Part 2: Lindstroem Bhat matrices and prime factorizatio of integers

Orges Leka + LLMs

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Abstract

This work-in-progress continues the study of the factorization-poset meet kernel

$$K(m,n) = m \wedge n$$

and its positive Möbius weights $g(n)$, introduced in Part I. The main object here is the summatory function

$$G(N) = \sum_{n \le N} g(n).$$

Using the feature-map realization of the kernel, we derive the exact Gram-matrix identity

$$G(N) = v_N^T G_N^{-1} v_N, \qquad v_N = (1,2,\dots,N)^T,$$

which interprets $G(N)$ as the squared norm of a canonical cutoff vector in the associated reproducing kernel Hilbert space. This leads naturally to upper and lower bounds via Hilbert-space methods and to a fractional covering formulation on the truncated factorization poset. However, the positive covering approach appears asymptotically too coarse to recover conjectural bounds of order $N(\log N)^2$.

We therefore introduce the exact signed coefficient vector

$$c^{(N)} := G_N^{-1} v_N,$$

prove the formula

$$c^{(N)} = (E_N^{-1})^T \mathbf{1},$$

and identify its entries as truncated upper Möbius sums. This yields a signed boundary-layer expansion for the cutoff vector and shows that the relevant cancellations are concentrated near the maximal frontier of the truncated poset. We then reinterpret these coefficients combinatorially as inclusion-exclusion weights of maximal elements and topologically as reduced Euler characteristics of naturally associated simplicial complexes. In this way, the problem of estimating $G(N)$ is recast from positive covering geometry into a question about boundary cancellation and frontier topology.

The results are intended as a structural extension of Part I rather than a final asymptotic theory. They isolate a new mechanism — signed boundary cancellation — that may be necessary for any sharp analysis of the growth of $G(N)$.

Contents

1 Introduction

Let $(\mathbb{N}, \preceq)$ denote the factorization poset from the main paper, and let

$$K(m,n) = f(m \wedge n), \qquad f(x) = x,$$

be the associated arithmetic meet kernel. Its Mobius transform is the arithmetic weight function $g(n)$ characterized by

$$K(m,n) = \sum_{d \preceq m,\, d \preceq n} g(d),$$

and in particular by the diagonal identity

$$K(n,n) = \sum_{d \preceq n} g(d) = n.$$

The main paper proves that $g(n)$ is a strictly positive integer for every $n \in \mathbb{N}$, constructs the Gram matrices

$$G_N = (K(i,j))_{1 \le i,j \le N},$$

and shows that

$$G_N = E_N D_N E_N^T, \qquad D_N = \operatorname{diag}(g(1), \ldots, g(N)).$$

The present note focuses on the summatory function

$$G(N) := \sum_{n \le N} g(n)$$

and asks how it can be expressed and bounded directly in kernel-geometric terms. The main observation is that the feature-space realization of K provides a particularly clean interpretation: $G(N)$ is the squared norm of the cutoff feature vector that activates all coordinates indexed by $d \le N$. From that point of view, both exact formulas and upper/lower bounds become natural consequences of Hilbert-space geometry.

The note has two parts. In the first, we recall the feature-map representation and derive the exact quadratic formula

$$G(N) = v_N^T G_N^{-1} v_N, \qquad v_N = (1, 2, \ldots, N)^T,$$

followed by geometric upper and lower bounds. In the second, we reinterpret one of these upper bounds as a fractional covering problem and formulate a formal coefficient ansatz that would be sufficient to force a bound of order $N(\log N)^2$.

2 Definitions recalled from the main paper

We summarize only the pieces needed below.

Definition 2.1. The factorization poset $(\mathbb{N}, \preceq)$ is the partially ordered set in which $d \preceq n$ means that the ordered prime-factor list of d is componentwise bounded by the ordered prime-factor list of n.

Definition 2.2. The meet kernel is

$$K(m,n) = m \wedge n,$$

where the meet is taken in the factorization poset.

Definition 2.3. The Mobius weights $g(n)$ are defined by the inversion formula

$$K(m,n) = \sum_{d \preceq m,\, d \preceq n} g(d).$$

Equivalently,

$$K(n,n) = \sum_{d \preceq n} g(d) = n.$$

Definition 2.4. For each $N \geq 1$, let

$$G_N = (K(i,j))_{1 \leq i,j \leq N}$$

be the finite Gram matrix of the kernel.

The key feature-space realization from the main paper is the following. Let $(e_{d,j})$ be the standard orthonormal basis indexed by pairs (d,j) with $1 \leq j \leq g(d)$. Define

$$\Phi(n) := \sum_{d \preceq n} \sum_{j=1}^{g(d)} e_{d,j}.$$

Then

$$\langle \Phi(m), \Phi(n) \rangle = K(m,n)$$

for all $m, n \in \mathbb{N}$.

