A Liouville Function for the Pratt Exponents

Orges Leka

March 28, 2026

Abstract

This note reorganizes the Pratt-exponent viewpoint around linear observables on the Pratt space. Let $\Phi_{\mathrm{P}}(n) = (m_p(n))_{p\in\mathcal{P}}$ be the Pratt valuation vector of the integer n. For each $r \geq 1$ we define the linear observable

$$\Theta_r(n) := \sum_{p\in\mathcal{P}} \frac{m_p(n)}{p^r} = \left\langle \Phi_{\mathrm{P}}(n), u^{(r)} \right\rangle, \qquad u^{(r)} := \sum_{p\in\mathcal{P}} \frac{1}{p^r} e_p.$$

The truncated weighted Pratt functionals are then

$$W_R(n) := \sum_{r=1}^{R} \frac{\Theta_r(n)}{r} = \langle \Phi_{\mathrm{P}}(n), w^{(R)} \rangle, \qquad w^{(R)} := \sum_{r=1}^{R} \frac{u^{(r)}}{r}.$$

In the limit one recovers the global weight vector

$$w = \sum_{r\geq 1} \frac{u^{(r)}}{r} = \sum_{p\in\mathcal{P}} -\log\left(1 - \frac{1}{p}\right) e_p,$$

and therefore the exact reconstruction formula

$$\log n = \langle \Phi_{\mathrm{P}}(n), w \rangle.$$

This shows that the interval $1 \leq n \leq N$ is exactly the half-space cut $\langle \Phi_{\mathrm{P}}(n), w \rangle \leq \log N$. The Pratt-Liouville sign remains the natural parity-valued character

$$\lambda_{\mathrm{P}}(n) := (-1)^{\Omega_{\mathrm{P}}(n)} = (-1)^{\|\Phi_{\mathrm{P}}(n)\|_2^2},$$

but the new weighted viewpoint leads to a family of completely multiplicative phase functions

$$\lambda_R(n) := \exp\bigl(2\pi i W_R(n)\bigr).$$

We study these truncations geometrically and analytically. Geometrically, the cuts $W_R(n) \leq T$ define real half-spaces in the finite Pratt semigroup and hence a new convex geometry of interval sections of the natural numbers. Analytically, the truncated prime-counting functions

$$\Pi_R(T) := \#\{p \in \mathcal{P} : W_R(p) \leq T\}$$

interpolate between the genuine Pratt-height counting function and the exact logarithmic cutoff $\pi(e^T)$. We show that W_R is additive, that $0 \leq W_R(n) \leq \log n$, and that the tail $\log n - W_R(n)$ decays exponentially in R with an explicit bound in terms of $\Omega_{\mathrm{P}}(n)$. We also introduce the corresponding bivariate Euler products

$$Z_R(s, y) := \sum_{n\geq 1} y^{W_R(n)} n^{-s} = \prod_{p} \bigl(1 - y^{W_R(p)} p^{-s}\bigr)^{-1},$$

which provide the natural truncated analogue of the analytic side of the Pratt-Ehrhart program. Finally, we add empirical plots of the summatory trajectories $\sum_{n\leq N} \lambda_R(n)$ in the complex plane and illustrations of the convex geometry of the cuts $W_R(n) \leq T$. Numerically, for $R = 1, 2, 3$ the prime weights $W_R(p)$ already track constant multiples of $\log p$ rather closely, suggesting a family of exponential prime-counting laws of the form $\Pi_R(T) \approx e^{T/c_R}/T$.

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1 Introduction

The Pratt valuation vector replaces the ordinary exponent vector of an integer by a recursively enriched object. This retains additivity under multiplication while encoding the entire recursive prime structure of the Pratt forest. The usual first observable on this state space is the total Pratt height

$$\Omega_{\mathrm{P}}(n) = \sum_p m_p(n) = \|\Phi_{\mathrm{P}}(n)\|_1.$$

It is natural, but it is not exact as a size parameter. By contrast, the global Pratt weight vector

$$w_p = -\log\left(1 - \frac{1}{p}\right)$$

gives the exact reconstruction formula

$$\log n = \sum_p m_p(n) w_p.$$

This immediately suggests that the right way to cut the natural numbers may be not by a discrete radial shell such as $\Omega_{\mathrm{P}}(n) \leq r$, but by a weighted linear inequality on the Pratt space.

