, the vertex 1 has degree exactly 1 in Γ_N^{P}. More precisely, its unique neighbor is 2:*

$$\deg_{\Gamma_N^{\mathrm{P}}}(1) = 1.$$

Proof. First, $1 <_{\mathrm{P}} 2$, because all Pratt valuations of 1 are zero while $m_2(2) = 1$. Since there is no positive integer strictly between 1 and 2, this shows that 1 and 2 are adjacent in Γ_N^{P}.

Now let $b > 2$. By Lemma 5.2, we have $m_2(b) \geq 1$. For every prime $p \neq 2$, one has $m_p(2) = 0 \leq m_p(b)$. Also $m_2(2) = 1 \leq m_2(b)$. Therefore

$$2 \leq_{\mathrm{P}} b.$$

This inequality is strict because $b \neq 2$, and the Pratt valuation map is injective. Hence for every $b > 2$,

$$1 <_{\mathrm{P}} 2 <_{\mathrm{P}} b.$$

So no such b can cover 1. Therefore the only cover of 1 is 2, and 1 has degree exactly 1. □

Proposition 5.4. *For every $N \geq 1$, the graph Γ_N^{P} is connected.*

Proof. The poset $(\{1, \dots, N\}, \leq_{\mathrm{P}})$ is finite. Let $n \in \{1, \dots, N\}$. If $n = 1$, there is nothing to prove. If $n > 1$, then $1 <_{\mathrm{P}} n$ by Lemma 5.2. In a finite poset, every element above the minimum can be connected to the minimum by a saturated chain of covers. Thus there exist

$$1 = a_0 \prec_P a_1 \prec_P \cdots \prec_P a_k = n.$$

Each cover relation gives an edge of the Hasse graph, so this yields a path from 1 to n in Γ_N^{P}. Hence the graph is connected. □

5.4 Shortest paths in the Pratt graph

We now imitate the argument from the first MathOverflow graph as closely as possible.

Definition 5.5. *For $a \in \mathbb{N}$, define the finite set*

$$S_a^{\mathrm{P}} := \{(p, i) : p \text{ prime}, \ 1 \leq i \leq m_p(a)\}.$$

Thus the p-th Pratt coordinate contributes exactly $m_p(a)$ formal units $(p, 1), \dots, (p, m_p(a))$.

Lemma 5.6. *For all positive integers a, b one has*

$$|S_a^{\mathrm{P}} \,\Delta\, S_b^{\mathrm{P}}| = \sum_p |m_p(a) - m_p(b)|.$$

Proof. For each prime p, the contribution of the p-th coordinate to the symmetric difference is

$$\big|\{1, \dots, m_p(a)\} \,\Delta\, \{1, \dots, m_p(b)\}\big| = |m_p(a) - m_p(b)|.$$

Summing over all primes gives the formula. □

Lemma 5.7. *For every prime q one has*

$$\phi(q) - \phi(q-1) = e_q,$$

where e_q denotes the standard basis vector supported at the coordinate indexed by q.

Proof. The Pratt tree of the prime q is obtained from the Pratt forest of $q - 1$ by adding one new root labeled q. Therefore the number of vertices labeled q increases by 1, while for every other prime $p \neq q$ the number of vertices labeled p stays unchanged. This is exactly the displayed formula. □

Theorem 5.8. *Let $a, b \in \mathbb{N}$ with $a <_{\mathrm{P}} b$. Then the following are equivalent:*