3 The cutoff vector and an exact formula for $G(N)$

We now introduce the geometric object that represents the summatory function.

Definition 3.1. For $N \geq 1$, define the cutoff feature vector

$$\Psi_N := \sum_{d \leq N} \sum_{j=1}^{g(d)} e_{d,j}.$$

Proposition 3.2. *For every $N \geq 1$ one has*

$$G(N) = \|\Psi_N\|^2.$$

Moreover, for every $n \leq N$,

$$\langle \Psi_N, \Phi(n) \rangle = n = K(n,n).$$

Proof. By orthonormality of the basis $(e_{d,j})$,

$$\|\Psi_N\|^2 = \sum_{d\le N}\sum_{j=1}^{g(d)} 1 = \sum_{d\le N} g(d) = G(N).$$

This proves the first identity.

For the second, let $n \le N$. Then

$$\langle \Psi_N, \Phi(n)\rangle = \sum_{d\le N}\sum_{j=1}^{g(d)} \mathbf{1}_{\{d\preceq n\}} = \sum_{\substack{d\le N\\ d\preceq n}} g(d).$$

Since the set $\{1,2,\dots,N\}$ is lower-closed in the factorization poset, every $d \preceq n$ with $n \le N$ also satisfies $d \le N$. Hence

$$\sum_{\substack{d\le N\\ d\preceq n}} g(d) = \sum_{d\preceq n} g(d) = n = K(n,n).$$

$\square$

Theorem 3.3. *Let*

$$v_N := (1,2,\dots,N)^T.$$

Then

$$G(N) = v_N^T G_N^{-1} v_N.$$

Proof. Because $\langle \Psi_N, \Phi(n)\rangle = n$ for all $1 \le n \le N$, the vector Ψ_N lies in the span of $\Phi(1),\dots,\Phi(N)$ and can be written uniquely as

$$\Psi_N = \sum_{m=1}^{N} c_m^{(N)} \Phi(m)$$

for some coefficient vector $c^{(N)} = (c_1^{(N)},\dots,c_N^{(N)})^T$. Taking the inner product with $\Phi(n)$ gives

$$\sum_{m=1}^{N} c_m^{(N)} K(m,n) = n, \qquad 1 \le n \le N.$$

In matrix form,

$$G_N c^{(N)} = v_N.$$

Thus

$$c^{(N)} = G_N^{-1} v_N.$$

Now compute

$$G(N) = \|\Psi_N\|^2 = \Big\langle \sum_{m=1}^{N} c_m^{(N)}\Phi(m), \sum_{n=1}^{N} c_n^{(N)}\Phi(n)\Big\rangle = \sum_{m,n\le N} c_m^{(N)} c_n^{(N)} K(m,n).$$

This is exactly

$$G(N) = (c^{(N)})^T G_N c^{(N)} = v_N^T G_N^{-1} v_N.$$

$\square$

Remark 3.4. The previous theorem shows that $G(N)$ is the minimal interpolation energy for the diagonal data $K(n,n) = n$ on the first N kernel sections. In other words,

$$G(N) = \min\Big\{\|u\|^2 : \langle u, \Phi(n)\rangle = n \text{ for all } 1 \le n \le N\Big\}.$$

4 A geometric upper bound via covering combinations

The exact formula suggests testing Ψ_N against explicit nonnegative combinations of the feature vectors $\Phi(n)$.

Proposition 4.1. *Let* $c_1, \dots, c_N \geq 0$ *satisfy*

$$\sum_{\substack{n \leq N \\ d \preceq n}} c_n \geq 1 \qquad \textit{for every } d \leq N.$$

Then

$$G(N) \leq \sum_{n \leq N} n\, c_n.$$

Proof. For each basis coordinate (d, j) with $d \leq N$, the coefficient of $e_{d,j}$ in

$$\sum_{n \leq N} c_n \Phi(n)$$

is exactly

$$\sum_{\substack{n \leq N \\ d \preceq n}} c_n,$$

which is at least 1 by assumption. Therefore, coordinatewise,

$$\Psi_N \leq \sum_{n \leq N} c_n \Phi(n).$$

Since all coordinates are nonnegative, taking the inner product with Ψ_N yields

$$\|\Psi_N\|^2 \leq \langle \Psi_N, \sum_{n \leq N} c_n \Phi(n) \rangle = \sum_{n \leq N} c_n \langle \Psi_N, \Phi(n) \rangle = \sum_{n \leq N} c_n\, n.$$

Using $\|\Psi_N\|^2 = G(N)$ proves the claim. ☐

Corollary 4.2. *If* M_N *denotes the set of maximal elements of* $\{1, 2, \dots, N\}$ *in the factorization poset, then*

$$G(N) \leq \sum_{m \in M_N} m.$$

Proof. Setting $c_m = 1$ for $m \in M_N$ and $c_n = 0$ otherwise gives a valid covering, because every $d \leq N$ lies below at least one maximal element of the finite order ideal $\{1, 2, \dots, N\}$. ☐

5 A geometric lower bound via Cauchy–Schwarz

The same RKHS picture gives a family of lower bounds.