The present note makes this linear perspective systematic. The first idea is to view each

$$\Theta_r(n) := \sum_p \frac{m_p(n)}{p^r}$$

as a linear observable on the Pratt space. The second is to sum them with the rational coefficients $1/r$, leading to the truncated functionals

$$W_R(n) = \sum_{r=1}^{R} \frac{\Theta_r(n)}{r}.$$

The third is to exponentiate those additive functionals, which produces the completely multiplicative phases

$$\lambda_R(n) = e^{2\pi i W_R(n)}.$$

These are the weighted analogues of the Pratt-Liouville sign.

The note has four intertwined themes. First, we recast the whole construction in the language of linear functionals on the Pratt lattice. Second, we interpret the interval $1 \le n \le N$ as a convex half-space cut on the Pratt semigroup and study its truncated approximants. Third, we introduce the truncated prime-counting functions

$$\Pi_R(T) = \#\{p : W_R(p) \le T\}$$

and analyze what can be shown rigorously and what remains heuristic. Fourth, we record empirical evidence for the summatory phase functions and for the geometry of the cuts.

2 Pratt vectors and the finite Pratt lattice

Let $\mathcal{P}$ denote the set of prime numbers. For each prime q, the Pratt tree T_q is defined recursively. The tree T_2 consists of a single vertex labelled 2. If $q > 2$ is prime, then the root is labelled q, and for each prime divisor $r \mid (q-1)$ one attaches exactly $v_r(q-1)$ children labelled r, each carrying a copy of T_r. If

$$n = \prod_{q \in \mathcal{P}} q^{v_q(n)},$$

then the Pratt forest of n is the disjoint union of $v_q(n)$ copies of T_q over all primes q. For each prime p, let $m_p(n)$ be the number of vertices labelled p in this forest. The Pratt valuation vector is

$$\Phi_{\mathrm{P}}(n) = (m_p(n))_{p \in \mathcal{P}}.$$

The fundamental property is additivity:

$$\Phi_{\mathrm{P}}(xy) = \Phi_{\mathrm{P}}(x) + \Phi_{\mathrm{P}}(y) \qquad (x, y \in \mathbb{N}).$$

Consequently every linear functional on the Pratt space becomes an additive arithmetic function on $\mathbb{N}$. The total Pratt height is

$$\Omega_{\mathrm{P}}(n) := \sum_p m_p(n),$$

so $\Omega_{\mathrm{P}}(xy) = \Omega_{\mathrm{P}}(x) + \Omega_{\mathrm{P}}(y)$.

Fix $N \in \mathbb{N}$ and write the primes up to N as

$$p_1 < p_2 < \cdots < p_d, \qquad d = \pi(N).$$

Define the finite Pratt matrix

$$A_N = (a_{ij})_{1 \le i,j \le d}, \qquad a_{ij} = m_{p_i}(p_j).$$

Its j-th column is the finite Pratt vector of p_j. By construction, A_N is lower triangular with all diagonal entries equal to 1. Hence $A_N \in GL_d(\mathbb{Z})$ and

$$\Lambda_N^{\mathrm{P}} := A_N \mathbb{Z}^d$$

is a unimodular lattice. The semigroup of finite Pratt vectors supported on primes $\le N$ is

$$\mathcal{S}_N := A_N \mathbb{Z}_{\ge 0}^d \subset \Lambda_N^{\mathrm{P}}.$$

This semigroup is the geometric home of all finite-dimensional constructions below.