(i) $a \prec_P b$*;*

(ii) $\phi(b) - \phi(a) = e_q$ *for some prime* q*;*

(iii) $b/a = q/(q-1)$ *for some prime* q*.*

In particular, if $a \prec_P b$, then

$$|S_a^{\mathrm{P}} \,\Delta\, S_b^{\mathrm{P}}| = 1.$$

Proof. The equivalence of (ii) and (iii) follows from Proposition 4.3 and Lemma 5.7. Indeed,

$$\phi(b) - \phi(a) = A(v(b) - v(a)),$$

and by Lemma 5.7 we also have

$$e_q = A(v(q) - v(q-1)).$$

Since A is invertible over finitely supported sequences, condition (ii) is equivalent to

$$v(b) - v(a) = v(q) - v(q-1),$$

which is equivalent to

$$\frac{b}{a} = \frac{q}{q-1}.$$

Next we prove that (ii) implies (i). If $\phi(b) - \phi(a) = e_q$ and $a \leq_{\mathrm{P}} c \leq_{\mathrm{P}} b$, then coordinatewise

$$\phi(a) \leq \phi(c) \leq \phi(a) + e_q.$$

Since all coordinates are integers, either $\phi(c) = \phi(a)$ or $\phi(c) = \phi(b)$. By injectivity of the Pratt valuation map, either $c = a$ or $c = b$. Hence there is no strict intermediate element, so $a \prec_P b$.

Finally, assume (i) and put

$$\delta := \phi(b) - \phi(a).$$

Then δ is a nonzero vector with nonnegative integer coordinates. Suppose that $\delta \neq e_q$ for every prime q. Let q be the largest prime with $\delta_q > 0$. Since A is lower triangular with diagonal entries 1, the q-th coordinate of $v(b) - v(a)$ is also strictly positive. Hence q divides b/a in the ordinary valuation sense, and therefore

$$c := b\frac{q-1}{q}$$

is a positive integer.

Using Lemma 5.7, we get

$$\phi(c) = \phi(b) - e_q.$$

Because $\delta_q > 0$, all coordinates of $\phi(c) - \phi(a) = \delta - e_q$ are still nonnegative, so $a \leq_{\mathrm{P}} c \leq_{\mathrm{P}} b$. Since $\delta \neq e_q$, we have $c \neq a$, and clearly $c \neq b$. Thus

$$a <_{\mathrm{P}} c <_{\mathrm{P}} b,$$

contradicting the assumption that $a \prec_P b$. Therefore $\delta = e_q$ for some prime q, which is (ii).

The final statement follows from Lemma 5.6. □

Corollary 5.9. *If $\{a, b\}$ is an edge of the Pratt cover graph, then*

$$\sum_p |m_p(a) - m_p(b)| = 1.$$

Equivalently, along every edge exactly one Pratt coordinate changes, and it changes by ± 1.

Definition 5.10. *Define the total Pratt weight by*

$$\Omega_{\mathrm{P}}(n) := \sum_p m_p(n).$$

Proposition 5.11. *If $a \le_{\mathrm{P}} b$, then the shortest-path distance in the Pratt cover graph is*

$$d_{\Gamma_N^{\mathrm{P}}}(a,b) = \Omega_{\mathrm{P}}(b) - \Omega_{\mathrm{P}}(a) = \sum_p (m_p(b) - m_p(a)).$$

Proof. By Corollary 5.9, each edge changes Ω_{P} by exactly 1. Hence any path from a to b has length at least $\Omega_{\mathrm{P}}(b) - \Omega_{\mathrm{P}}(a)$.

For the reverse inequality, if $a = b$ there is nothing to prove. Assume $a <_{\mathrm{P}} b$. If $a \prec_P b$, then Theorem 5.8 gives a path of length 1, and the formula is clear. Otherwise, by the proof of Theorem 5.8, there exists an integer c with

$$a <_{\mathrm{P}} c <_{\mathrm{P}} b$$

and

$$\Omega_{\mathrm{P}}(c) = \Omega_{\mathrm{P}}(b) - 1.$$

Applying induction on $\Omega_{\mathrm{P}}(b) - \Omega_{\mathrm{P}}(a)$ to the interval $[a, c]$, we obtain a path from a to c of length $\Omega_{\mathrm{P}}(c) - \Omega_{\mathrm{P}}(a)$. Appending the edge from c to b yields a path from a to b of length

$$\Omega_{\mathrm{P}}(c) - \Omega_{\mathrm{P}}(a) + 1 = \Omega_{\mathrm{P}}(b) - \Omega_{\mathrm{P}}(a).$$