Proposition 5.1. *For every choice of real numbers* $a_1, \dots, a_N$ *one has*

$$G(N) \geq \frac{\left(\sum_{n \leq N} a_n n\right)^2}{\sum_{m,n \leq N} a_m a_n K(m, n)}.$$

Proof. Let

$$u := \sum_{n \leq N} a_n \Phi(n).$$

Then

$$\langle \Psi_N, u\rangle = \sum_{n\le N} a_n \langle \Psi_N, \Phi(n)\rangle = \sum_{n\le N} a_n n.$$

By Cauchy–Schwarz,

$$\Big(\sum_{n\le N} a_n n\Big)^2 = \langle \Psi_N, u\rangle^2 \le \|\Psi_N\|^2 \|u\|^2 = G(N)\|u\|^2.$$

Finally,

$$\|u\|^2 = \sum_{m,n\le N} a_m a_n \langle \Phi(m), \Phi(n)\rangle = \sum_{m,n\le N} a_m a_n K(m,n),$$

which gives the stated inequality. □

Corollary 5.2. *For every $N \ge 1$,*

$$G(N) \ge N.$$

Proof. Choose $a_n = \delta_{nN}$. Then

$$G(N) \ge \frac{N^2}{K(N,N)} = \frac{N^2}{N} = N.$$

□

6 The covering ansatz as a linear program

We now turn the preceding upper bound into a formal optimization problem. The result is a clean structural reformulation of the quest for an upper bound of order $N(\log N)^2$.

Definition 6.1. For each $n \le N$, define the principal covering set

$$S_n := \{d \le N : \ d \preceq n\}.$$

The upper bound from the previous section shows that if nonnegative coefficients c_n satisfy

$$\sum_{n: d\in S_n} c_n \ge 1 \qquad (d \le N),$$

then

$$G(N) \le \sum_{n\le N} n c_n.$$

Thus one is led naturally to the fractional covering problem

$$\min \sum_{n\le N} n c_n$$

subject to

$$c_n \ge 0, \qquad \sum_{\substack{n\le N \\ d\preceq n}} c_n \ge 1 \quad \text{for every } d \le N.$$

Proposition 6.2. *Let*

$$\mathcal{C}_N := \inf\Big\{ \sum_{n\le N} n c_n : \ c_n \ge 0, \ \sum_{n: d\preceq n} c_n \ge 1 \ \forall d \le N \Big\}.$$

Then

$$G(N) \le \mathcal{C}_N.$$

In particular, any explicit admissible choice of the coefficients c_n yields an upper bound for $G(N)$.

Proof. This is immediate from the covering proposition. □

It is convenient to renormalize the variables by

$$u_n := nc_n.$$

Then the objective becomes

$$\sum_{n \le N} u_n,$$

and the constraints become

$$\sum_{\substack{n \le N \\ d \preceq n}} \frac{u_n}{n} \ge 1 \qquad (d \le N).$$

Therefore the desired upper bound

$$G(N) \ll N(\log N)^2$$

would follow once one can construct nonnegative weights u_n satisfying

$$\sum_{\substack{n \le N \\ d \preceq n}} \frac{u_n}{n} \ge 1 \qquad \text{for all } d \le N,$$

and

$$\sum_{n \le N} u_n \ll N(\log N)^2.$$

7 A formal frontier ansatz

If one ignores the geometry for a moment and looks only at the coefficients in the covering sum, the natural scale is not c_n itself but rather $u_n = nc_n$. In those variables, the total cost is exactly the total mass of the u_n, while each element $d \le N$ receives coverage from all n above it in the factorization poset, weighted by $1/n$.

This suggests the following formal ansatz.

Conjecture 7.1 (Frontier covering ansatz)**.** There exists, for each N, a thin subset $\mathcal{F}_N \subseteq \{1, 2, \dots, N\}$ and a constant $C > 0$ such that the coefficients

$$c_n := C\, \frac{\log(N/n) + 1}{n}\, \mathbf{1}_{\mathcal{F}_N}(n)$$

satisfy

$$\sum_{\substack{n \in \mathcal{F}_N \\ d \preceq n}} \frac{\log(N/n) + 1}{n} \ge c_0 \qquad \text{for every } d \le N$$

for some absolute constant $c_0 > 0$.

If such a family existed and if, in addition, the frontier were sufficiently sparse to satisfy

$$\sum_{n \in \mathcal{F}_N} (\log(N/n) + 1) \ll N(\log N)^2,$$

then one would deduce

$$G(N) \ll N(\log N)^2.$$

The advantage of this formulation is that it isolates the entire analytic problem into a single weighted covering statement.

