Rational Embedding of Positive Definite Kernels on $\mathbb{N}$ via Minimal-Norm Argument

Orges Leka

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Abstract

We prove that every symmetric, rational, and positive definite kernel

$$k \,:\, \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q} \quad \text{with} \quad k(n,n) = 1,\ \forall\, n \in \mathbb{N}$$

admits an embedding $\phi \colon \mathbb{N} \to \ell_2(\mathbb{Q})$ such that

$$k(a,b) \;=\; \langle \phi(a),\, \phi(b) \rangle_{\ell_2} \quad \forall\, a, b \in \mathbb{N},$$

and every coordinate of $\phi(n)$ lies in $\mathbb{Q}$. The construction proceeds inductively: in the n-th step one solves a linear system with rational data and chooses the solution of minimal Euclidean norm. This ensures that all coordinates remain rational and that $\|\, w_{n-1}\|^2 < k(n,n) = 1$ at each stage.

1 Preliminaries

Let

$$k \colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}$$

be a symmetric, positive definite kernel normalized by

$$k(n,n) \;=\; 1 \qquad (\forall\, n \in \mathbb{N}).$$

Positive definiteness means: for every finite subset $I \subset \mathbb{N}$ and any rational coefficients $\{c_i\}_{i \in I} \subset \mathbb{Q}$, one has

$$\sum_{i,j \in I} c_i\, c_j\, k(i,j) \;> 0.$$

Our goal is to construct a mapping $\phi(n) \in \ell_2(\mathbb{Q})$ such that

$$\langle \phi(a),\, \phi(b) \rangle_{\ell_2} \;=\; k(a,b) \quad (\forall\, a, b \in \mathbb{N}),$$

and every component of $\phi(n)$ is rational.

We denote

$$\ell_2(\mathbb{Q}) = \big\{ (x_1, x_2, \ldots, x_N, 0, 0, \ldots) \;\big|\; x_i \in \mathbb{Q},\, N < \infty \big\},$$

with the standard inner product $\langle x, y \rangle = \sum_{r=1}^{\infty} x_r y_r$.

Base Case ($n = 1$). Define

$$\phi(1) \;=\; (1, 0, 0, \ldots) \in \ell_2(\mathbb{Q}).$$

Then $\langle \phi(1), \phi(1) \rangle = 1 = k(1,1)$. Clearly, $\phi(1)$ is rational and nonzero.

2 Induction Step

Assume we have already constructed $\phi(1), \dots, \phi(n-1) \in \ell_2(\mathbb{Q})$ such that

$$\langle \phi(i),\, \phi(j) \rangle = k(i,j) \quad \text{for all } 1 \le i,j \le n-1,$$

and $\phi(1), \dots, \phi(n-1)$ are linearly independent. We will show how to determine $\phi(n)$ with purely rational components.

2.1 Definition of C_{n-1} and v_{n-1}

Write each $\phi(i)$ as a finitely supported vector in $\mathbb{Q}^{n-1}$ (all other entries zero):

$$\phi(i) \;=\; \big(\phi(i)_1,\, \phi(i)_2,\, \dots,\, \phi(i)_{\,n-1},\, 0, 0, \dots\big), \qquad \phi(i)_r \in \mathbb{Q}.$$

Define the $(n-1) \times (n-1)$-matrix

$$C_{n-1} = \begin{pmatrix} \phi(1)_1 & \phi(1)_2 & \cdots & \phi(1)_{\,n-1} \\ \phi(2)_1 & \phi(2)_2 & \cdots & \phi(2)_{\,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi(n-1)_1 & \phi(n-1)_2 & \cdots & \phi(n-1)_{\,n-1} \end{pmatrix} \;\in\; \mathbb{Q}^{(n-1)\times(n-1)}.$$

Also set

$$v_{n-1} \;=\; \begin{pmatrix} k(1,n) \\ k(2,n) \\ \vdots \\ k(n-1,n) \end{pmatrix} \;\in\; \mathbb{Q}^{n-1}.$$