Thus equality holds. □

Theorem 5.12. *For all $a, b \in \{1, \dots, N\}$, the shortest-path distance in the Pratt cover graph is*

$$d_{\Gamma_N^{\mathrm{P}}}(a,b) = \sum_p |m_p(a) - m_p(b)|.$$

Equivalently, if

$$g := a \wedge_P b$$

denotes the Pratt meet, then

$$d_{\Gamma_N^{\mathrm{P}}}(a,b) = \Omega_{\mathrm{P}}(a) + \Omega_{\mathrm{P}}(b) - 2\Omega_{\mathrm{P}}(g).$$

Proof. Let $g = a \wedge_P b$. By definition of the coordinatewise meet,

$$m_p(g) = \min(m_p(a), m_p(b))$$

for every prime p.

By Proposition 5.11, there is a path from a to g of length $\Omega_{\mathrm{P}}(a) - \Omega_{\mathrm{P}}(g)$ and a path from g to b of length $\Omega_{\mathrm{P}}(b) - \Omega_{\mathrm{P}}(g)$. Concatenating them gives a path from a to b of length

$$\Omega_{\mathrm{P}}(a) + \Omega_{\mathrm{P}}(b) - 2\Omega_{\mathrm{P}}(g).$$

Hence

$$d_{\Gamma_N^{\mathrm{P}}}(a,b) \le \Omega_{\mathrm{P}}(a) + \Omega_{\mathrm{P}}(b) - 2\Omega_{\mathrm{P}}(g).$$

On the other hand, by Corollary 5.9, every edge changes exactly one Pratt coordinate by ± 1. Therefore any path from a to b must have length at least

$$\sum_p |m_p(a) - m_p(b)|.$$

Finally,

$$|m_p(a) - m_p(b)| = m_p(a) + m_p(b) - 2\min(m_p(a), m_p(b)),$$

and summing over p gives

$$\sum_p |m_p(a) - m_p(b)| = \Omega_{\mathrm{P}}(a) + \Omega_{\mathrm{P}}(b) - 2\Omega_{\mathrm{P}}(g).$$

So the upper and lower bounds coincide, proving both formulas. □

5.5 A Hamming embedding

The previous theorem gives an exact Hamming realization of the Pratt graph.

Definition 5.13. *Choose any ordering of the countable set*

$$\{(p, i) : p \text{ prime},\ i \in \mathbb{N}_{\geq 1}\},$$

and let $\beta_P(a)$ *be the corresponding* 0-1 *indicator vector of* S_a^{P}.

Corollary 5.14. *For all* $a, b \in \{1, \dots, N\}$,

$$d_{\Gamma_N^{\mathrm{P}}}(a, b) = d_H(\beta_P(a), \beta_P(b)),$$

where d_H *denotes Hamming distance. In particular,* $a \mapsto \beta_P(a)$ *is an isometric embedding of the Pratt cover graph into a hypercube.*

Proof. By definition of the Hamming metric,

$$d_H(\beta_P(a), \beta_P(b)) = |S_a^{\mathrm{P}} \,\Delta\, S_b^{\mathrm{P}}|.$$

Now apply Lemma 5.6 and Theorem 5.12. □

5.6 The distance from 1 and the graph-theoretic Liouville sign

Taking $a = 1$ in Theorem 5.12 gives the exact analogue of the classical valuation formula.

Corollary 5.15. *For every* $n \leq N$ *one has*

$$d_{\Gamma_N^{\mathrm{P}}}(1, n) = \Omega_{\mathrm{P}}(n).$$

Hence

$$(-1)^{d_{\Gamma_N^{\mathrm{P}}}(1,n)} = (-1)^{\Omega_{\mathrm{P}}(n)}.$$

Proof. Since $m_p(1) = 0$ for every prime p, the first formula is immediate from Theorem 5.12. The sign identity follows at once. □

5.7 Why $\deg(1) = 1$ does not settle the question by itself

At first sight, Proposition 5.3 may suggest that the graph-theoretic prime number theorem and Riemann hypothesis should be easy for the Pratt graph, just as they are easy for the path graph. However, the degree of the basepoint alone is far too weak to control the signed distance sum.

