Rail Networks with Timetables

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1 Cost Estimation based on Shortest Paths

Problem: Given a road graph and its shortest paths, we want to determine the costs on the edges that correspond to these shortest paths.

Literature: 'Burton and Toint - 1992 - On an instance of the inverse shortest paths problem.'

In the above paper, the following problem is solved: Given pre-determined edge costs and known shortest paths, we want to find additional edge costs that explain these shortest paths and, additionally, minimize the l_2-norm to the given costs. The problem is formulated as a quadratic optimization problem and is solved using a method that takes n^2 time, where n is the number of nodes. This method could be relevant for railways, where the spatial distance between two nodes could be used as the 'pre-determined cost'.

Another approach:

A directed, connected, and edge-weighted graph $G = (V, E, c)$ is called k.w.-Graph (shortest paths are most probable paths) if there exists a stochastic matrix P such that G and P form a Markov chain, and additionally:

$$-\log(P(i, j)) = c(i, j)$$

2 Rail Networks with Timetables

A rail network $\mathbf{N}$ is a tuple $\mathbf{N} := (\mathbf{Z}, \mathbf{S}, \mathbf{B})$ with a finite set $\mathbf{Z}$ of trains and a track space $\mathbf{S}$ (whose elements are called track points), which is initially a topological space and has a finite cover of closed sets $\mathbf{B} = (B_1, \cdots, B_b)$:

$$\cup_{1 \leq i \leq b} B_i = \mathbf{S} \tag{1}$$

The finite sets B_i are called block sections. A timetable $\mathbf{F}$ is a tuple $(\mathbf{N}, \beta, \mathbf{T})$ where $\mathbf{N}$ is a rail network, $\mathbf{T} = [t_S, t_E]$ is the closed time interval of the timetable, and β is a function called location curve, such that:

$$\beta : \mathbf{Z} \times \mathbf{T} \to \mathbf{S} \tag{2}$$

β is continuous in the second argument, i.e., it is a path.

$$\forall z \in \mathbf{Z} \text{ is } t \mapsto \beta(z,t) \text{ a continuous function of topological spaces} \tag{3}$$

β is injective in the first argument: (No two trains can be at the same track point at the same time.)

$$\forall z_1 \neq z_2, z_1, z_2 \in \mathbf{Z} \forall t \in \mathbf{T} \text{ is } \beta(z_1,t) \neq \beta(z_2,t) \tag{4}$$

However, even more is required: No two trains can be at the same block section simultaneously: Let:

$$\forall z \in \mathbf{Z} \text{ let } \beta_z : \mathbf{T} \to \mathbf{S}, z \mapsto \beta(z,t) \tag{5}$$

Then it should hold:

$$\forall B \in \mathbf{B} \forall z_1 \neq z_2, z_1, z_2 \in \mathbf{Z} : \beta_{z_1}^{-1}(B) \cap \beta_{z_2}^{-1}(B) = \emptyset \tag{6}$$

Two block sections are connected if they intersect at finitely many track points:

$$\forall B, C \in \mathbf{B} : B \equiv C :\iff 0 < |B \cap C| < \infty \tag{7}$$

We say: A train $z \in \mathbf{Z}$ is in block section B (according to timetable $\mathbf{F}$) in the time interval $[t_a, t_d] \subset [t_S, t_E]$, if:

$$\forall \beta_z([t_a, t_d]) \subset B \tag{8}$$

We also say: A train $z \in \mathbf{Z}$ travels over block section B (according to timetable $\mathbf{F}$) in the time interval $[t_a, t_d] \subset [t_S, t_E]$, if z is in B in the time interval $[t_a, t_d]$ and:

$$\forall \epsilon > 0 \text{ it is } \{\beta(z, t_a - \epsilon), \beta(z, t_d + \epsilon)\} \cap B = \emptyset \tag{9}$$

Here, t_a is the arrival time and t_d is the departure time. A train $z \in Z$ travels from B to C with $B, C \in \mathbf{B}$ if:

$$\exists B = B_1 \equiv B_2 \equiv \cdots \equiv B_n = C, B_i \in \mathbf{B} \tag{10}$$

and

$$\exists [t_{a_1}, t_{d_1}], \cdots, [t_{a_n}, t_{d_n}] \subset [t_S, t_E] : \forall i = 1, \cdots, n-1 : \beta(z, t_{d_i}) = \beta(z, t_{a_{i+1}}) \tag{11}$$

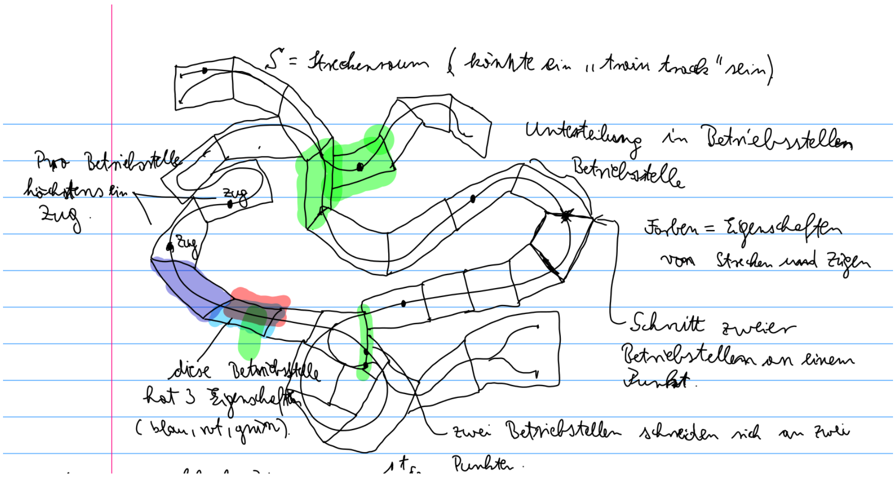
and z travels over B_i in the time interval $[t_{a_i}, t_{d_i}]$.
A timetable $\hat{\mathbf{F}}$ is an extension of $\mathbf{F}$ $:\iff$

$$\mathbf{T} \subset \hat{\mathbf{T}}, \hat{\beta}|_{\mathbf{T}} = \beta, N \subset \hat{N} \tag{12}$$

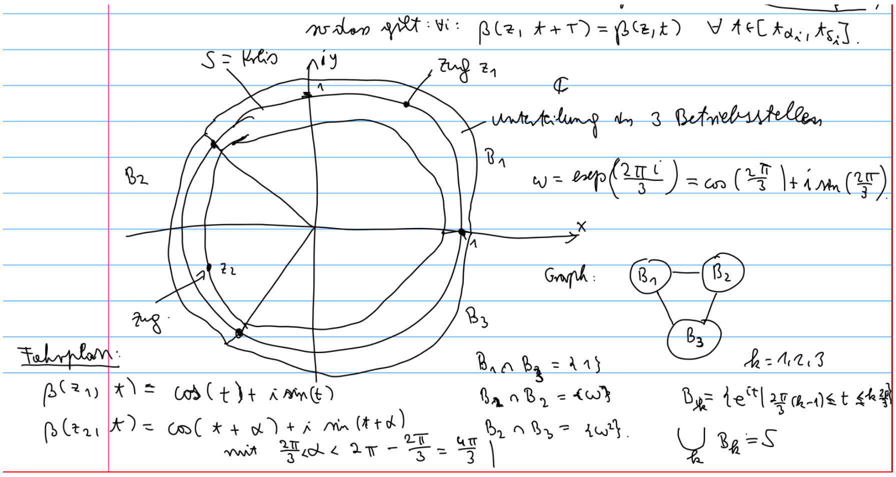
Here, we understand $N \subset \hat{N}$ as follows:

$$\mathbf{Z}_N \subset \mathbf{Z}_{\hat{N}} \text{ as sets}, \mathbf{S}_N \subset \mathbf{S}_{\hat{N}} \text{ as topological spaces} \tag{13}$$

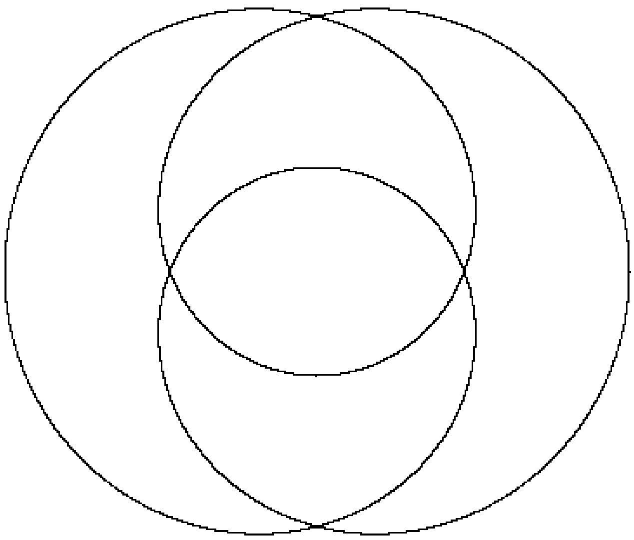
The above definitions may seem too general, but one should keep in mind the following image as inspiration, and the definitions can still be changed or are chosen in a way to generate mathematical examples that can be investigated for algorithmic questions:



A mathematical example is given by the following image:



We imagine that the above image corresponds to the track space of a railway network, which has not yet been divided into block sections and has no trains and timetable:

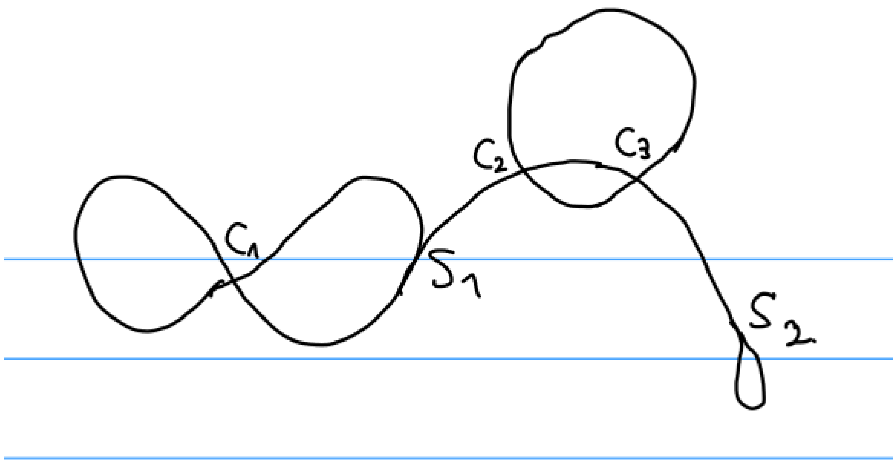


After some further consideration, the track space S should have the following properties:

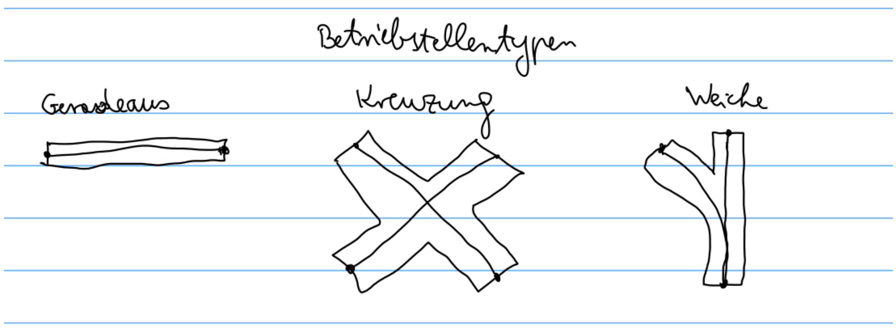
It consists of a finite family of curves in $\mathbf{R}^2$ that satisfy the following conditions:

- The families intersect at a finite set of nodes.
- Each node can be either a 'switch' or a 'crossing.'
- At each 'switch,' three curves touch with the same tangent, with two curves entering from one direction and one from the other.
- At each 'crossing,' two curves intersect or one curve intersects itself. (At each crossing, there are exactly two non-parallel tangents that intersect.)
- Away from the finite set of nodes, the curves are smooth and do not touch or intersect each other.

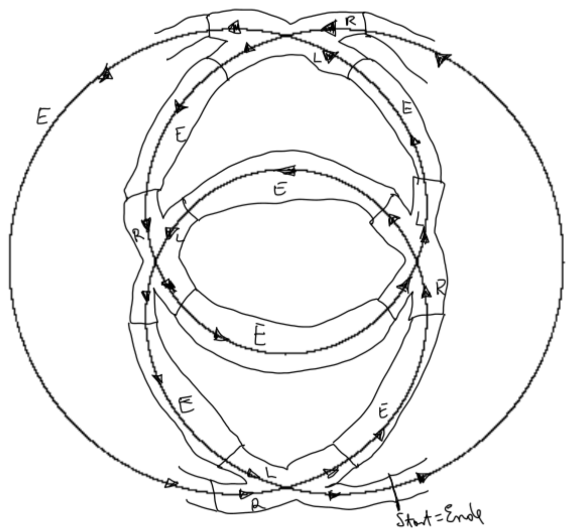
I have added an image to describe this geometric construction more precisely. C_1, C_2, and C_3 are crossings, and S_1 and S_2 are switches.



Next, we consider decompositions of the track space consisting of three types of block sections (switch, crossing, straight):



Example:



Start $\mathbb{1} = E,\ R,\ R \cdot E = R,\ R^2,\ R^2 L,\ R^2 L^2,\ R^2 L^2 R,$
$\ldots, R^2 L^2 R^2 L^3 = $ Ende.

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow R^2 L^2 R^2 L^3 = \begin{pmatrix} 41 & 12 \\ 17 & 5 \end{pmatrix}$$

$$\frac{41}{17} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3}}} = [2; 2, 2, 3]$$

Idee: Stelle eine Verbindung her zwischen Pfaden und rationalen Zahlen in Form von Kettenbrüchen.

We can parameterize the function β_z by arc length if we assume that for all $z \in \mathbf{Z}$ and $\forall t \in \mathbf{T}$:

$$\forall z \in \mathbf{Z}, \forall t \in \mathbf{T} : \beta_z'(t) \neq 0 \tag{14}$$

Let $\alpha_z(t)$ be the arc length of train z at time t. Then:

$$\alpha_z(t) = \int_{t_S}^{t} |\beta_z'(\tau)| d\tau \tag{15}$$

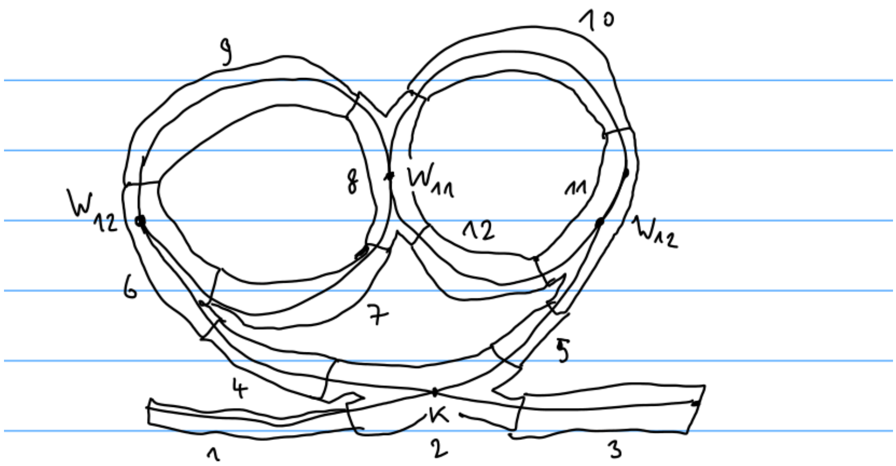
Since we have assumed in Equation (18) that $\beta_z'(\tau) \neq 0$, $\alpha_z(t)$ is strictly monotonically increasing. Thus, it has an inverse function (or we can solve for t given α), and we can write:

$$\beta_z(\alpha) = \beta_z(t_z(\alpha)), \text{ for } \alpha \geq 0 \tag{16}$$

3 Definition of the Track Space $\mathbb{S}$

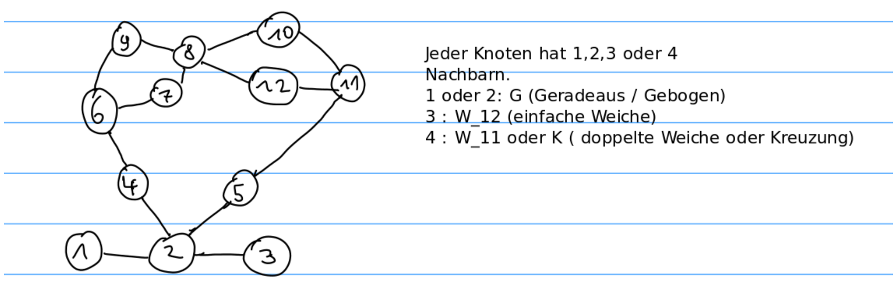
Bemerkung 3.1. *In the railway network, there are, among other things, simple switches, double switches, crossings, and straight or curved track sections. We want to attempt a mathematical formulation of a railway network, called track space, which includes simple switches, double switches, and crossings. Afterwards, we aim to divide this track space into block sections of types: G (straight/curved), W_{11} (double switches), W_{21} (simple switches), K (crossings). In each block section, only one train is allowed during a specific time period, following Oskar Happel's (1959) theory of time-locking staircases. In the next image, an example of a track space divided into block sections is shown.*

Beispiel eines Streckenraums S unterteilt in Blockabschnitte:



We want to define an associated graph for each track space:

zugehöriger ungerichteter zusammenhängender planarer Graph:



The track space **S** is defined by:

$$\mathbf{S} = \cup_{i=1}^{c} r_i([0,1]) \tag{17}$$

where the differentiable paths $r_i : [0,1] \to \mathbb{R}^2, i = 1, \cdots, c$ are subject to the following properties (Eqs. $18-24$):

The paths are simple, i.e., they do not intersect each other:

$$r_i(t_1) \neq r_i(t_2) \forall i = 1, \cdots, c \forall t_1 \neq t_2, t_1, t_2 \in [0,1) \tag{18}$$

(However, a path r_i can be closed, i.e., $r_i(0) = r_i(1)$.)