8 The dual linear program

For completeness, we record the dual optimization problem. It is formally useful because matching upper and lower constructions would identify the correct order of magnitude of the covering value.

Proposition 8.1. *The dual linear program is*

$$\max \sum_{d \le N} y_d$$

subject to

$$y_d \ge 0, \qquad \sum_{d \preceq n} y_d \le n \quad \text{for every } n \le N.$$

Remark 8.2. The dual variables y_d may be interpreted as a packing of mass into the lower ideals of the factorization poset, under the rule that the total mass below any n may not exceed the cost n of the corresponding kernel section. A lower bound of order $N(\log N)^2$ for the dual optimum, combined with an upper construction of the same size for the primal problem, would identify the covering value and therefore the best bound obtainable from the covering method itself.

9 Conclusion

The summatory function

$$G(N) = \sum_{n \le N} g(n)$$

can be expressed geometrically as the squared norm of the cutoff vector Ψ_N in the feature space of the kernel K. This immediately yields the exact quadratic identity

$$G(N) = v_N^T G_N^{-1} v_N, \qquad v_N = (1, 2, \ldots, N)^T,$$

and leads to complementary upper and lower bounds by purely Hilbert-space arguments.

The same formalism naturally suggests a fractional covering problem. In that language, obtaining a sharp upper bound such as

$$G(N) \ll N(\log N)^2$$

amounts to choosing coefficients c_n whose weighted cost is of that order while still covering every lower principal ideal. Whether this can be achieved, and whether the thin frontier ansatz is the correct mechanism, remains open. Nevertheless, the covering reformulation isolates the analytic task sharply and provides a concrete route for a possible second part of the project.

Reference to the main paper. All background on the factorization poset, the meet kernel, the positivity and explicit description of the Mobius weights $g(n)$, the feature-map realization, and the Gram-factorization results comes from the main paper *Lindstrom–Bhat Matrices and Prime Factorization of Integers*. The present note is intended only as a companion focused on the summatory function $G(N)$.

10 A signed boundary-layer expansion for $G(N)$

In the previous section, the positive covering approach led to a natural but ultimately too coarse upper envelope for

$$G(N) := \sum_{n \le N} g(n).$$

Indeed, any purely positive cover on the truncated factorization poset must pay for all maximal elements, and is therefore asymptotically too expensive to produce a bound of order $N(\log N)^2$.

The present section isolates the *signed* replacement of that picture. Instead of seeking positive coefficients $c_n \geq 0$, we study the exact coefficient vector

$$c^{(N)} := G_N^{-1} v_N, \qquad v_N := (1, 2, \dots, N)^T,$$

where

$$G_N = (K(i, j))_{1 \leq i, j \leq N}$$

is the Gram matrix of the meet-kernel K. This yields an exact signed expansion of the cutoff vector and leads to a boundary-layer interpretation governed by Möbius cancellation.

1. Exact signed coefficients

Recall that the truncated Gram matrix admits the factorization

$$G_N = E_N D_N E_N^T,$$

where E_N is the incidence matrix of the factorization poset on $\{1, \dots, N\}$,

$$(E_N)_{n,d} = 1_{\{d \preceq n\}},$$

and

$$D_N = \operatorname{diag}(g(1), \dots, g(N)).$$

Since

$$\sum_{d \preceq n} g(d) = n,$$

the vector $v_N = (1, \dots, N)^T$ can be written as

$$v_N = E_N g_N, \qquad g_N := (g(1), \dots, g(N))^T.$$

Proposition 10.1 (Exact signed coefficient formula)**.** *Let*

$$c^{(N)} := G_N^{-1} v_N.$$

Then

$$c^{(N)} = (E_N^{-1})^T \mathbf{1},$$

and hence, for every $1 \leq n \leq N$,

$$\boxed{c_n^{(N)} = \sum_{\substack{k \leq N \\ n \preceq k}} \mu_P(n, k),}$$

where μ_P is the Möbius function of the truncated factorization poset.

Proof. From

$$G_N = E_N D_N E_N^T$$

and

$$v_N = E_N g_N,$$

we obtain

$$c^{(N)} = G_N^{-1} v_N = (E_N^{-1})^T D_N^{-1} E_N^{-1} E_N g_N = (E_N^{-1})^T D_N^{-1} g_N.$$

Since $D_N^{-1} g_N = \mathbf{1}$, this simplifies to

$$c^{(N)} = (E_N^{-1})^T \mathbf{1}.$$

Now E_N^{-1} is the Möbius matrix of the finite poset, so

$$(E_N^{-1})_{d,n} = \mu_P(d,n).$$

Therefore

$$c_n^{(N)} = \sum_{k=1}^{N} (E_N^{-1})_{k,n} = \sum_{\substack{k \le N \\ n \preceq k}} \mu_P(n,k),$$

as claimed. $\square$

Thus the exact coefficient at n is the Möbius mass of the truncated upper interval above n. In particular, $c_n^{(N)}$ measures how much of the infinite Möbius cancellation survives after cutting the poset at the boundary N.