3 Linear observables on the Pratt space

3.1 The observables Θ_r

Definition 3.1. For each integer $r \geq 1$ define the vector

$$u^{(r)} := \sum_{p \in \mathcal{P}} \frac{1}{p^r} e_p$$

and the associated Pratt observable

$$\Theta_r(n) := \left\langle \Phi_{\mathrm{P}}(n), u^{(r)} \right\rangle = \sum_{p \in \mathcal{P}} \frac{m_p(n)}{p^r}.$$

Proposition 3.2. *For each fixed $r \geq 1$, the function Θ_r is additive:*

$$\Theta_r(mn) = \Theta_r(m) + \Theta_r(n) \qquad (m, n \in \mathbb{N}).$$

Proof. This follows immediately from the additivity of Φ_{P} and linearity of the inner product. □

Remark 3.3. The first observable is

$$\Theta_1(n) = \sum_{p} \frac{m_p(n)}{p},$$

which already sees much more than the ordinary Pratt height. The higher observables penalize large prime labels more strongly and therefore resolve the Pratt tree on finer and finer scales.

3.2 Truncated and full weights

Definition 3.4. For $R \geq 1$ define the truncated weight vector

$$w^{(R)} := \sum_{r=1}^{R} \frac{u^{(r)}}{r}$$

and the corresponding truncated weighted Pratt functional

$$W_R(n) := \langle \Phi_{\mathrm{P}}(n), w^{(R)} \rangle = \sum_{r=1}^{R} \frac{\Theta_r(n)}{r}.$$

The full weight vector is

$$w := \sum_{r \geq 1} \frac{u^{(r)}}{r}.$$

Proposition 3.5. *For every prime p one has*

$$w_p = \sum_{r \geq 1} \frac{1}{r p^r} = -\log\left(1 - \frac{1}{p}\right).$$

Consequently

$$w = \sum_{p \in \mathcal{P}} -\log\left(1 - \frac{1}{p}\right) e_p.$$

Moreover,

$$\log n = \langle \Phi_{\mathrm{P}}(n), w \rangle = \sum_{r \geq 1} \frac{\Theta_r(n)}{r}.$$

Proof. The Taylor expansion

$$-\log(1-x) = \sum_{r\geq 1} \frac{x^r}{r} \qquad (|x| < 1)$$

with $x = 1/p$ gives the formula for w_p. The identity $\log n = \langle \Phi_P(n), w\rangle$ is the global reconstruction formula. Substituting the series for w_p and interchanging the finite and infinite sums yields

$$\log n = \sum_p m_p(n) \sum_{r\geq 1} \frac{1}{rp^r} = \sum_{r\geq 1} \frac{1}{r} \sum_p \frac{m_p(n)}{p^r} = \sum_{r\geq 1} \frac{\Theta_r(n)}{r}.$$

□

Corollary 3.6. *For every $R \geq 1$ and every $n \in \mathbb{N}$ one has*

$$0 \leq W_R(n) \leq \log n, \qquad W_1(n) \leq W_2(n) \leq \cdots \leq \log n.$$

Proof. Each $\Theta_r(n)$ is nonnegative, so the truncated sums increase with R. The formula of Proposition 3.5 then implies the upper bound and monotonicity. □

Proposition 3.7. *Let*

$$E_R(n) := \log n - W_R(n) = \sum_{r>R} \frac{\Theta_r(n)}{r}.$$

Then

$$0 \leq E_R(n) \leq \Omega_P(n) \sum_{r>R} \frac{1}{r2^r}.$$

In particular there exists an absolute constant $C > 0$ such that

$$E_R(n) \leq C\,\Omega_P(n) 2^{-R} \qquad (R \geq 1,\ n \in \mathbb{N}).$$

Proof. Since $p \geq 2$ for every prime p,

$$\Theta_r(n) = \sum_p \frac{m_p(n)}{p^r} \leq \sum_p \frac{m_p(n)}{2^r} = \frac{\Omega_P(n)}{2^r}.$$

Substituting this into the tail sum gives the result. □

4 Parity and the unweighted Pratt-Liouville sign

The weighted functionals do not replace the original parity picture; they sit above it. The canonical parity-valued character on the Pratt space is still the one attached to the total Pratt height.