Since $k(i,n) \in \mathbb{Q}$, v_{n-1} is rational. Furthermore, define

$$G_{n-1} \;:=\; C_{n-1}\, C_{n-1}^T \;=\; [\,k(i,j)\,]_{1 \le i,j \le n-1} \;\in\; \mathbb{Q}^{(n-1)\times(n-1)}.$$

By the induction hypothesis ($\phi(1), \dots, \phi(n-1)$ are linearly independent), one has $\det(G_{n-1}) \neq 0$. Hence G_{n-1} is invertible and $G_{n-1}^{-1} \in \mathbb{Q}^{(n-1)\times(n-1)}$.

2.2 Moore–Penrose Solution w_{n-1}

Define the *Moore–Penrose pseudo-inverse*

$$M_{n-1} \;=\; C_{n-1}^T\, G_{n-1}^{-1} \;\in\; \mathbb{Q}^{(n-1)\times(n-1)}.$$

Then set

$$w_{n-1} \;:=\; M_{n-1}\, v_{n-1} \;\in\; \mathbb{Q}^{n-1}.$$

Properties of w_{n-1}:

1. $C_{n-1}\, w_{n-1} \;=\; v_{n-1}$. Hence w_{n-1} satisfies $\langle \phi(i),\, (w_{n-1}, 0, 0, \dots) \rangle = k(i,n)$ for $1 \le i \le n-1$.

2. w_{n-1} is the unique solution of $C_{n-1}\, x = v_{n-1}$ with minimal Euclidean norm.

3. Since C_{n-1}, v_{n-1}, G_{n-1}^{-1} are all rational, every component of w_{n-1} lies in $\mathbb{Q}$.

2.3 Strict Inequality $\| w_{n-1}\|^2 < k(n,n)$

We first show

$$\| w_{n-1}\|^2 \; < \; k(n,n) \; = \; 1.$$

Consider the Gram matrix

$$G_n \; = \; \begin{pmatrix} G_{n-1} & v_{n-1} \\ v_{n-1}^T & k(n,n) \end{pmatrix} \; = \; \begin{pmatrix} G_{n-1} & v_{n-1} \\ v_{n-1}^T & 1 \end{pmatrix}.$$

Since k is positive definite, $\det(G_n) > 0$ and $\det(G_{n-1}) > 0$. The Schur complement yields

$$\det(G_n) \; = \; \det(G_{n-1}) \left(1 - v_{n-1}^T \, G_{n-1}^{-1} \, v_{n-1}\right).$$

Because $\det(G_n) > 0$ and $\det(G_{n-1}) > 0$, it follows that

$$1 - v_{n-1}^T \, G_{n-1}^{-1} \, v_{n-1} \; > \; 0.$$

But

$$v_{n-1}^T \, G_{n-1}^{-1} \, v_{n-1} \; = \; w_{n-1}^T \, w_{n-1} \; = \; \| w_{n-1}\|^2.$$

Hence

$$\| w_{n-1}\|^2 \; < \; 1 = k(n,n).$$

This shows $0 < 1 - \| w_{n-1}\|^2 \in \mathbb{Q}$.

2.4 Choice of $\phi(n)$ Using Four Squares

We now choose $\phi(n)$ so that

$$\langle \phi(i), \, \phi(n) \rangle = k(i,n) \; (i = 1, \ldots, n-1), \quad \langle \phi(n), \, \phi(n) \rangle = k(n,n) = 1,$$

and all components of $\phi(n)$ lie in $\mathbb{Q}$. First, set the partial vector

$$\phi(n)^{(1 \ldots \, n-1)} \; = \; w_{n-1} \; = \; \left(w_{n-1,1}, \, w_{n-1,2}, \, \ldots, \, w_{n-1,n-1}\right)^T.$$