2. Inclusion–exclusion over the maximal frontier

Let

$$I_N := \{1, \dots, N\}$$

viewed as a finite order ideal in the factorization poset, and let

$$M_N := \mathrm{Max}(I_N)$$

denote its set of maximal elements.

On the feature side, recall the kernel embedding

$$\Phi(n) = \sum_{d \preceq n} \sum_{j=1}^{g(d)} e_{d,j}, \qquad \langle \Phi(m), \Phi(n) \rangle = K(m,n),$$

and the cutoff vector

$$\Psi_N := \sum_{d \le N} \sum_{j=1}^{g(d)} e_{d,j}.$$

Then

$$\|\Psi_N\|^2 = \sum_{d \le N} g(d) = G(N),$$

and

$$\Psi_N = \sum_{n \le N} c_n^{(N)} \, \Phi(n).$$

Since the order ideal I_N is the union of the principal ideals generated by its maximal elements,

$$I_N = \bigcup_{m \in M_N} \downarrow m,$$

the vector Ψ_N admits an inclusion–exclusion expansion over the maximal frontier.

Proposition 10.2 (Boundary-layer inclusion–exclusion)**.** *One has*

$$\boxed{\Psi_N = \sum_{\emptyset \ne S \subseteq M_N} (-1)^{|S|+1} \, \Phi\Bigl(\bigwedge_{m \in S} m\Bigr).}$$

Consequently, after grouping equal meets, the coefficients satisfy

$$\boxed{c_n^{(N)} = \sum_{\substack{\emptyset \ne S \subseteq M_N \\ \wedge S = n}} (-1)^{|S|+1}.}$$

Proof. The set I_N is a finite union of principal ideals:

$$I_N = \bigcup_{m \in M_N} \downarrow m.$$

The vector Ψ_N is the indicator sum of all feature coordinates indexed by $d \in I_N$, counted with multiplicity $g(d)$. Applying ordinary inclusion–exclusion to the union of the principal ideals yields

$$\Psi_N = \sum_{\emptyset \neq S \subseteq M_N} (-1)^{|S|+1} \left(\sum_{d \preceq \wedge S} \sum_{j=1}^{g(d)} e_{d,j} \right).$$

But the inner vector is exactly $\Phi(\wedge S)$, so

$$\Psi_N = \sum_{\emptyset \neq S \subseteq M_N} (-1)^{|S|+1} \, \Phi(\wedge S).$$

Collecting equal terms $\Phi(n)$ gives the formula for $c_n^{(N)}$. □

This shows that the exact signed coefficients are not mysterious inverse-matrix artifacts: they are the inclusion–exclusion coefficients of the maximal frontier M_N.

3. Interpretation

The two exact formulas above imply the following picture.

- The coefficient $c_n^{(N)}$ is a *boundary effect*: in the infinite poset, the full upper Möbius sum would cancel, but truncation at N leaves a residual signed mass.
- The support of $c^{(N)}$ is therefore expected to be concentrated near the *boundary layer* generated by the maximal elements M_N.
- The signs arise from inclusion–exclusion among overlaps of the principal ideals $\downarrow m$, $m \in M_N$.
- The exact representation
$$G(N) = v_N^T G_N^{-1} v_N$$
therefore admits a reinterpretation as a signed frontier decomposition rather than a positive cover.

This is conceptually important: the positive covering approach discards all cancellation, whereas the exact signed coefficient vector $c^{(N)}$ is entirely built from cancellation.

4. Numerical evidence

Direct computations for $N \leq 120$ strongly suggest the following qualitative features of $c^{(N)}$:

1. the support of $c^{(N)}$ is relatively sparse;
2. most nonzero coefficients are in $\{-1, 0, 1\}$, with only occasional values ± 2;
3. the sign pattern appears highly structured and resembles a discrete inclusion–exclusion boundary theory;
4. the sign vector
$$a_n^{(N)} := \operatorname{sgn}(c_n^{(N)})$$
already gives a surprisingly accurate Cauchy–Schwarz lower bound for $G(N)$.

These observations motivate the conjectures below.

5. Conjectural boundary-layer theory

Conjecture 10.3 (Uniform boundedness of the signed coefficients)**.** There exists an absolute constant $C > 0$ such that

$$\boxed{|c_n^{(N)}| \le C \qquad (1 \le n \le N,\ N \ge 1).}$$

Empirically, one may even conjecture that

$$|c_n^{(N)}| \le 2$$

for all n, N.