Definition 4.1. For $j \geq 1$ define the Pratt moments

$$\Omega_j^*(n) := \sum_p m_p(n)^j.$$

In particular,

$$\Omega_1^*(n) = \Omega_P(n), \qquad \Omega_2^*(n) = \|\Phi_P(n)\|_2^2.$$

Define the Pratt-Liouville sign by

$$\lambda_P(n) := (-1)^{\Omega_P(n)}.$$

Lemma 4.2. *For every integer vector $x \in \mathbb{Z}^d$ one has*

$$\sum_i x_i^2 \equiv \sum_i x_i \pmod 2.$$

Hence for every $n \in \mathbb{N}$,

$$\Omega_2^*(n) \equiv \Omega_{\mathrm{P}}(n) \pmod 2.$$

Proof. For every integer u, one has $u^2 \equiv u \pmod 2$. Apply this coordinatewise. □

Corollary 4.3. *The Pratt-Liouville sign also admits the quadratic description*

$$\lambda_{\mathrm{P}}(n) = (-1)^{\Omega_2^*(n)}.$$

Moreover λ_{P} is completely multiplicative.

Proof. The first assertion follows from the parity lemma. Complete multiplicativity follows from additivity of Ω_{P}. □

5 The weighted phase functions λ_R

Definition 5.1. For $R \geq 1$ define the weighted Pratt phase function

$$\lambda_R(n) := \exp\bigl(2\pi i W_R(n)\bigr).$$

Its summatory function is

$$S_R(N) := \sum_{1 \leq n \leq N} \lambda_R(n).$$

Proposition 5.2. *For each $R \geq 1$, the phase function λ_R is completely multiplicative.*

Proof. Since W_R is additive,

$$\lambda_R(mn) = e^{2\pi i W_R(mn)} = e^{2\pi i W_R(m)} e^{2\pi i W_R(n)} = \lambda_R(m)\lambda_R(n).$$

□

Remark 5.3. In the limit $R \to \infty$, one obtains

$$\lambda_\infty(n) := e^{2\pi i \log n} = n^{2\pi i}.$$

Thus the full weighted phase is a classical logarithmic Mellin phase, while the finite truncations are genuinely new Pratt objects.

6 Convex geometry of weighted interval cuts

6.1 Half-spaces in the finite Pratt semigroup

Fix a prime cutoff N and write $x = (x_1, \ldots, x_d) \in \mathbb{Z}_{\geq 0}^d$ for the ordinary exponent vector of an integer supported on primes $\leq N$. If

$$n = \prod_{j=1}^d p_j^{x_j},$$

then by additivity one has

$$W_R(n) = \sum_{j=1}^d b_j^{(R)} x_j, \qquad b_j^{(R)} := W_R(p_j).$$

Hence the weighted cut $W_R(n) \le T$ corresponds to the half-space

$$\mathcal{P}_{N,R}(T) := \left\{ x \in \mathbb{R}^d_{\ge 0} : \sum_{j=1}^{d} b_j^{(R)} x_j \le T \right\}.$$

This is a real weighted simplex in exponent coordinates, or equivalently a half-space cut in the Pratt semigroup $\mathcal{S}_N = A_N \mathbb{Z}^d_{\ge 0}$.

Proposition 6.1. *For the full weight vector, the cut is exact:*

$$1 \le n \le X \iff \langle \Phi_{\mathrm{P}}(n), w \rangle \le \log X.$$

Proof. By Proposition 3.5, the left-hand side is equivalent to $\log n \le \log X$, which is equivalent to $n \le X$. □

Remark 6.2. Thus the interval of natural numbers is a genuine convex half-space slice of the Pratt semigroup. The truncated cuts $W_R(n) \le T$ are under-approximating half-spaces which converge monotonically to the exact interval cut. In this sense the family W_R gives a convex-geometric resolution of the ordinary size interval.

6.2 Interval sections and slabs

For a fixed R and $T_0 < T_1$, the set

$$\{n \in \mathbb{N} : T_0 \le W_R(n) < T_1\}$$

is the arithmetic shadow of a slab between two parallel hyperplanes in the finite Pratt semigroup. When $R = 2$, one has

$$W_2(n) = \Theta_1(n) + \frac{1}{2}\Theta_2(n),$$

so in the (Θ_1, Θ_2)-plane the cuts are the affine lines

$$\Theta_2 = 2(T - \Theta_1).$$

This gives a concrete two-dimensional picture of the new convex geometry.