Proposition 5.16. *There exists an increasing sequence of finite connected graphs* $(\Delta_N, 1)$ *such that*

$$\deg_{\Delta_N}(1) = 1 \qquad (N \geq 2),$$

but

$$\frac{\sum_{v \in V(\Delta_N)} (-1)^{d_{\Delta_N}(1,v)}}{|V(\Delta_N)|} \not\to 0.$$

In particular, degree 1 *at the basepoint does not imply the prime number theorem in the sense of the MathOverflow question.*

Proof. Let Δ_N be the graph on vertices $\{1, 2, \dots, N\}$ with edges

$$\{1,2\} \quad \text{and} \quad \{2,v\} \text{ for every } v \in \{3,\dots,N\}.$$

So Δ_N is a star centered at 2, with one distinguished leaf 1. It is connected, and clearly

$$\deg_{\Delta_N}(1) = 1.$$

Now

$$d_{\Delta_N}(1,1) = 0, \qquad d_{\Delta_N}(1,2) = 1, \qquad d_{\Delta_N}(1,v) = 2 \text{ for } v \geq 3.$$

Therefore

$$\sum_{v \in V(\Delta_N)} (-1)^{d_{\Delta_N}(1,v)} = 1 - 1 + (N-2) = N - 2.$$

Dividing by $|V(\Delta_N)| = N$, we get

$$\frac{\sum_{v \in V(\Delta_N)} (-1)^{d_{\Delta_N}(1,v)}}{|V(\Delta_N)|} = \frac{N-2}{N} \to 1.$$

So the prime number theorem in the MathOverflow sense fails completely. □

Thus Proposition 5.3 is an interesting structural fact about the Pratt graph, but by itself it does not prove either the prime number theorem or the Riemann hypothesis in the sense of the MathOverflow question.

6 The analogue of the MathOverflow question for the Pratt graph

The second MathOverflow question suggests that for any natural graph sequence (Γ_N, v_0), one should study the signed sums

$$\sum_{v \in V(\Gamma_N)} (-1)^{d_{\Gamma_N}(v_0,v)}.$$

We now apply exactly this point of view to the Pratt cover graph.

Question 6.1 (Prime number theorem for the Pratt graph, in the sense of the MathOverflow question)**.** *Does the sequence* $(\Gamma_N^{\mathrm{P}}, 1)$ *satisfy*

$$\lim_{N\to\infty} \frac{\sum_{v=1}^{N} (-1)^{d_{\Gamma_N^{\mathrm{P}}}(1,v)}}{N} = 0?$$

Question 6.2 (Riemann hypothesis for the Pratt graph, in the sense of the MathOverflow question)**.** *For every* $\varepsilon > 0$*, does one have*

$$\lim_{N\to\infty} \frac{\sum_{v=1}^{N} (-1)^{d_{\Gamma_N^{\mathrm{P}}}(1,v)}}{N^{1/2+\varepsilon}} = 0?$$

By Corollary 5.15, these become the purely arithmetic questions

$$\lim_{N\to\infty} \frac{\sum_{n \leq N} (-1)^{\Omega_{\mathrm{P}}(n)}}{N} = 0$$

and

$$\forall \varepsilon > 0: \qquad \lim_{N\to\infty} \frac{\sum_{n \leq N} (-1)^{\Omega_{\mathrm{P}}(n)}}{N^{1/2+\varepsilon}} = 0.$$

Thus the corrected Pratt graph leads to a genuine new sign pattern on the natural numbers, obtained from the total Pratt valuation.