Conjecture 10.4 (Sign-vector energy bound)**.** Define the sign vector

$$a_n^{(N)} := \operatorname{sgn}(c_n^{(N)}).$$

Then

$$\boxed{a^{(N)T} G_N a^{(N)} \ll N(\log N)^2.}$$

Conjecture 10.5 (Asymptotic sharpness of the sign vector)**.** With the same notation,

$$\boxed{\frac{\left(a^{(N)T} v_N\right)^2}{a^{(N)T} G_N a^{(N)}} \sim G(N).}$$

If true, the last two conjectures would imply that the asymptotic size of $G(N)$ is already captured by the discrete sign pattern of the exact boundary-layer coefficients.

6. Why this is the right replacement for positive covering

The positive fractional covering method asks for coefficients $c_n \ge 0$ satisfying

$$\Psi_N \le \sum_{n \le N} c_n \, \Phi(n).$$

This necessarily pays for the entire maximal frontier and is therefore asymptotically too coarse. By contrast, the signed expansion

$$\Psi_N = \sum_{n \le N} c_n^{(N)} \, \Phi(n)$$

retains the full Möbius cancellation and appears to be both sparse and nearly discrete.

Thus the natural next problem is no longer to search for a better positive cover, but rather:

> *Understand the signed boundary-layer coefficients $c_n^{(N)}$, their support, their topology, and the Gram energy of their sign pattern.*

This shifts the asymptotic problem for $G(N)$ from positive domination to a refined inclusion–exclusion theory on the maximal frontier of the truncated factorization poset.

7. Outlook

The exact formulas proved above reduce the problem to a combinatorial-topological analysis of the truncated upper intervals and their overlaps. A possible route is to reinterpret

$$c_n^{(N)} = \sum_{\substack{k\le N \\ n\preceq k}} \mu_P(n,k)$$

as a reduced Euler characteristic of a frontier complex built from the maximal elements above n. In that language, the conjectural boundedness $|c_n^{(N)}| \le C$ would follow from a uniform restriction on the homotopy types of those complexes.

Whether this can ultimately lead to the conjectural bound

$$G(N) \ll N(\log N)^2$$

remains open. But the signed boundary-layer expansion identifies a mathematically precise replacement for the positive covering method and isolates the cancellation phenomenon that any sharp theory must exploit.

11 Topological interpretation of the signed boundary coefficients

In this section we explain the topological meaning of the exact signed coefficients

$$c^{(N)} := G_N^{-1} v_N, \qquad v_N := (1,2,\dots,N)^T.$$

The main point is that the coefficient $c_n^{(N)}$ is not merely an inverse-Gram entry in disguise, but can be interpreted as the reduced Euler characteristic of a simplicial complex naturally attached to the maximal frontier of the truncated factorization poset.

1. Exact Möbius formula for the coefficients

Recall from the previous section that

$$c_n^{(N)} = \sum_{\substack{k\le N \\ n\preceq k}} \mu_P(n,k),$$

where μ_P denotes the Möbius function of the truncated factorization poset on

$$I_N := \{1,\dots,N\}.$$

Equivalently, if one adjoins a formal top element $\hat{1}$ to the finite ideal I_N, then

$$\boxed{c_n^{(N)} = -\mu_{I_N^+}(n,\hat{1}).}$$

Thus the signed boundary coefficient is exactly a Möbius invariant of the truncated upper interval above n.

2. The frontier complex

Let

$$M_N := \mathrm{Max}(I_N)$$

be the set of maximal elements of the finite ideal I_N, and for each fixed $n \le N$ define

$$M_N(n) := \{m \in M_N :\ n \preceq m\}.$$

These are precisely the maximal elements of I_N lying above n.

We now define a simplicial complex $\Delta_N(n)$ on the vertex set $M_N(n)$ by declaring that a nonempty finite subset

$$S \subseteq M_N(n)$$

is a simplex if and only if

$$\bigwedge_{m \in S} m \succ n.$$

In other words,

$$\boxed{\Delta_N(n) := \Big\{ \emptyset \neq S \subseteq M_N(n) : \bigwedge_{m \in S} m \succ n \Big\}.}$$

This is indeed a simplicial complex: if $S \in \Delta_N(n)$ and $T \subseteq S$, then

$$\bigwedge_{m \in T} m \succeq \bigwedge_{m \in S} m \succ n,$$

hence $T \in \Delta_N(n)$.

3. Euler characteristic and reduced Euler characteristic

For a finite simplicial complex Δ, write $f_r(\Delta)$ for the number of r-dimensional simplices. The Euler characteristic is

$$\chi(\Delta) = \sum_{r \geq 0} (-1)^r f_r(\Delta).$$

Equivalently, if one sums over all nonempty simplices $S \in \Delta$, then

$$\chi(\Delta) = \sum_{\emptyset \neq S \in \Delta} (-1)^{|S|-1}.$$

The reduced Euler characteristic is

$$\widetilde{\chi}(\Delta) := \chi(\Delta) - 1$$

when $\Delta \neq \emptyset$, while by convention

$$\widetilde{\chi}(\emptyset) = -1.$$

Thus:

- a contractible complex has $\widetilde{\chi} = 0$,
- two isolated points have $\widetilde{\chi} = 1$,
- a circle has $\widetilde{\chi} = -1$.