Different paths intersect at most at one point:

$$\forall i \neq j : |r_i([0,1]) \cap r_j([0,1])| \leq 1 \tag{19}$$

There exists a finite set $P \subset \mathbf{S}$, such that:

$$\begin{aligned} P &= K \cup W_{11} \cup W_{12}, \\ |P| &= |K| + |W_{11}| + |W_{12}| < \infty, \text{ (as shown below)} \\ K \cap W_{11} &= K \cap W_{12} = W_{11} \cap W_{12} = \emptyset \text{ (as shown below)} \end{aligned} \tag{20}$$

where the definition of K, W_{11}, W_{12} is given by:

$$K = \{p \in S | \exists! \{i,j\}, 0 < t_1, t_2 < 1 : r_i(t_1) = r_j(t_2) = p, r_i'(t_1) \neq r_j'(t_2)\} \tag{21}$$

$$W_{11} = \{p \in S | \exists! \{i,j\}, 0 < t_1, t_2 < 1 : r_i(t_1) = r_j(t_2) = p, r_i'(t_1) = r_j'(t_2)\} \tag{22}$$

$$\begin{aligned} W_{12} = \{p \in S | \exists! \{i,j,k\}, \\ \exists! (t_1, t_2, t_3) \in \{(0,0,1), (0,1,0), (1,0,0)\} : \\ r_i(t_1) = r_j(t_2) = r_k(t_3) = p, r_i'(t_1) = r_j'(t_2) = r_k'(t_3)\} \end{aligned} \tag{23}$$

Furthermore, at the finitely many points of P, at least two but at most three paths should intersect:

$$\begin{aligned} \forall p \in P : \\ 2 \leq |\{i : 1 \leq i \leq c, \exists t \in [0,1] : r_i(t) = p\}| \leq 3 \end{aligned} \tag{24}$$

Outside of P, there are no further intersections:

$$\begin{aligned} \forall i \neq j : \\ \text{If } q := r_i(t_1) = r_j(t_2) \text{ for } t_1, t_2 \in (0,1), \\ \text{then it follows that } q \in P \\ \text{However, the case } r_i(1) = r_j(0) \text{ cannot occur in } S \text{ but not in } P. \end{aligned} \tag{25}$$

Proposition 3.2. *Let* $U := \{q \in S | \exists i \neq j : r_i(t_1) = r_j(t_2)$ *for* $t_1, t_2 \in [0,1]\}$. *Then,* $P \subset U$ *and* $0 \leq |P| \leq |U| \leq \frac{c(c-1)}{2} < \infty$.

Proof. Let $p \in P$.

If $p \in K$, there exist $i \neq j : r_i(t_1) = r_j(t_2)$ for $t_1, t_2 \in [0,1]$, so $p \in U$.

If $p \in W_{11}$, there exist $i \neq j : r_i(t_1) = r_j(t_2)$ for $t_1, t_2 \in [0,1]$, so $p \in U$.

If $p \in W_{12}$, there exist $i, j, k : r_i(t_1) = r_j(t_2) = r_k(t_3)$ for $t_1, t_2, t_3 \in \{(0,0,1), (0,1,0), (1,0,0)\}$, so $p \in U$. Overall, $p \in U$, and hence, $P \subset U$.

According to equation 19, two paths intersect at most at one point, and since there are $c < \infty$ paths, and each path can intersect with the other paths at most once, the number of intersection points is at most $\frac{c(c-1)}{2}$, thus $|U| \leq \frac{c(c-1)}{2} < \infty$.

Thus, we have

$$0 \leq |P| \leq |U| \leq \frac{c(c-1)}{2} < \infty.$$

□

Bemerkung 3.3. *Therefore, the finite points in* P *can be classified into the following categories:*

$p \in K$ *is a 'crossing point': There are exactly two different paths* r_i, r_j *and* $0 < t_1, t_2 < 1$ *with* $r_i(t_1) = r_j(t_2) = p$, $r_i'(t_1) \neq r_j'(t_2)$.

$p \in W_{11}$ *is a 'double switch point': There are exactly two different paths* r_i, r_j *and* $0 < t_1, t_2 < 1$ *with* $r_i(t_1) = r_j(t_2) = p$, $r_i'(t_1) = r_j'(t_2)$.

$p \in W_{12}$ *is a 'single switch point': There are exactly three pairwise different paths* r_i, r_j, r_k *and* $(t_1, t_2, t_3) \in \{(0,0,1), (0,1,0), (1,0,0)\}$ *with* $r_i(t_1) = r_j(t_2) = r_k(t_3) = p$, $r_i'(t_1) = r_j'(t_2) = r_k'(t_3)$.

Proposition 3.4. *With the above definitions of* K, W_{11}, W_{12}, *it holds that* $K \cap W_{11} = K \cap W_{12} = W_{11} \cap W_{12} = \emptyset$

Proof. Assume $p \in K \cap W_{11}$. Then there exist $i \neq j$ and t_1, t_2 with $r_i'(t_1) = r_j'(t_2)$ since $p \in W_{11}$, but also $r_i'(t_1) \neq r_j'(t_2)$ since $p \in K$, which is a contradiction. Thus, $K \cap W_{11} = \emptyset$.

Assume $p \in K \cap W_{12}$. Then there exist $i \neq j$ and $0 < t_1, t_2 < 1$ with $p = r_i(t_1) = r_j(t_2)$ and $r_i'(t_1) \neq r_j'(t_2)$ since $p \in K$. Furthermore, there exist pairwise different a, b, c and $(t_a, t_b, t_c) \in \{(0,0,1), (0,1,0), (1,0,0)\}$ with $p = r_a(t_a) = r_b(t_b) = r_c(t_c)$ and $r_a'(t_a) = r_b'(t_b) = r_c'(t_c)$. According to equation 24, the set $\{i, j, a, b, c\}$ contains two or three elements. As a, b, c are pairwise different, this set must contain three elements, i.e., $\{i, j, a, b, c\} = \{a, b, c\}$. Without loss of generality, assume $i = a$ and $j = b$. Then $r_a(t_a) = p = r_i(t_1) = r_a(t_1)$, and $t_a \in \{0,1\}, 0 < t_1 < 1$, which implies $t_a \neq t_1$. This leaves the following possibilities according to equation 18: $(t_a, t_1) = (0,1)$, which contradicts $t_1 < 1$, or $(t_a, t_1) = (1,0)$, which contradicts $t_1 > 0$. Hence, our assumption $K \cap W_{12} \neq \emptyset$ cannot hold, and it must be $K \cap W_{12} = \emptyset$.

The case $W_{11} \cap W_{12} = \emptyset$ is proven analogously:

Assume $p \in W_{11} \cap W_{12}$. Then there exist $i \neq j$ and $0 < t_1, t_2 < 1$ with $p = r_i(t_1) = r_j(t_2)$ and $r_i'(t_1) = r_j'(t_2)$ since $p \in K$. Furthermore, there exist pairwise different a, b, c

and $(t_a, t_b, t_c) \in \{(0,0,1), (0,1,0), (1,0,0)\}$ with $p = r_a(t_a) = r_b(t_b) = r_c(t_c)$ and $r'_a(t_a) = r'_b(t_b) = r'_c(t_c)$. According to equation 24, the set $\{i, j, a, b, c\}$ contains two or three elements. As a, b, c are pairwise different, this set must contain three elements, i.e., $\{i, j, a, b, c\} = \{a, b, c\}$. Without loss of generality, assume $i = a$ and $j = b$. Then $r_a(t_a) = p = r_i(t_1) = r_a(t_1)$, and $t_a \in \{0, 1\}, 0 < t_1 < 1$, which implies $t_a \neq t_1$. This leaves the following possibilities according to equation 18: $(t_a, t_1) = (0, 1)$, which contradicts $t_1 < 1$, or $(t_a, t_1) = (1, 0)$, which contradicts $t_1 > 0$. Hence, our assumption $W_{11} \cap W_{12} \neq \emptyset$ cannot hold, and it must be $W_{11} \cap W_{12} = \emptyset$.

□

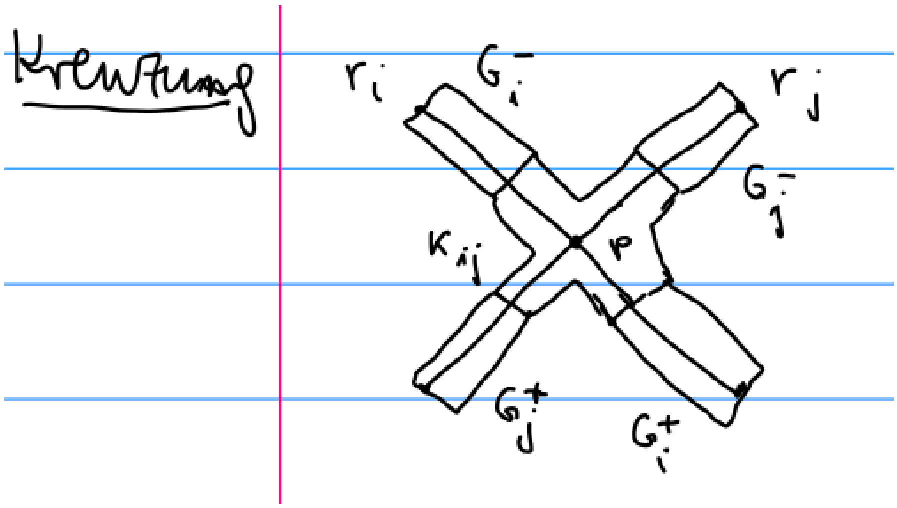
Proposition 3.5. *At every crossing point $p \in K$ and at every point of a double switch $p \in W_{11}$, there exist $i \neq j, 1 \leq i, j \leq c$ and closed sets $K_{ij}, G_i^-, G_i^+, G_j^-, G_j^+$ of S such that:*

$$r_i([0,1]) \cup r_j([0,1]) = K_{ij} \cup G_i^- \cup G_i^+ \cup G_j^- \cup G_j^+ \tag{26}$$

and it is

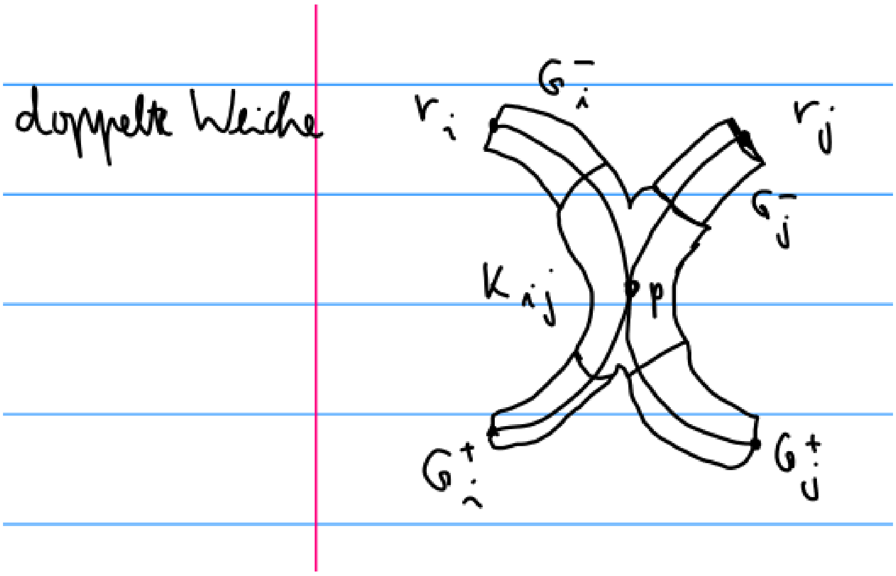
$$|K_{ij} \cap G| = 1 \text{ for } G \in \{G_i^-, G_i^+, G_j^-, G_j^+\} \tag{27}$$

$$|G \cap H| = 0 \text{ for } G \neq H, G, H \in \{G_i^-, G_i^+, G_j^-, G_j^+\} \tag{28}$$



Proof.