7 Truncated prime-counting functions

Definition 7.1. For $R \ge 1$ define the truncated weighted prime-counting function by

$$\Pi_R(T) := \#\{p \in \mathcal{P} : W_R(p) \le T\}.$$

Proposition 7.2. *For every $R \ge 1$, the function $\Pi_R(T)$ is finite for each fixed T, provided one has a lower growth estimate*

$$W_R(p) \ge c \log p - C$$

with constants $c > 0$ and C independent of p.

Proof. Such an inequality implies

$$\log p \le \frac{T + C}{c}$$

whenever $W_R(p) \le T$, hence

$$p \le \exp\left(\frac{T + C}{c}\right).$$

So only finitely many primes contribute. □

The central analytic problem is therefore to understand the size of $W_R(p)$ in terms of $\log p$. The full weighted function gives exactly $\log p$. For fixed R this is no longer true, but the numerical evidence suggests a very stable proportionality.

Question 7.3. *For each fixed $R \geq 1$, does there exist a constant $c_R \in (0,1)$ such that*

$$W_R(p) \sim c_R \log p \qquad (p \to \infty,\ p \text{ prime})?$$

If so, can one identify c_R explicitly or characterize it recursively through the Pratt trees?

If one assumes such a law, then the weighted prime-counting function should satisfy

$$\Pi_R(T) \approx \pi\big(e^{T/c_R}\big).$$

Using the prime number theorem heuristically would then give

$$\Pi_R(T) \approx c_R \frac{e^{T/c_R}}{T}.$$

The purpose of the present note is not to claim this rigorously, but to isolate the right geometric and analytic framework in which such a statement would live.

8 A truncated Euler-product theory in the style of Riemann

The classical move in prime-counting theory is to package arithmetic information into a Dirichlet series and then to study its factorization over primes. The truncations W_R permit exactly the same strategy.

Definition 8.1. For $R \geq 1$ define the bivariate truncated Pratt zeta function by

$$Z_R(s,y) := \sum_{n\geq 1} y^{W_R(n)} n^{-s}.$$

Whenever the series converges absolutely, it admits the Euler product

$$Z_R(s,y) = \prod_p \left(1 - y^{W_R(p)} p^{-s}\right)^{-1}.$$

The corresponding prime series is

$$P_R(s,y) := \sum_p y^{W_R(p)} p^{-s}.$$

Proposition 8.2. *For fixed R and real $y > 0$, if there exists σ_0 such that*

$$\sum_p y^{W_R(p)} p^{-\sigma}$$

converges for every $\sigma > \sigma_0$, then $Z_R(s,y)$ converges absolutely in the same half-plane and has the displayed Euler product there.

Proof. This is the standard Euler-product argument for a completely multiplicative weight, using the additivity of W_R and absolute convergence to justify rearrangement. $\square$

Remark 8.3. Assuming heuristically that $W_R(p) \sim c_R \log p$, one gets

$$y^{W_R(p)} p^{-s} \approx p^{c_R \log y - s},$$

so the prime series looks like

$$\sum_p p^{-(s - c_R \log y)}.$$

This suggests a candidate singular line

$$\Re s = 1 + c_R \log |y|,$$

which is the natural truncation-analogue of the singular-line heuristics already present in the broader Pratt-zeta program.

9 Empirical observations

The figures in this section were generated directly from the truncations W_R for $R = 1, 2, 3$. The goal is not to prove asymptotic laws, but to visualize the new geometry and the new summatory phase functions.

Numerically, for primes $p \leq 1200$ one finds that the ratios $W_R(p)/\log p$ are already remarkably stable:

$$W_1(p) \approx 0.74 \log p, \qquad W_2(p) \approx 0.91 \log p, \qquad W_3(p) \approx 0.97 \log p.$$

These values should be read as empirical constants rather than theorems, but they strongly support the heuristic law $W_R(p) \sim c_R \log p$ with $c_R \uparrow 1$.