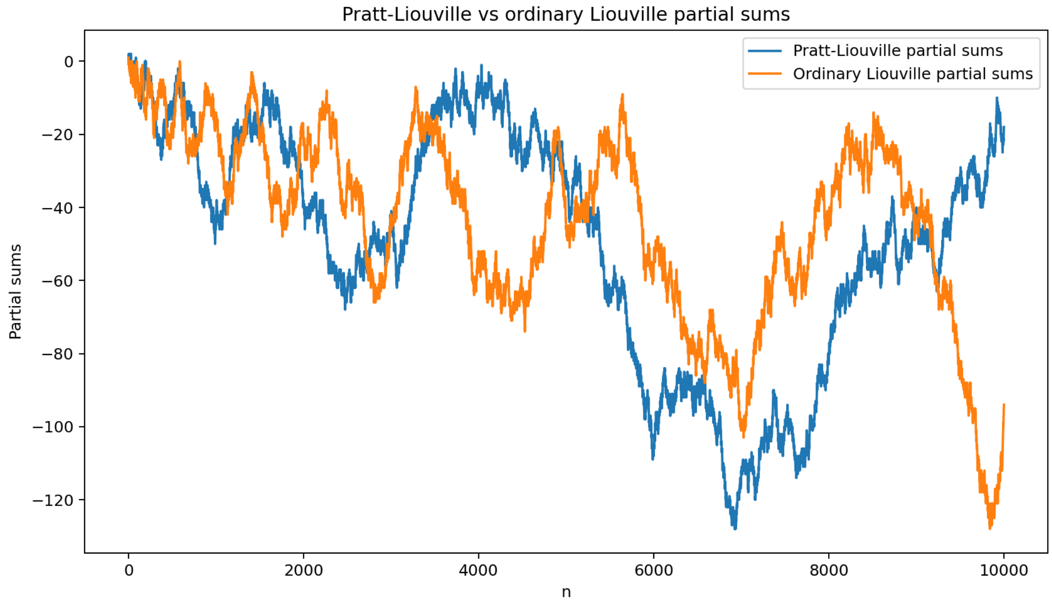

Figure 1: Pratt-Liouville vs. ordinary Liouville partial sums

7 Distance to powers of 2 and the OEIS sequence

The OEIS sequence [3] suggests looking at the distance from an integer to the chain of powers of 2 inside the Pratt graph. In the language of the present note, define

$$a(n) := \min_{k\geq 0} d_{\Gamma^{\mathrm{P}}}(n, 2^k),$$

where Γ^{P} denotes the infinite Pratt cover graph. Since Theorem 5.12 expresses distance purely in terms of the Pratt coordinates, this quantity has an especially simple closed form.

Proposition 7.1. *For every $n \geq 1$,*

$$a(n) = \min_{k\geq 0} d_{\Gamma^{\mathrm{P}}}(n, 2^k) = \Omega_{\mathrm{P}}(n) - m_2(n) = \sum_{p>2} m_p(n).$$

Proof. Fix $n \geq 1$. For each $k \geq 0$, Theorem 5.12 gives

$$d_{\Gamma^{\mathrm{P}}}(n, 2^k) = \Omega_{\mathrm{P}}(n) + \Omega_{\mathrm{P}}(2^k) - 2\,\Omega_{\mathrm{P}}(n \wedge_P 2^k).$$

We now compute the two terms involving 2^k.

First, by additivity of Pratt valuations,

$$\phi(2^k) = k\phi(2).$$

Since the Pratt tree of 2 consists of a single vertex labeled 2, one has $m_2(2) = 1$ and $m_p(2) = 0$ for every prime $p > 2$. Hence

$$m_2(2^k) = k, \qquad m_p(2^k) = 0 \quad (p > 2),$$

so in particular

$$\Omega_{\mathrm{P}}(2^k) = k.$$

Second, the Pratt meet is computed coordinatewise. Because 2^k has no Pratt coordinates away from $p = 2$, we get

$$m_p(n \wedge_P 2^k) = 0 \quad (p > 2),$$

and

$$m_2(n \wedge_P 2^k) = \min\{m_2(n), k\}.$$

Therefore

$$\Omega_{\mathrm{P}}(n \wedge_P 2^k) = \min\{m_2(n), k\}.$$

Substituting these identities into the distance formula yields

$$d_{\Gamma^{\mathrm{P}}}(n, 2^k) = \Omega_{\mathrm{P}}(n) + k - 2\min\{m_2(n), k\}.$$