First, we prove the case for a crossing point: Since $p \in K$, there exist $t_1, t_2, 0 < t_1, t_2 < 1$ and $i \neq j$ such that $r_i(t_1) = p = r_j(t_2)$. Set $G_i^- := \{r_i(t) | 0 \leq t \leq \frac{t_1}{2}\}$, $G_j^- := \{r_j(t) | 0 \leq t \leq \frac{t_2}{2}\}$, $G_i^+ := \{r_i(t) | t_1 + \frac{1-t_1}{2} \leq t \leq 1\}$, $G_j^+ := \{r_j(t) | t_2 + \frac{1-t_2}{2} \leq t \leq 1\}$, and $K_{ij} := \{r_i(t) | \frac{t_1}{2} \leq t \leq t_1 + \frac{1-t_1}{2}\} \cup \{r_j(t) | \frac{t_2}{2} \leq t \leq t_2 + \frac{1-2_1}{2}\}$. Then, $G_i^- \cap K_{ij} = \{r_i(\frac{t_1}{2})\}$, $G_j^- \cap K_{ij} = \{r_j(\frac{t_2}{2})\}$, $G_i^+ \cap K_{ij} = \{r_i(t_1 + \frac{1-t_1}{2})\}$, $G_j^+ \cap K_{ij} = \{r_j(t_2 + \frac{1-t_2}{2})\}$, and $r_i([0,1]) \cup r_j([0,1]) = K_{ij} \cup G_i^- \cup G_i^+ \cup G_j^- \cup G_j^+$, which shows equations 26 and 27. For equation 28, we observe that r_i and r_j already intersect at p, so $G \cap H = \emptyset$ for $G \neq H, G, H \in \{G_i^-, G_i^+, G_j^-, G_j^+\}$; otherwise, there would be another point q as an intersection of r_i and r_j, which is not possible according to equation 19.



In the case of a double switch $p \in W_{11}$, there exist $t_1, t_2, 0 < t_1, t_2 < 1$ and $i \neq j$ such that $r_i(t_1) = p = r_j(t_2)$, and we proceed analogously as in a crossing point. $\square$

Proposition 3.6. *At every point of a single switch $p \in W_{12}$, there exist $i, j, k, 1 \leq i, j, k \leq c$ pairwise different and closed sets B, B_i, B_j, B_k of S, such that:*

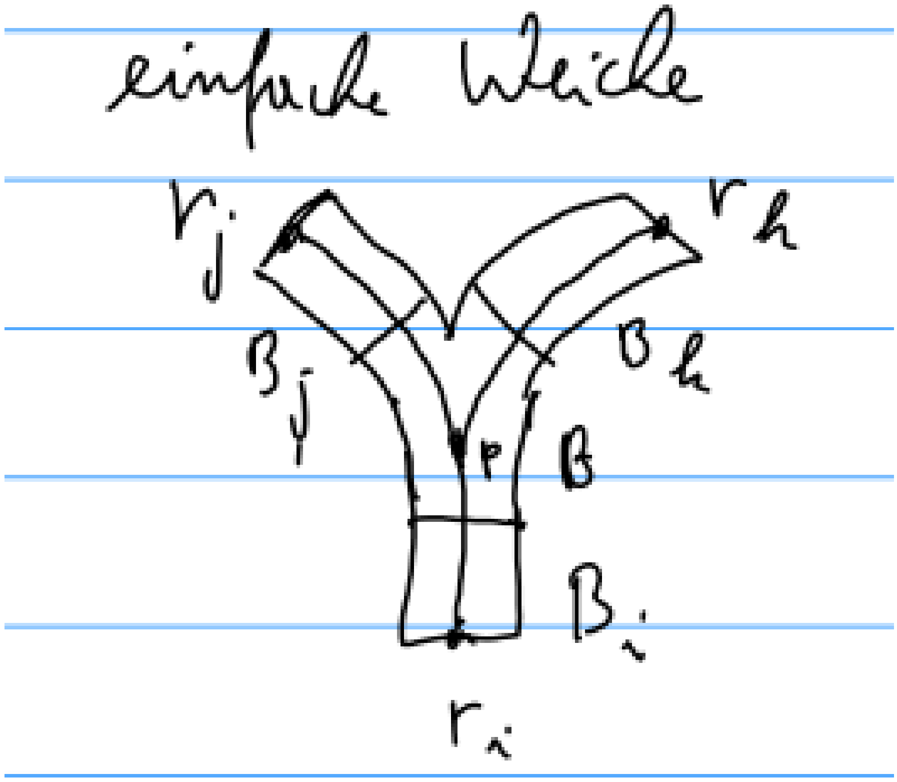
$$r_i([0,1]) \cup r_j([0,1]) \cup r_k([0,1]) = B \cup B_i \cup B_j \cup B_k \tag{29}$$

and it is

$$|X \cap Y| = 0 \text{ for } X, Y \in \{B_i, B_j, B_k\} \tag{30}$$

and

$$|B \cap X| = 1 \text{ for } X \in \{B_i, B_j, B_k\} \tag{31}$$



Proof.
Since $p \in W_{12}$, there exist three pairwise different paths r_i, r_j, r_k and $(t_1, t_2, t_3) \in \{(0,0,1), (0,1,0), (1,0,0)\}$ such that $r_i(t_1) = r_j(t_2) = r_k(t_3) = p$, $r_i'(t_1) = r_j'(t_2) = r_k'(t_3)$. Without loss of generality, assume $(t_1, t_2, t_3) = (1,0,0)$. We set $B_i := \{r_i(t)|0 \leq t \leq \frac{1}{2}\}, B_j := \{r_j(t)|1/2 \leq t \leq 1\}, B_k := \{r_k(t)|1/2 \leq t \leq 1\}$, and $B := \{r_i(t)|1/2 \leq t \leq 1\} \cup \{r_j(t)|0 \leq t \leq 1/2\} \cup \{r_k(t)|0 \leq t \leq 1/2\}$, and equation 29 follows. Equation 30 follows because r_i, r_j, r_k already intersect at p and since two different paths intersect at most at one point according to equation 19, we have $|X \cap Y| = 0$ for $X, Y \in \{B_i, B_j, B_k\}$, and equation 30 is shown. It holds that $B \cap B_x = \{r_x(1/2)\}$ for $x \in \{i, j, k\}$. □

Proposition 3.7. *It is $S = \cup_{k=1}^{b} B_k$ for a natural number b, and $\mathbf{B} := \{B_1, \cdots, B_b\}$. The sets B_k are closed, non-empty, path-connected, and it holds*

$$|B_i \cap B_j| \in \{0, 1\}$$

Proof. TODO □

Definition 3.1. *The associated undirected graph $G_S := (\mathbf{B}, E)$ is defined by:*

$$B_i \approx B_j :\Longleftrightarrow |B_i \cap B_j| = 1 \tag{32}$$

We divide the block sections into 'Straight' and 'Bridges':

$$\mathbf{G} := \{B \in \mathbf{B} | B \cap P = \emptyset\} =: \text{'Straight'} \tag{33}$$

$$\mathbf{G}^c := \mathbf{B} - \mathbf{G} = \{B \in \mathbf{B} | B \cap P \neq \emptyset\} =: \text{'Bridges'} \tag{34}$$

The bridges $\mathbf{G}^c$ are only those block sections of type simple switch, double switch, and crossing, while in $\mathbf{G}$, the block sections are of type straight.