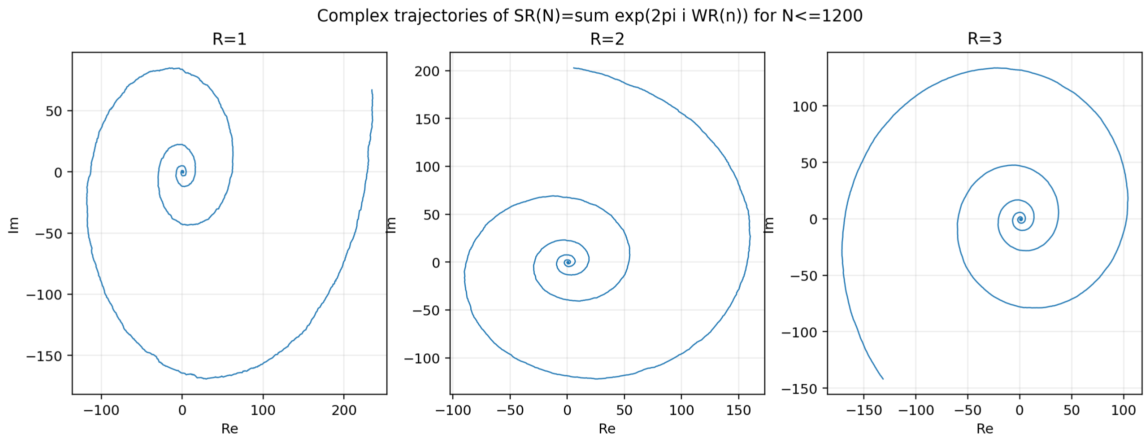


Figure 1: Complex trajectories of the summatory functions $S_R(N) = \sum_{n \leq N} e^{2\pi i W_R(n)}$ for $R = 1, 2, 3$ and $N \leq 1200$. The curves are visibly structured and become closer to the logarithmic Mellin spiral as R increases.