Thus

$$d_{\Gamma^{\mathrm{P}}}(n, 2^k) = \begin{cases} \Omega_{\mathrm{P}}(n) - k, & 0 \le k \le m_2(n), \\ \Omega_{\mathrm{P}}(n) + k - 2m_2(n), & k \ge m_2(n). \end{cases}$$

In the first range this is strictly decreasing in k, and in the second range it is weakly increasing in k. Hence the minimum is attained at

$$k = m_2(n),$$

and its value is

$$a(n) = \Omega_{\mathrm{P}}(n) - m_2(n).$$

Finally,

$$\Omega_{\mathrm{P}}(n) = \sum_p m_p(n) = m_2(n) + \sum_{p>2} m_p(n),$$

so subtracting $m_2(n)$ gives

$$a(n) = \sum_{p>2} m_p(n).$$

This proves the formula. □

Remark 7.2. *Proposition 7.1 explains why the OEIS sequence is computationally practical once the Pratt exponents are known: one does not need to search over all powers of* 2*. The nearest point on the* 2*-chain is exactly* $2^{m_2(n)}$*, and the distance is obtained simply by discarding the* 2*-coordinate from the total Pratt weight.*

Equivalently, the sequence measures how much of the Pratt forest of n *lies away from the distinguished prime* 2*. In that sense,* $m_2(n)$ *records the part of* n *already aligned with the canonical* 2*-direction in the Pratt graph, while* $\sum_{p>2} m_p(n)$ *measures the remaining displacement.*

8 An excursion: iterates of Euler's function and Erdős Problem 408

The recursive definition of the Pratt valuations originates in the repeated passage from a prime q to $q-1$ in the paper of Erdős, Granville, Pomerance, and Spiro [1], so we now return to that origin and look at a function connected with Erdős Problem 408. It is therefore natural to compare the Pratt order with iterates of Euler's totient function.

Following Erdős Problem 408, define

$$\phi_1(n) := \phi(n), \qquad \phi_k(n) := \phi(\phi_{k-1}(n)) \quad (k \ge 2),$$

and

$$f(n) := \min\{k \ge 0 : \phi_k(n) = 1\}.$$

The problem asks for the typical size and distribution of $f(n)$; see [4]. In the present note we only need a very simple lower bound for the Pratt distance from 1.

Proposition 8.1. *For every prime p, one has*

$$p - 1 \prec_P p.$$

In particular,

$$p - 1 <_{\mathrm{P}} p.$$

Proof. Apply Theorem 5.8 with $a = p - 1$ and $b = p$. Since

$$\frac{b}{a} = \frac{p}{p-1},$$

condition (iii) of that theorem holds, and therefore $p - 1 \prec_P p$. □

Corollary 8.2. *For every prime p and every integer $n \ge 1$,*

$$p^n - p <_{\mathrm{P}} p^n.$$

Proof. By Proposition 8.1, we have $p - 1 <_{\mathrm{P}} p$. Since the Pratt valuation map is additive under multiplication, the Pratt order is multiplicatively monotone: if $a \le_P b$, then for every $c \in \mathbb{N}$,

$$ac \le_P bc.$$

Applying this with $a = p - 1$, $b = p$, and $c = p^{n-1}$ gives

$$p^{n-1}(p-1) <_{\mathrm{P}} p^{n-1}p,$$

that is,

$$p^n - p <_{\mathrm{P}} p^n.$$

□

Theorem 8.3. *For every integer $m > 1$, one has*

$$\phi(m) <_{\mathrm{P}} m.$$

Proof. Write

$$m = \prod_{j=1}^{r} p_j^{\alpha_j}$$

with distinct primes p_j and exponents $\alpha_j \ge 1$. Euler's product formula gives

$$\phi(m) = m \prod_{j=1}^{r} \left(1 - \frac{1}{p_j}\right) = \prod_{j=1}^{r} p_j^{\alpha_j - 1}(p_j - 1).$$