Let

$$N_B := \cup_{B \approx B_j, B \neq B_j} B_j = B \cap \overline{B^c} = B \cap \overline{(S - B)} \tag{35}$$

be the set of 'direction points of B'. Then, $|N_B| < \infty$ and $\deg(B) = |N_B|$, where deg *is the degree of B in G_S. For each $B \in \mathbf{B}$, there exists a 'direction function' $\rho_B : N_B \to \mathcal{P}(N_B)$, which describes the possible direction points $\rho_B(x)$ that are a subset of N_B when a train enters the block section B at a direction point x in N_B. This function is determined / computed based on the paths r_i and the block section type (G, W_{11}, W_{12}, K). Using this function, one can represent how a train can move at a simple / double switch or crossing.*

A bridge $D \in \mathbf{G}^c$ 'leads from B to C' (with $B, C \in \mathbf{G}$) $:\Longleftrightarrow$

$$\begin{aligned} |B \cap C| = 0 \text{ and } |B \cap D| = 1 \text{ and } |D \cap C| = 1 \\ \text{and } (N_C \cap N_D) \subset \rho_B(N_B \cap N_D) \end{aligned} \tag{36}$$

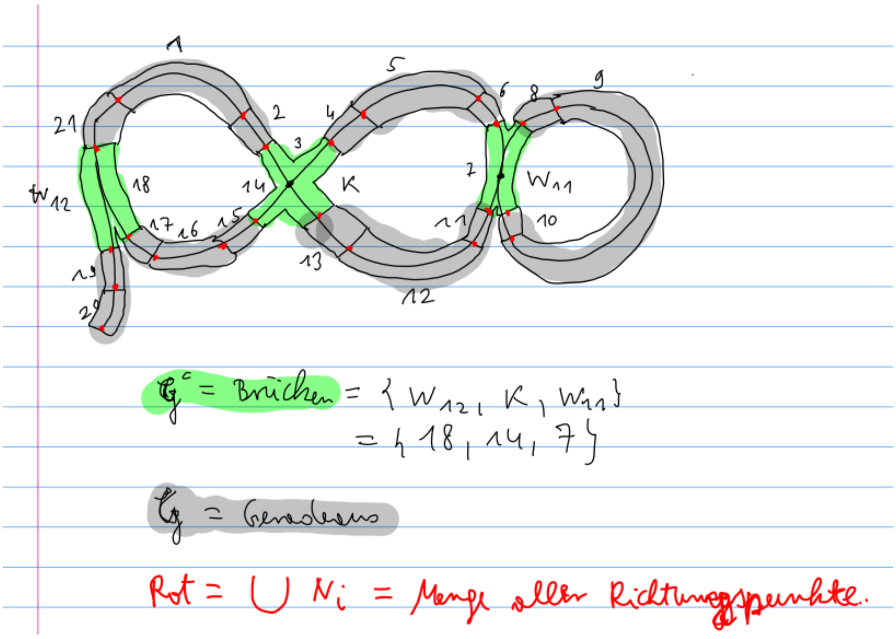
(Each block section / node B in G_S has a 'weight $\mu(B)$' which corresponds to the sum of the arc lengths of the respective r_i's. Each edge $\{B, C\}$ has a 'weight $\mu(\{B, C\}) := \mu(B) + \mu(C)$'.) :

Let $H_S = (\mathbf{G}, E_H)$, where the edges are defined as:

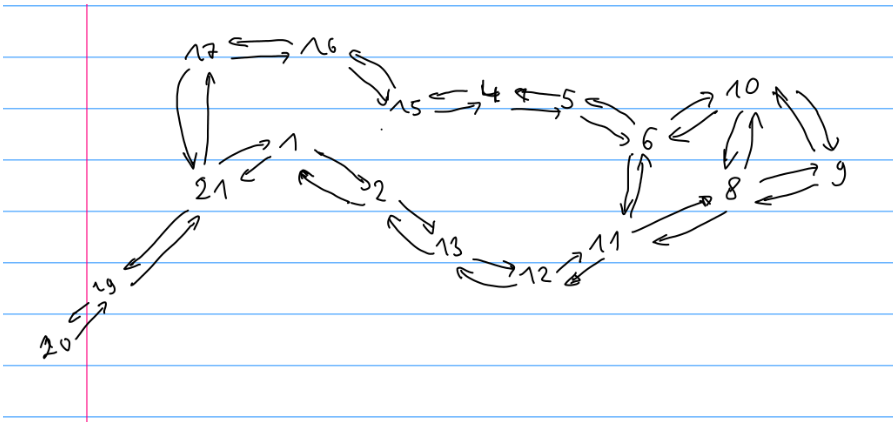
$$\begin{aligned} B \to D :\Longleftrightarrow \\ \text{Either } B \text{ and } D \text{ are connected in } G_S \\ \text{or they are not connected in } G_S \\ \text{and there is a bridge } C \in \mathbf{G}^c \text{, leading from } B \text{ to } D. \end{aligned} \tag{37}$$

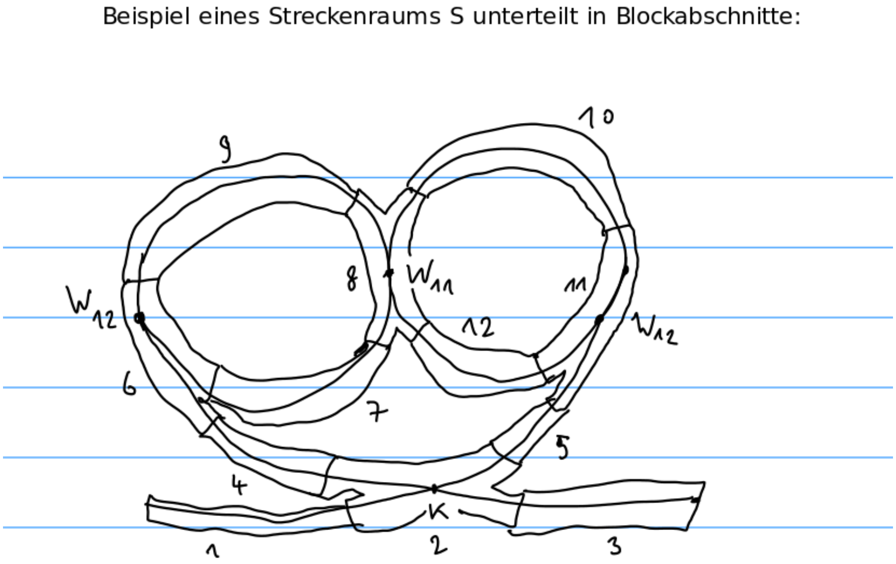
In principle, this directed graph H_S is created by removing the bridges from G^c and considering the direction function.

Examples for the last definition:

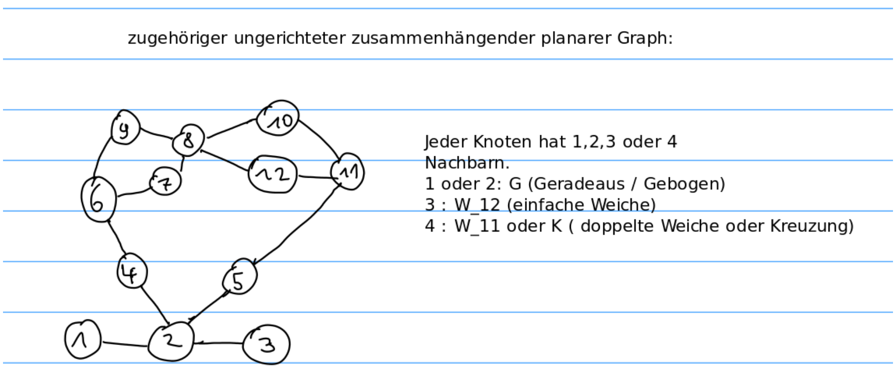


The associated directed graph H_S is given by:



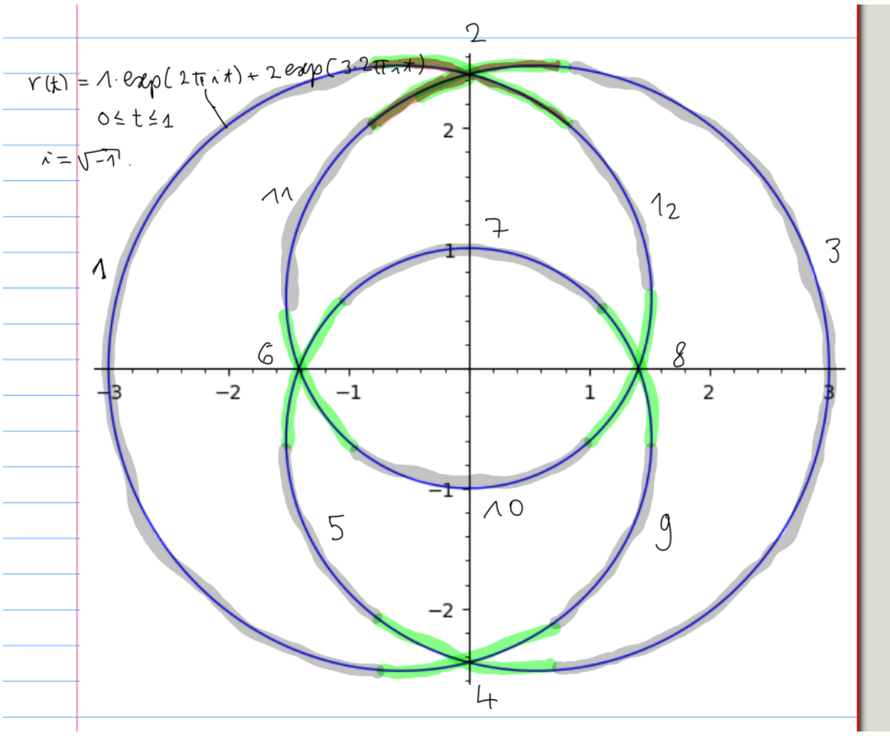


We want to define an associated graph for each track space:

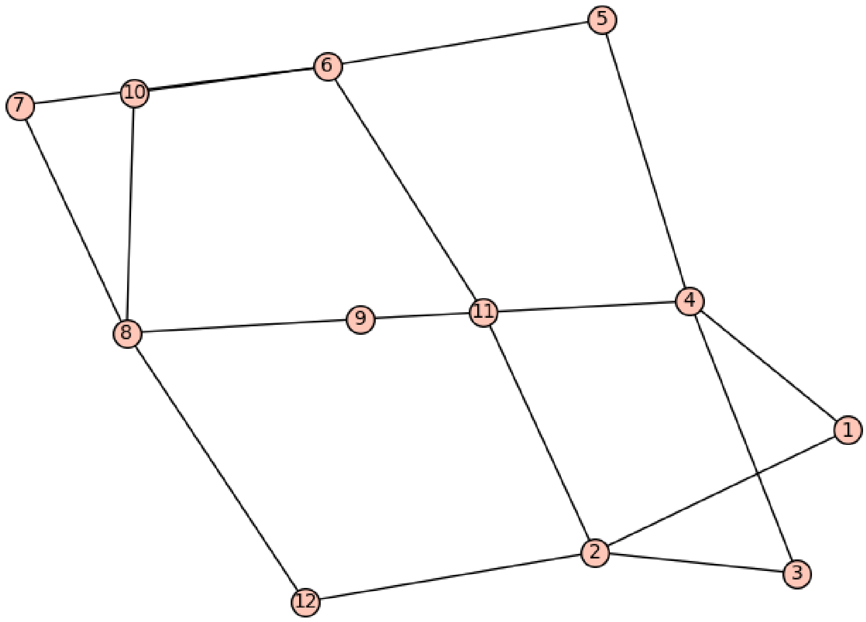


Further examples:

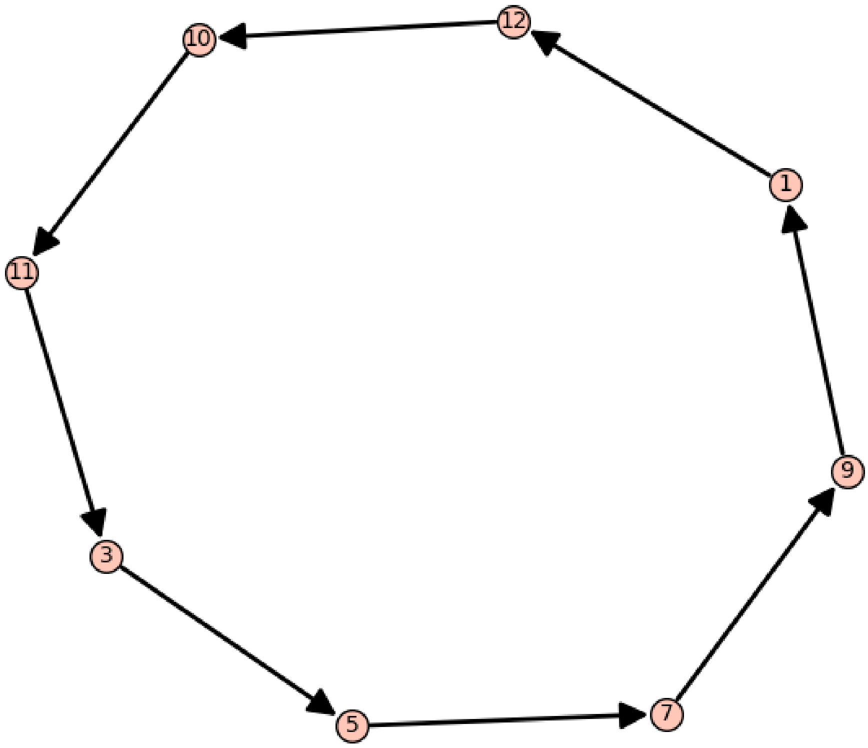
Let $r(t) := 1 \cdot \exp(1 \cdot 2\pi\sqrt{-1}t) + 2 \cdot \exp(3 \cdot 2\pi\sqrt{-1}t), 0 \leq t \leq 1$. We decompose $r(t)$ into 'smaller paths' and then into block sections:



The undirected graph is then given by:



The directed graph is given by:



Here, it can be observed that in principle, there can be no branching during the passage of the track space, as a train at crossings can only go straight ahead without diverging. This graph reflects that the track space has been defined by a closed path. Therefore, the graph is cyclic.

Definition 3.2. *The perspective plane P_z of the train $z \in \mathbf{Z}$ is $\mathbb{R}^2$. For each train z, there are mappings:*

$$\hat{\alpha}_z : \mathbf{T} \to P_z, t \mapsto (\int_{t_S}^{t} |\beta_z'(\tau)| d\tau, t) = (\alpha_z(t), t)$$

and

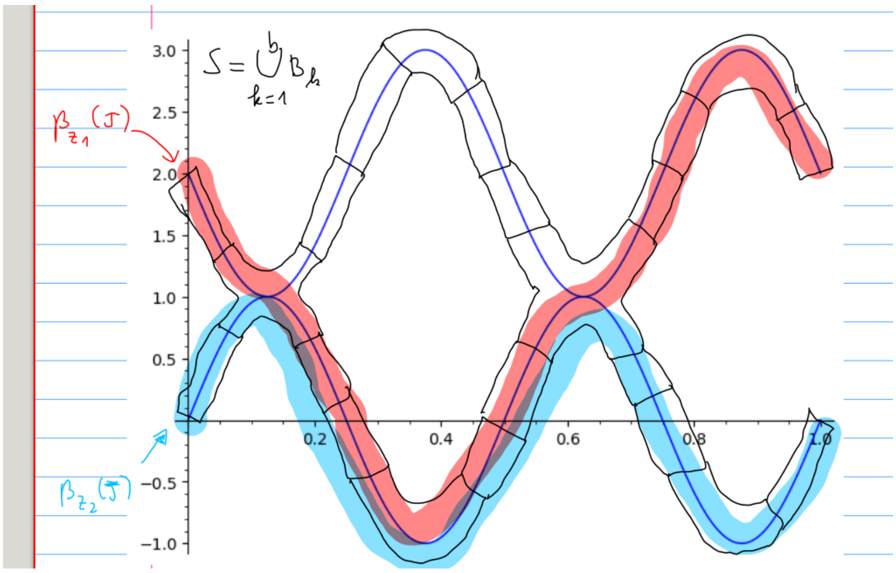
$$\hat{t}_z : [0, \alpha_{E,z}] \to P_z, \alpha \mapsto (\alpha, t_z(\alpha))$$

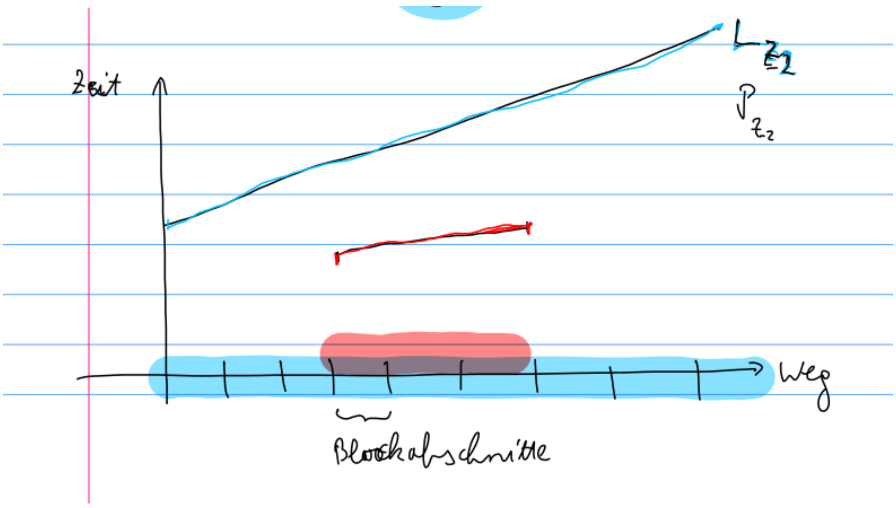
.

where $\alpha_{E,z} := \int_{t_S}^{t_E} |\beta_z'(\tau)| d\tau$ is the total length of the track that train z travels during the time period $\mathbf{T} = [t_S, t_E]$ and $t_z(\alpha)$ indicates the time point if the arc length α of train z is given. Furthermore, $\mathbf{T} = [t_S, t_E]$ is the time interval of the given timetable. Here, we identify the track section $\{\beta_z(t) \in S | t_S \leq t \leq t_E\}$ that the train z travels on S with the beginning of the x-axis of $\mathbb{R}^2$ starting at the origin and extending to the right.

The worldline L_z of the train $z \in \mathbf{Z}$ is defined as $L_z := \{(\alpha, t_z(\alpha)) \in P_z | 0 = \alpha_S \leq \alpha \leq \alpha_{E,z}\} = \hat{\alpha}_z(\mathbf{T})$. The x-coordinate of a point on the worldline L_z represents the arc *length of train z, and the y-coordinate represents the time at which the train has traveled the arc length.*

Bemerkung 3.8. *The perspective plane P_z of train z represents the plane containing the worldline L_z, and other worldlines of other trains could be drawn in this plane. In manual timetable planning, this perspective is used to schedule trains collision-free. It is important to note that the definition of P_z and L_z depends on the path that train z travels on the track space S using β_z. In other words, two trains that have disjoint paths will also have disjoint perspective planes. Only for trains that share a common path (partially), the rectangles of their respective perspective planes can intersect, and one worldline may appear/partially appear in the perspective of the other train.*