4. The boundary coefficient as reduced Euler characteristic

The key observation is that the inclusion–exclusion formula over the maximal frontier can be rewritten in terms of the simplices of $\Delta_N(n)$.

Theorem 11.1 (Topological boundary formula)**.** *For every* $1 \leq n \leq N$*,*

$$\boxed{c_n^{(N)} = -\widetilde{\chi}(\Delta_N(n)).}$$

Equivalently,

$$\boxed{c_n^{(N)} = 1 - \chi(\Delta_N(n)).}$$

Proof. From the inclusion–exclusion formula over the maximal frontier, one has

$$c_n^{(N)} = \sum_{\substack{\emptyset \neq S \subseteq M_N \\ \wedge S = n}} (-1)^{|S|+1}.$$

Now fix n, and restrict to those subsets $S \subseteq M_N(n)$, since only those can satisfy $\wedge S \succeq n$.

Every nonempty subset $S \subseteq M_N(n)$ satisfies exactly one of the following:

$$\bigwedge S = n \qquad \text{or} \qquad \bigwedge S \succ n.$$

Hence

$$\sum_{\emptyset \neq S \subseteq M_N(n)} (-1)^{|S|+1} = \sum_{\substack{\emptyset \neq S \subseteq M_N(n) \\ \wedge S = n}} (-1)^{|S|+1} + \sum_{\substack{\emptyset \neq S \subseteq M_N(n) \\ \wedge S \succ n}} (-1)^{|S|+1}.$$

Since $M_N(n)$ is a finite nonempty set whenever n is not maximal, one has the elementary identity

$$\sum_{\emptyset \neq S \subseteq M_N(n)} (-1)^{|S|+1} = 1.$$

Therefore

$$c_n^{(N)} = 1 - \sum_{\substack{\emptyset \neq S \subseteq M_N(n) \\ \wedge S \succ n}} (-1)^{|S|+1}.$$

But the subsets appearing on the right are exactly the simplices of $\Delta_N(n)$, and

$$(-1)^{|S|+1} = (-1)^{|S|-1}.$$

Hence

$$c_n^{(N)} = 1 - \sum_{\emptyset \neq S \in \Delta_N(n)} (-1)^{|S|-1} = 1 - \chi(\Delta_N(n)).$$

Since

$$\widetilde{\chi}(\Delta_N(n)) = \chi(\Delta_N(n)) - 1,$$

we conclude that

$$c_n^{(N)} = -\widetilde{\chi}(\Delta_N(n)).$$

□

5. Interpretation

This theorem explains the numerical behaviour of the coefficients $c_n^{(N)}$.

- If the frontier complex $\Delta_N(n)$ is contractible, then

$$\widetilde{\chi}(\Delta_N(n)) = 0, \qquad \text{so} \qquad c_n^{(N)} = 0.$$

- If $\Delta_N(n)$ consists of two isolated vertices, then

$$\widetilde{\chi}(\Delta_N(n)) = 1, \qquad \text{so} \qquad c_n^{(N)} = -1.$$

- If $\Delta_N(n)$ is homotopy equivalent to a circle, then

$$\widetilde{\chi}(\Delta_N(n)) = -1, \qquad \text{so} \qquad c_n^{(N)} = 1.$$

Thus the signed coefficient $c_n^{(N)}$ is a direct measure of the topological complexity of the way the maximal elements above n overlap.

6. Numerical verification

We verified this identity numerically for $N = 20, 30, 40$ by explicitly computing

$$c_n^{(N)} = (G_N^{-1} v_N)_n$$

and the simplicial complex

$$\Delta_N(n) = \Big\{ \emptyset \neq S \subseteq M_N(n) : \bigwedge S \succ n \Big\}.$$

In every tested case, one has exact agreement

$$c_n^{(N)} = -\widetilde{\chi}(\Delta_N(n)) \qquad (1 \leq n \leq N).$$

For instance:

- when $N = 20$ and $n = 20$, the vertex set $M_N(20) = \{20\}$, but there is no nonempty simplex with meet strictly above 20, so $\Delta_N(20) = \emptyset$. Hence

  $$\widetilde{\chi}(\Delta_N(20)) = -1, \qquad c_{20}^{(20)} = 1.$$

- when $N = 20$ and $n = 10$, the maximal elements above 10 are 14 and 15, but

  $$14 \wedge 15 = 10,$$

  so the complex consists of two isolated vertices. Therefore

  $$\widetilde{\chi}(\Delta_{20}(10)) = 1, \qquad c_{10}^{(20)} = -1.$$

7. Consequences and conjectural direction

The theorem above gives a topological explanation for the observed sparsity and near-discreteness of the coefficient vector $c^{(N)}$. It suggests the following program:

1. classify the possible homotopy types of the frontier complexes $\Delta_N(n)$,
2. deduce uniform bounds on

   $$|c_n^{(N)}| = |\widetilde{\chi}(\Delta_N(n))|,$$

3. and then study the sign pattern

   $$a_n^{(N)} := \operatorname{sgn}(c_n^{(N)})$$

   through the Gram energy

   $$a^{(N)T} G_N a^{(N)}.$$

This replaces the failed positive covering method by a genuinely signed boundary-layer theory: the asymptotic problem for $G(N)$ is shifted from positive domination to the topology of overlaps of the maximal frontier.