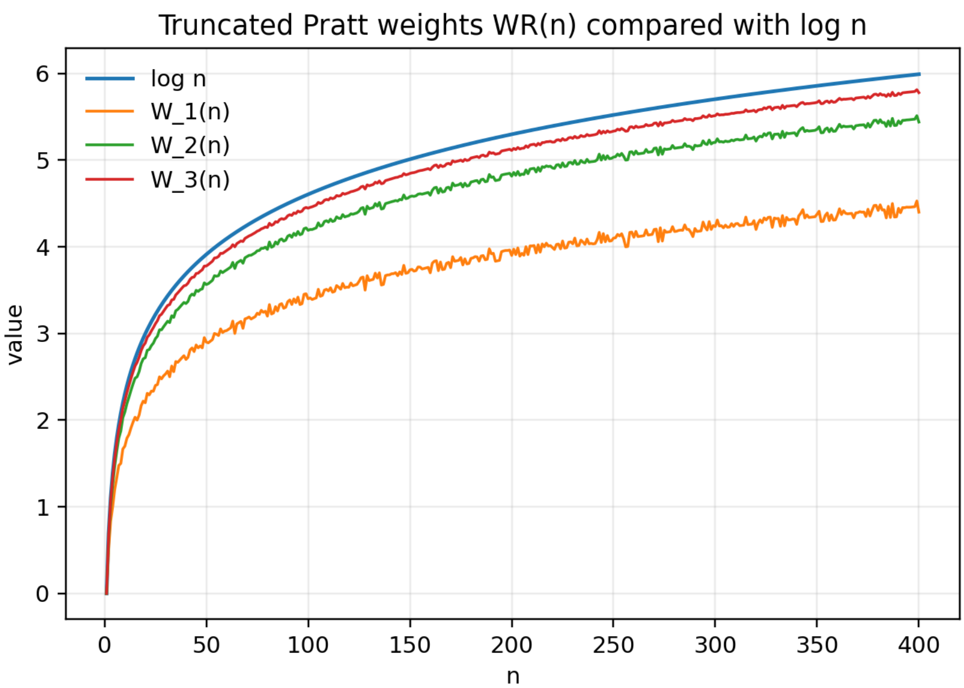
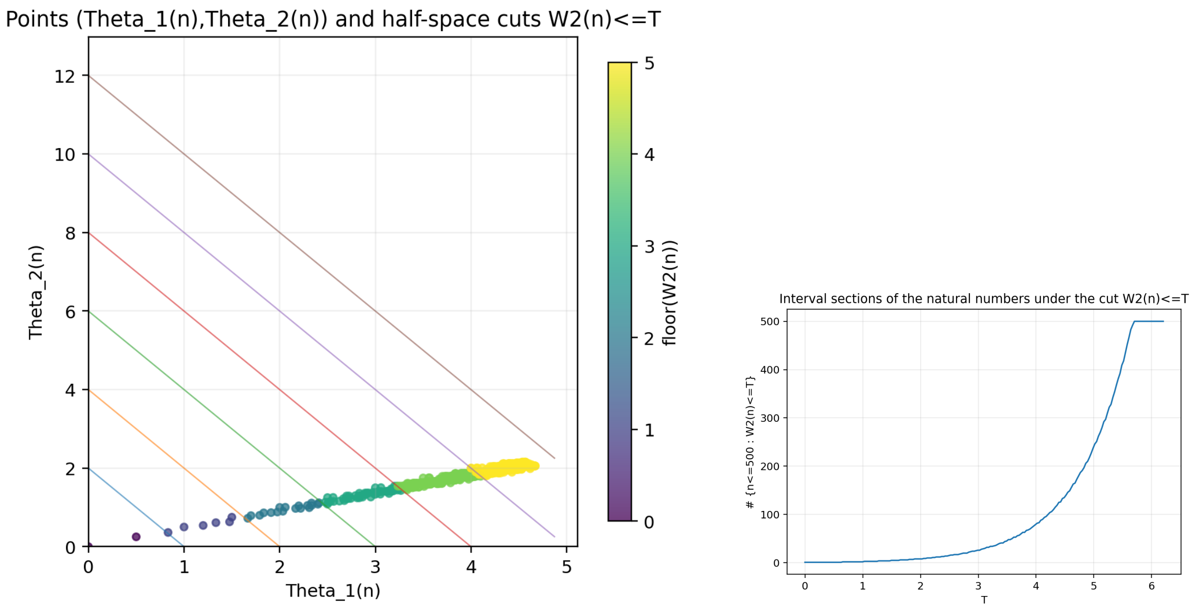


Figure 2: The truncated weights $W_R(n)$ for $R = 1, 2, 3$ compared with the exact quantity $\log n$. The monotone convergence $W_R(n) \uparrow \log n$ is visible even on short ranges.



(a) The observables $(\Theta_1(n), \Theta_2(n))$ for $n \leq 500$, colored by $\lfloor W_2(n) \rfloor$. The lines shown are the half-space boundaries $W_2(n) = T$.

(b) The counting function $\#\{n \leq 500 : W_2(n) \leq T\}$ as a function of T.

Figure 3: Convex geometry of interval sections under the cut $W_2(n) \leq T$. The left panel displays the half-space geometry in the (Θ_1, Θ_2)-plane; the right panel shows the induced slicing of the natural numbers.

10 Concluding remarks

The linear-observable perspective clarifies the role of the weighted Pratt functions. The unweighted parity remains the natural source of the Pratt-Liouville sign, but the weighted vectors $u^{(r)}$ and their partial sums $w^{(R)}$ produce a genuinely new analytic and geometric hierarchy. The key exact identity is

$$\log n = \sum_{r \geq 1} \frac{\Theta_r(n)}{r},$$

which shows that the ordinary interval of natural numbers is already a half-space cut in the Pratt semigroup. The truncations W_R then interpolate between the coarse discrete radial world of Ω_{P} and the exact logarithmic size parameter.

From this point of view, the most natural next questions are these. First, can one prove an asymptotic law $W_R(p) \sim c_R \log p$ for fixed R? Second, what is the precise geometry of the slabs cut out by W_R in the finite Pratt semigroups? Third, how much of the global logarithmic phase is already visible in the low layers $\Theta_1, \Theta_2, \Theta_3$? The present note does not settle these questions, but it isolates a framework in which they can now be asked cleanly.