Then already $\langle \phi(i), \, (\, w_{n-1}, 0, 0, \ldots) \rangle = k(i,n)$ for $1 \le i \le n-1$. The norm of w_{n-1} is strictly less than 1. Let

$$\alpha \; := \; \| w_{n-1}\|^2 \; < \; 1, \quad \alpha \in \mathbb{Q} \text{ by induction.}$$

Thus for $\phi(n)$ we must have:

$$\| \phi(n)\|^2 \; = \; \| w_{n-1}\|^2 \; + \; \sum_{\ell=1}^{m} \beta_\ell^2 \; = \; 1,$$

where we introduce $m \ge 2$ additional coordinates $\beta_1, \ldots, \beta_m \in \mathbb{Q}$ to realize the difference $1 - \alpha$ as a sum of squares. By Lagrange's four-squares theorem, there exist rational numbers $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Q}$ such that

$$\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 \; = \; 1 - \alpha \; > \; 0.$$

Then define

$$\phi(n) \; = \; \left(w_{n-1,\,1}, \, w_{n-1,\,2}, \, \ldots, \, w_{n-1,\,n-1}, \; \beta_1, \, \beta_2, \, \beta_3, \, \beta_4, \; 0, 0, \ldots \right).$$

Clearly, all entries of $\phi(n)$ are rational. Moreover,

$$\big\langle \phi(i), \, \phi(n) \big\rangle = \sum_{r=1}^{n-1} \phi(i)_r \, w_{n-1,\,r} = (C_{n-1} w_{n-1})_{\,i} = v_{n-1,i} = k(i,n), \quad i = 1, \ldots, n-1,$$

and

$$\|\phi(n)\|^2 = \| w_{n-1}\|^2 + (\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2) = \alpha + \big(1 - \alpha\big) = 1 = k(n,n).$$

Thus the Gram matrix $\big[\, \langle \phi(i), \phi(j) \rangle \big]_{1 \le i,j \le n}$ coincides with $\big[k(i,j)\big]_{1 \le i,j \le n}$.

2.5 Linear Independence

It remains to show $\phi(n) \notin \mathrm{span}\{\phi(1), \ldots, \phi(n-1)\}$, so that the dimension indeed increases. Suppose, for the sake of contradiction, that $\phi(n)$ lies in the $(n-1)$-dimensional subspace $\mathrm{span}\{\phi(1), \ldots, \phi(n-1)\}$. Then, in particular, the "residual" $(n-1)$-coordinate block w_{n-1} would be exactly the representation of $\phi(n)$ in that basis, and hence

$$\|\phi(n)\|^2 \;=\; \|\,w_{n-1}\|^2.$$

But by construction, w_{n-1} is the *minimum-norm* solution of $C_{n-1}x = v_{n-1}$. Therefore, for any other solution x of $C_{n-1}x = v_{n-1}$, in particular for $x = \phi(n)^{(1\ldots n-1)}$, one has

$$\|\,w_{n-1}\|^2 \;\le\; \|\phi(n)^{(1\ldots n-1)}\|^2.$$

However, $\|\phi(n)\|^2 = 1$, whereas $\|\,w_{n-1}\|^2 < 1$. If $\phi(n)$ lay in $\mathrm{span}(\phi(1), \ldots, \phi(n-1))$, then $\phi(n)$ would have to be represented entirely by w_{n-1} (with no additional coordinates), i.e. all $\beta_j = 0$, and thus $\|\phi(n)\|^2 = \|\,w_{n-1}\|^2 < 1$, contradicting $\|\phi(n)\|^2 = 1$. Therefore $\phi(n) \notin \mathrm{span}\{\phi(1), \ldots, \phi(n-1)\}$.