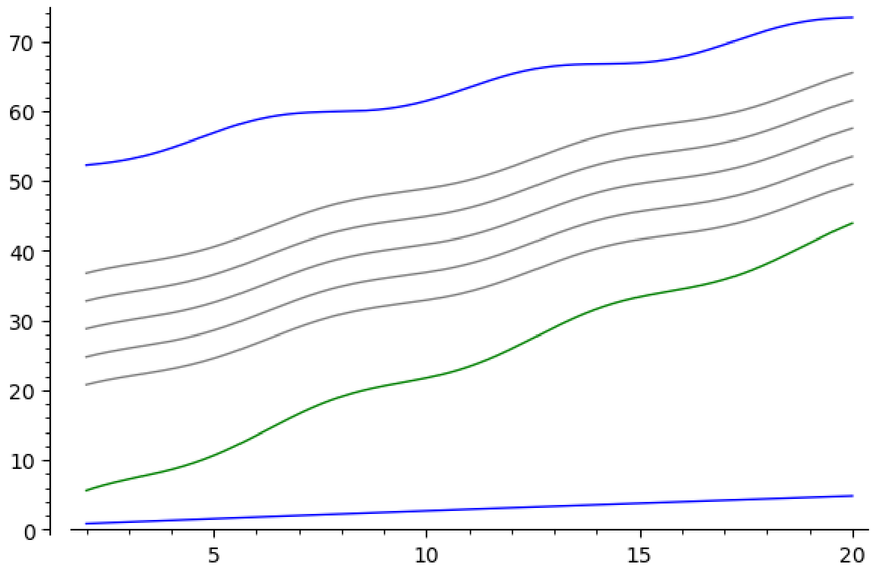


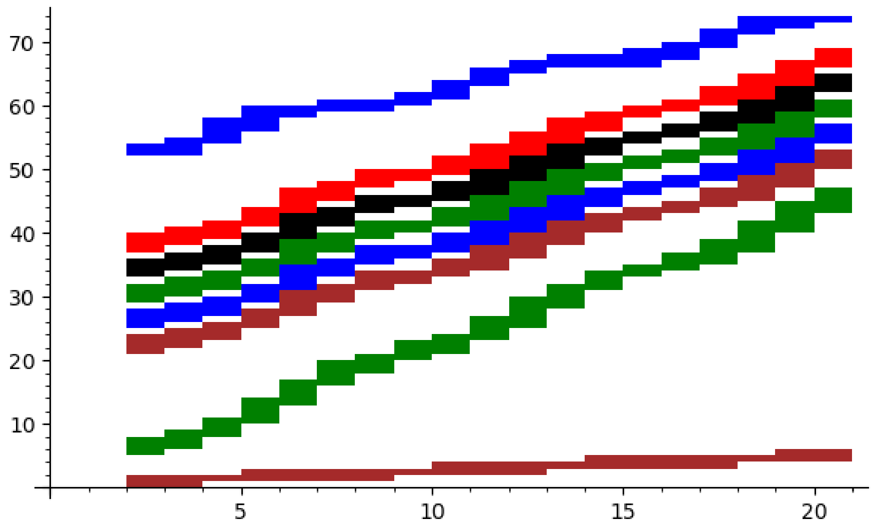
Definition 3.3. *For each straight block section $B \in \mathbf{G}$, the perspective of B is defined*

as:

$$P_B := \cup_{\beta_z(\mathbf{T})\cap B\neq\emptyset}\hat{\alpha}_z(\beta_z^{-1}(B))$$

(With this definition, we aim to describe parts of the worldlines of all trains that can be observed from the perspective of an individual straight block section. The idea is to combine the perspectives of several straight block sections that form a path in the graph H_S to consider the worldlines of all trains that travel on this path in the graph H_S.)





Proposition 3.9. *For each straight block section $B \in \mathbf{G}$ and two distinct trains $z_1 \neq z_2$, it holds:*

$$\hat{\alpha}_{z_1}(\beta_{z_1}^{-1}(B)) \cap \hat{\alpha}_{z_2}(\beta_{z_2}^{-1}(B)) = \emptyset$$

(In other words: The worldlines of different trains at a straight block section do not intersect.)

Proof. Assume there exists a point $(\alpha_{z_1}(t_1), t_1) = (\alpha_{z_2}(t_2), t_2)$ in the intersection. Then, we have $t := t_1 = t_2$, and it follows that $t = t_2 = t_1 \in \beta_{z_1}^{-1}(B)$ and $t = t_1 = t_2 \in \beta_{z_2}^{-1}(B)$. Thus, $\beta_{z_1}^{-1}(B) \cap \beta_{z_2}^{-1}(B)$ contains the time point t, which contradicts the assumption $\beta_{z_1}^{-1}(B) \cap \beta_{z_2}^{-1}(B) = \emptyset$ (Eq. 5) in a timetable. □

4 Quadratic Optimization on a Perspective Plane

In this section, we want to define a density for a vector and examine when this density is minimized. Based on this, we want to extend this density to functions that correspond to the worldlines of trains and attempt to describe an algorithm that shifts an existing worldline of a train in a given perspective plane so that the density becomes as small as possible.

Definition 4.1. *For a vector $x = (x_1, \cdots, x_n)$, we define the density $\delta(x)$ as:*

$$\delta(x) := \sqrt{\sum_{i=1}^{n-1} |x_i - x_{i+1}|^2} \tag{38}$$

Proposition 4.1. *For all $x = (x_1, \cdots, x_n)$: The density is minimized if:*

$$x_{i+1} - x_i = \frac{x_n - x_1}{n-1} \text{ for all } i = 1, \cdots, n-1 \tag{39}$$

In this case, it holds:

$$\delta(x) = \frac{|x_n - x_1|}{\sqrt{n-1}} \tag{40}$$

while for any x:

$$\delta(x) \geq \frac{|x_n - x_1|}{\sqrt{n-1}} \tag{41}$$

Proof. Assume $x_{i+1} - x_i = \frac{x_n - x_1}{n-1}$ for all $i = 1, \cdots, n-1$. Then we have:

$$\delta(x) = \sqrt{\sum_{i=1}^{n-1} |x_i - x_{i+1}|^2} = \sqrt{\sum_{i=1}^{n-1} \frac{(x_n - x_1)^2}{(n-1)^2}}$$
$$= \sqrt{\frac{(n-1)(x_n - x_1)^2}{(n-1)^2}} = \frac{|x_n - x_1|}{\sqrt{n-1}}$$

Now, for an arbitrary x, set $a_i := x_{i+1} - x_i, b_i := 1$ for $i = 1, \cdots, n-1$, and $a = (a_1, \cdots, a_{n-1}), b = (b_1, \cdots, b_{n-1})$. By Cauchy-Schwarz, we have:

$$\langle a, b \rangle^2 \leq \langle a, a \rangle \cdot \langle b, b \rangle$$

$$\iff (a_1^2 + \cdots + a_{n-1}^2)(b_1^2 + \cdots + b_{n-1}^2) \geq (a_1 b_1 + \cdots + a_{n-1} b_{n-1})^2$$
$$\iff ((x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2) \cdots (n-1) \geq$$
$$((x_1 - x_2) \cdot 1 + (x_2 - x_3) \cdot 1 + \cdots + (x_{n-1} - x_n) \cdot 1)^2 = (x_n - x_1)^2$$

Hence:

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2 \geq \frac{(x_n - x_1)^2}{n-1}$$

or equivalently:

$$\delta(x) = \sqrt{(x_1 - x_2)^2 + \cdots + (x_{n-1} - x_n)^2} \geq \frac{|x_n - x_1|}{\sqrt{n-1}}$$

□

Definition 4.2. *Let $f = (f_1, \cdots, f_n)$ be a vector of functions that are quadratically integrable over $[x_1, x_2]$. Define the 'density of' f as:*

$$\delta(f) := \sqrt{\sum_{i=1}^{n-1} |f_i - f_{i+1}|^2} \tag{42}$$

where $|g| := \sqrt{\int_{x_1}^{x_2} g(x)^2 dx}$ is the $L_2([x_1, x_2])$ norm.

Proposition 4.2. *For all $f = (f_1, \cdots, f_n)$: (Conjecture) The density of f is minimized if:*

$$f_{i+1} - f_i = \frac{f_n - f_1}{n-1} \text{ for all } i = 1, \cdots, n-1 \tag{43}$$

(Conjecture) In this case, it holds:

$$\delta(f) = \frac{|x_n - x_1|}{\sqrt{n-1}} \tag{44}$$

while for any f:

$$\delta(f) \geq \frac{|f_n - f_1|}{\sqrt{n-1}} \tag{45}$$

Proof. We prove 45: For $x_1 \leq x \leq x_2$, let $f(x) := (f_1(x), \cdots, f_n(x))$. Then, according to 41, we have:

$$\delta(f(x))^2 = \sum_{i=1}^{n-1} (f_{i+1}(x) - f_i(x))^2 \geq^{41} \frac{|f_n(x) - f_1(x)|^2}{n-1}$$

Thus, we have (∗):

$$\delta(f(x))^2 \geq \frac{|f_n(x) - f_1(x)|^2}{n-1}$$

. Integrating over x, we obtain:

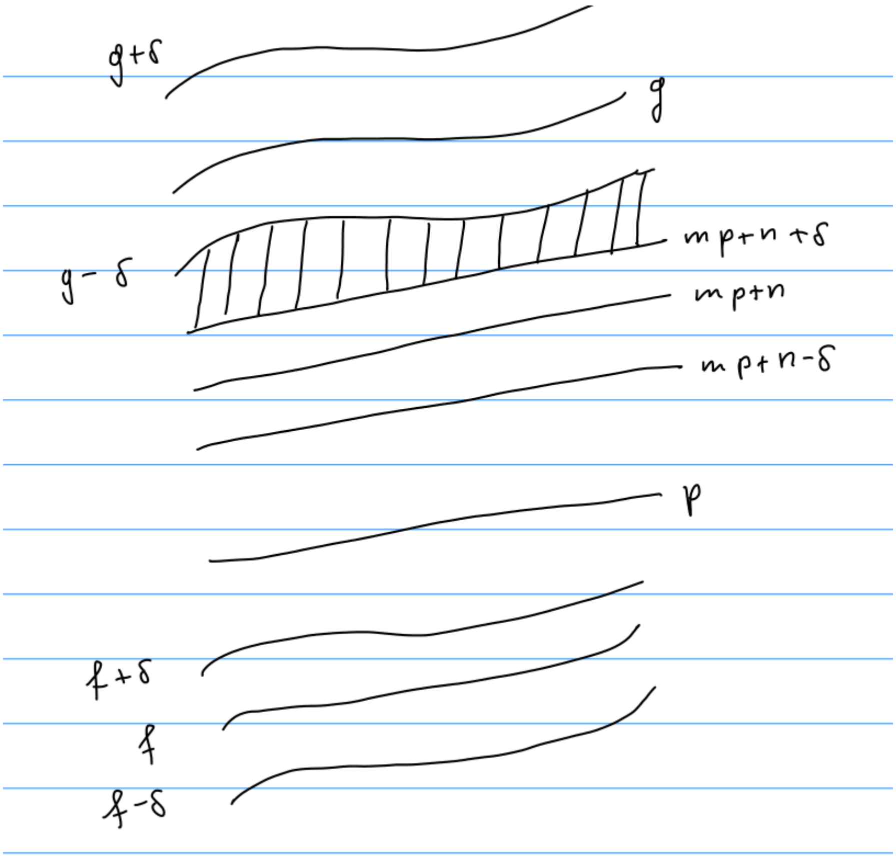
$$\delta(f)^2 = \sum_{i=1}^{n-1} |f_{i+1} - f_i|^2 = \sum_{i=1}^{n-1} \int_{x_1}^{x_2} (f_{i+1}(x) - f_i(x))^2 dx \geq^{(*)}$$
$$\int_{x_1}^{x_2} \frac{|f_n(x) - f_1(x)|^2}{n-1} dx = \frac{1}{n-1} \cdot |f_n - f_1|^2$$