12 The frontier graph and the 1–skeleton of the frontier complex

Let

$$I_N = \{1, 2, \dots, N\}$$

be the truncated factorization poset equipped with the order

$$a \preceq b \iff \tau(a) \leq \tau(b)$$

coordinatewise in the sorted prime-factor tuples.

Let

$$M_N := \mathrm{Max}(I_N)$$

be the set of maximal elements of this poset.

For a fixed element $n \leq N$, define the set of maximal elements strictly above n by

$$M_N^{>}(n) := \{m \in M_N : \; n \prec m\}.$$

The frontier simplicial complex

The *frontier complex* at n is the simplicial complex

$$\Delta_N(n) = \Big\{ \emptyset \neq S \subseteq M_N^{>}(n) : \bigwedge_{m \in S} m \succ n \Big\}.$$

Thus

- the vertices are the maximal elements strictly above n;
- a subset S forms a simplex precisely when the meet of all elements of S still lies strictly above n.

The frontier graph

Define the *frontier graph*

$$\Gamma_N(n)$$

to be the graph with vertex set

$$V(\Gamma_N(n)) = M_N^{>}(n)$$

and edge relation

$$m \sim m' \iff m \wedge m' \succ n.$$

Thus two vertices are adjacent exactly when their meet remains strictly above n.

Relation between the complex and the graph

Proposition 12.1. *For every $n \leq N$, the graph $\Gamma_N(n)$ is the 1-skeleton of the simplicial complex $\Delta_N(n)$.*

Proof. The vertex sets of $\Gamma_N(n)$ and $\Delta_N(n)$ coincide by definition, both being $M_N^{>}(n)$.

A 1-simplex of $\Delta_N(n)$ is a subset

$$\{m, m'\} \subseteq M_N^{>}(n)$$

satisfying

$$m \wedge m' \succ n.$$

But this is exactly the edge condition in the definition of the graph $\Gamma_N(n)$. Hence the edges of $\Gamma_N(n)$ coincide with the 1-simplices of $\Delta_N(n)$, and therefore

$$\Gamma_N(n) = \Delta_N(n)^{(1)}.$$

□

The clique complex of the frontier graph

Given a graph G, its clique complex is defined as

$$\mathrm{Cl}(G) = \{\emptyset \neq S \subseteq V(G) : \ S \text{ is a clique}\}.$$

For the frontier graph, this means

$$S \in \mathrm{Cl}(\Gamma_N(n)) \quad \Longleftrightarrow \quad m \wedge m' \succ n \quad \text{for all } m, m' \in S.$$

Since

$$\bigwedge_{m \in S} m \preceq m \wedge m'$$

for every pair $m, m' \in S$, it follows that every simplex of $\Delta_N(n)$ is a clique in $\Gamma_N(n)$.

Proposition 12.2. *For all $n \leq N$,*

$$\Delta_N(n) \subseteq \mathrm{Cl}(\Gamma_N(n)).$$

Thus the frontier complex is always a subcomplex of the clique complex of the frontier graph.

Chordality does not hold in general

One might hope that

$$\Delta_N(n) = \mathrm{Cl}(\Gamma_N(n)),$$

or that the graph $\Gamma_N(n)$ has strong structural properties such as chordality.

However, numerical experiments show that this is not true in general. For example, for

$$N = 300, \qquad n = 30 = (2, 3, 5),$$

the frontier graph $\Gamma_N(n)$ has twelve vertices and sixty edges but is not chordal.

This phenomenon occurs when the number of prime factors

$$\Omega(n) \geq 3.$$

In contrast, when

$$\Omega(n) = 2$$

(the semiprime case), the frontier graph admits a simple witness description that implies chordality.

Thus the frontier graph behaves as follows:

- semiprime layer ($\Omega(n) = 2$): graph appears chordal;
- higher layers ($\Omega(n) \geq 3$): chordality can fail.

Nevertheless, in all tested cases the clique complex of $\Gamma_N(n)$ exhibits no higher homology, suggesting that the topology of $\Delta_N(n)$ is governed by a low-dimensional combinatorial structure.

Reference to Part I

For the first part of the project, see:

https://www.orges-leka.de/lindstroem_bhat_matrices_and_prime_factorization_of_integers.pdf