By Proposition 8.1, we have $p_j - 1 <_{\mathrm{P}} p_j$ for each j. Multiplying these inequalities by the common factor

$$\prod_{i \ne j} p_i^{\alpha_i}\, p_j^{\alpha_j - 1}$$

and using multiplicative monotonicity of $\leq_P$, we obtain

$$\prod_{j=1}^{r} p_j^{\alpha_j - 1}(p_j - 1) \leq_P \prod_{j=1}^{r} p_j^{\alpha_j} = m.$$

This is exactly $\phi(m) \leq_P m$. Since $m > 1$, at least one prime divisor p_j occurs, and for that prime the inequality $p_j - 1 <_{\mathrm{P}} p_j$ is strict. Hence the product inequality is strict as well, so

$$\phi(m) <_{\mathrm{P}} m.$$

□

Corollary 8.4. *For every* $n \geq 1$,

$$\Omega_P(n) = d_{\Gamma_N^{\mathrm{P}}}(1, n) \geq f(n)$$

whenever $N \geq n$.

Proof. If $n = 1$, then both sides are 0. Assume $n > 1$, and put

$$n_0 := n, \qquad n_j := \phi_j(n) \quad (j \geq 1).$$

By Theorem 8.3, we have a strict Pratt descent

$$n_0 >_P n_1 >_P n_2 >_P \cdots >_P n_{f(n)} = 1.$$

Since strict inequality in the coordinatewise Pratt order forces strict decrease of the total weight, we get

$$\Omega_P(n_{j+1}) < \Omega_P(n_j) \qquad (0 \leq j < f(n)).$$

Thus each iterate lowers Ω_P by at least 1, and because $\Omega_P(1) = 0$, it follows that

$$\Omega_P(n) \geq f(n).$$

Finally, Corollary 5.15 gives

$$\Omega_P(n) = d_{\Gamma_N^{\mathrm{P}}}(1, n).$$

This proves the claim. □

Remark 8.5. *The inequality in Corollary 8.4 is best possible in small cases: for instance*

$$\Omega_P(2) = f(2) = 1, \qquad \Omega_P(3) = f(3) = 2, \qquad \Omega_P(4) = f(4) = 2.$$

So one cannot replace "≥" by a strict inequality in general.

9 Conclusion

The two MathOverflow questions lead to a very concrete program.

For the classical valuation graph, one has a clean and exact picture:

- graph distance is given by Ω-difference,
- the Liouville function is the parity of distance from 1,

- and the prime number theorem and the Riemann hypothesis can be rewritten as graph-theoretic cancellation statements.

For the Pratt graph, the correct object is the undirected Hasse graph of the finite Pratt poset. We proved in detail that this graph is connected and that

$$\deg_{\Gamma_N^{\mathrm{P}}}(1) = 1 \qquad (N \geq 2).$$

This makes the Pratt graph look superficially close to the easy path-graph example from MathOverflow. Nevertheless, Proposition 5.16 shows that having only one neighbor at the basepoint does not by itself force the graph-theoretic prime number theorem or the graph-theoretic Riemann hypothesis.

So the honest conclusion is this: the corrected Pratt graph gives a natural new sequence of arithmetic graphs, and it is completely reasonable to ask whether it satisfies the prime number theorem or the Riemann hypothesis in the sense of the MathOverflow question. But this is still a genuine question, not something that follows immediately from $\deg(1) = 1$.

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[6] MathOverflow question: *Example of sequence of graphs which satisfy the Riemann hypothesis or the prime number theorem?* https://mathoverflow.net/questions/432283/example-of-sequence-of-graphs-which-satisfy-the-riemann-hypothesis-or-the-prime