Thus, we have:

$$\delta(f) \geq \frac{|f_n - f_1|}{\sqrt{n-1}}$$

□

Bemerkung 4.3. *Suppose we consider two trains whose worldlines are given by the functions $g > f$ in a perspective plane and we want to shift the worldline of a third train p with a linear transformation $m \cdot p(x) + n$ such that this worldline 'fits well between $f < m \cdot p + n < g$' and 'does not waste too much space'. Multiplying p by m corresponds to changing the speed of the train, while adding n to mp corresponds to shifting departure or arrival times. To simulate the lock stair times, we choose a positive number δ.*



One possible task could be: Choose m, n such that the square of the hatched area $|(g - \delta) - (mp + n + \delta)|^2$ is minimized, subject to the constraints:

$$\delta > 0, \forall x \in [x_1, x_2], g(x) - \delta \geq mp(x) + n + \delta$$

Proposition 4.4. *The task: Choose m, n such that the square of the hatched area $|(g-\delta) - (mp + n + \delta)|^2$ is minimized, subject to the constraints:*

$$\delta > 0, \forall x \in [x_1, x_2], g(x) - \delta \geq mp(x) + n + \delta$$

leads to a quadratic optimization problem.

Proof. It is

$$\begin{aligned} |(g-\delta) - (mp + n + \delta)|^2 &= \int_{x_1}^{x_2} (g(x) - mp(x) - n - 2\delta)^2 dx = \\ &= am^2 + 2bm + 2cmn + 2dn + en^2 + f \end{aligned} \tag{46}$$

where

- $a = \int_{x_1}^{x_2} p(x)^2 dx$
- $b = \int_{x_1}^{x_2} 2\delta p(x) - g(x)p(x) dx$
- $c = \int_{x_1}^{x_2} p(x) dx$
- $d = \int_{x_1}^{x_2} 4\delta - 2g(x) dx$
- $e = x_2 - x_1$
- $f = \int_{x_1}^{x_2} g(x)^2 - 4\delta g(x) + 4\delta^2 dx$

Constraints:

$$\forall x \in [x_1, x_2] : g(x) - \delta \geq mp(x) + n + \delta$$

Idea: Divide the interval $[x_1, x_2]$ into r parts:

$$\hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_r : \hat{x}_i \in [x_1, x_2]$$

$$\text{Constraints :} \tag{47}$$

- $g(\hat{x}_1) - \delta \geq mp(\hat{x}_1) + n + \delta$
- $g(\hat{x}_1) - \delta \geq mp(\hat{x}_1) + n + \delta$
- $\cdots$
- $g(\hat{x}_{r-1}) - \delta \geq mp(\hat{x}_{r-1}) + n + \delta$
- $g(\hat{x}_r) - \delta \geq mp(\hat{x}_r) + n + \delta$

Let $h = (g(\hat{x}_1) - \delta, \cdots, g(\hat{x}_r) - \delta)^T$, $G = ((p(\hat{x}_1, 1), \cdots (p(\hat{x}_r, 1))^T$, $Q = ((2a, 2c), (2c, 2e))^T$, $v = (2b, 2d)^T$, $w = (m, n)^T$.

Then, minimizing 46 subject to the constraints 47 is equivalent to:

$$\begin{aligned} \min_w 1/2 w^T Q w + w^T v \\ G w \leq h \end{aligned} \tag{48}$$

and that is a quadratic optimization problem. □

5 TODOs / Ideas

Idea:

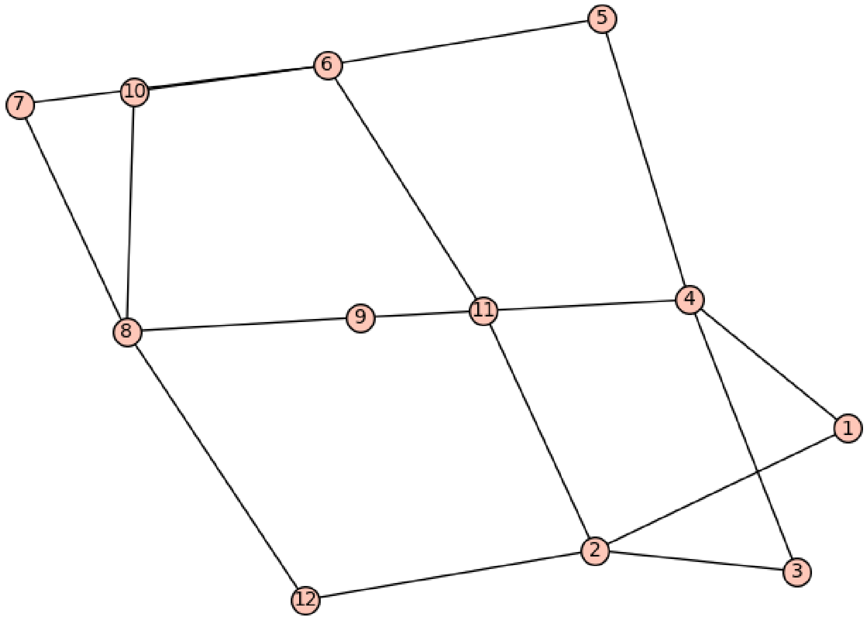
- Use path integrals in the last proposition.
- Analogy to physics: No two trains on the same block section at the same time

  $$\iff$$

  Pauli exclusion principle: No two particles at the same location at the same time.
- This analogy may seem far-fetched, but path integrals are also used in physics (Feynman integral).

Desired properties of the track space S, the set of block sections $\mathbf{B}$, direction points N_i, and bridges G^c:

- $S = \cup_{i=1}^{b} B_i, \mathbf{B} = \{B_1, \cdots, B_b\}$
- $|B_i \cap B_j| \in \{0, 1\}$ for $i \neq j$
- S is closed and should be path-connected.
- $\forall i : B_i$ is closed, path-connected, and non-empty.
- Definition of the undirected graph G_S: $B_i \approx B_j :\iff |B_i \cap B_j| = 1$
- Definition of direction points: $N_i := B_i \cap \overline{(S - B_i)}$ and in the graph G_S: $\deg(B_i) = |N_i| \in \{1, 2, 3, 4\}$
- Definition of block sections of type Straight: $\mathbf{G} := \{B \in \mathbf{B} | \deg(B) = |N_B| \leq 2\}$
- Definition of block sections of type Bridge: $\mathbf{G}^c := \mathbf{B} - \mathbf{G} = \{B \in \mathbf{B} | \deg(B) = |N_B| \in \{3, 4\}\}$
- $\forall B \in \mathbf{G} : \exists l > 0$ and there exists a homeomorphism $\phi_B : B \to [0, l]$
- Existence of a direction function: $\forall B \in \mathbf{B} : \exists \rho_B : N_B \to \mathcal{P}(N_B)$



It is interesting that the automorphism group of the graph, which is generated by

$$[(5, 9)(6, 8)(11, 12), (2, 4)(5, 11)(9, 12), (1, 3), (10, 1)(2, 8)(3, 7)(4, 6)(9, 11)]$$

has order 32 and is non-abelian SmallGroup(32,27), as can be computed with SAGE-MATH:

```
#
G = Graph({1:[2,4],2:[1,3,11,12],
           3:[2,4],4:[1,3],5:[4,6],
           6:[11,7,10,5],7:[6,8],
           8:[9,10,12],9:[4,8],10:[6,8],
           11:[2,6],12:[2,8]})
plot(G).show()
GG = G.automorphism_group()
print(list(GG.gens()))
print(GG.id())
```

- Define on the set of trains $\mathbf{Z}$ in the schedule $\mathbf{F}$ a quasi-order:

$$z_1 \leq_{\mathbf{F}} z_2 :\Longleftrightarrow \beta_{z_1}(\mathbf{T}) \subset \beta_{z_2}(\mathbf{T})$$

- This could potentially be used to draw the world lines of all trains $y \leq x$ in the perspective plane P_x.
- A quasi-order defines a finite topology on the set of all trains, where for example, open sets are defined as: $O_z := \{y \in \mathbf{Z} | y \leq z\}$. Continuous functions f are then

precisely the monotonic functions: f is continuous / monotonic $\iff \forall x, y \in \mathbf{Z} : x \leq y \Rightarrow f(x) \leq f(y)$.

- Subdivision of S into disjoint block sections of types: Straight, Intersection, Simple Switch, Double Switch. [x]
- Prove that this subdivision covers the whole S and is pairwise disjoint ('modulo a finite number of intersection points'.)
- Define graphs based on the subdivision. [x]
- Explore: 2D splines / Euler spirals / Clothoids / Spiro-Curve by Ralph Levin Python module.
- Cost estimation for the block sections based on the weights of the graph and the method of Burton and consider how to incorporate it here.