In summary, for each $n \in \mathbb{N}$ we have found a vector

$$\phi(n) \in \ell_2(\mathbb{Q})$$

such that

$$\langle \phi(i),\, \phi(j) \rangle = k(i,j) \quad \forall\, 1 \le i, j \le n, \quad \phi(n) \notin \mathrm{span}\{\phi(1), \ldots, \phi(n-1)\}.$$

It follows immediately:

Theorem 2.1. Let

$$k \colon \mathbb{N} \times \mathbb{N} \;\longrightarrow\; \mathbb{Q}$$

be a symmetric, rational, positive definite kernel with $k(n,n) = 1$. Then there exists a mapping $\phi \colon \mathbb{N} \to \ell_2(\mathbb{Q})$ such that

$$k(a,b) \;=\; \langle \phi(a),\, \phi(b) \rangle_{\ell_2} \quad \forall\, a, b \in \mathbb{N},$$

and every coordinate of $\phi(n)$ lies in $\mathbb{Q}$. Moreover, for all n, the set $\{\phi(1), \ldots, \phi(n)\}$ is linearly independent.

□

3 SageMath Implementation

Below is a SageMath script that implements the inductive minimal-norm embedding for a positive-definite kernel $k \colon \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$. The function `phi(n,kernel)` returns the vector $\phi(n) \in \ell_2(\mathbb{Q})$ whose inner products reproduce $k(i,j)$ for $1 \le i, j \le n$. If the shortfall $1 - \|w_{n-1}\|^2$ is not already a perfect square in $\mathbb{Q}$, we use Lagrange's four-squares theorem (via `four_squares_rational`) to write it as a sum of four rational squares.

```
from sage.all import *
from functools import lru_cache

def four_squares_rational(r):
```

```
    """
    Given a positive rational r = p/q in lowest terms,
    find (b1, b2, b3, b4) in Q^4 so that
        b1^2 + b2^2 + b3^2 + b4^2 = r.
    Achieved by writing n = p*q and using Sage's four_squares() on integer n.
    """
    if r <= 0:
        raise ValueError("r must be a positive rational.")
    p, q = Integer(r.numerator()), Integer(r.denominator())
    n = p * q
    wxyz = four_squares(n)  # returns [w, x, y, z] with w^2+x^2+y^2+z^2 = n
    if len(wxyz) != 4:
        raise RuntimeError(f"four_squares returned {wxyz} instead of 4 integers")
    w, x, y, z = map(Integer, wxyz)
    # Convert to rationals by dividing each by q
    return (QQ(w) / QQ(q),
            QQ(x) / QQ(q),
            QQ(y) / QQ(q),
            QQ(z) / QQ(q))

@lru_cache(maxsize=None)
def phi(n, kernel):
    """
    Recursively construct phi(n) in l2(Q) for the given `kernel(i,j)`.
    Returns phi(n) as a Python list of QQ-coordinates.
    Uses memoization so phi(k) for k < n is computed only once.
    """
    if n == 1:
        # Base case: phi(1) = [1]
        return [QQ(1)]

    # Recursively obtain phi(1), phi(2), ..., phi(n-1)
    prev_vectors = [phi(k, kernel) for k in range(1, n)]

    # Determine current embedding dimension d = max length among phi(1..n-1)
    d = max(len(vec) for vec in prev_vectors)

    # Build C_{n-1} as an (n-1)×d QQ-matrix; missing entries are 0
    def entry(i, j):
        row = prev_vectors[i]  # phi(i+1)
        return row[j] if j < len(row) else QQ(0)

    C = Matrix(QQ, n-1, d, entry)

    # Build v_{n-1} = (kernel(1,n), ..., kernel(n-1,n))^T in QQ^(n-1)
    v = vector(QQ, [ kernel(i+1, n) for i in range(n-1) ])

    # Compute Gram G_{n-1} = C * C^T
    G = C * C.transpose()
```

```
    Ginv = G.inverse()  # invertible over QQ by positive definiteness

    # Moore-Penrose minimal-norm solution w_{n-1} = C^T * G^{-1} * v
    M = C.transpose() * Ginv
    w = M * v  # length-d vector in QQ

    # Compute a = ||w||^2 = w*w in QQ
    alpha = w.dot_product(w)
    if not (alpha < QQ(1)):
        raise AssertionError(f"Expected a < 1 but got a = {alpha} at n = {n}")

    # Let r = 1 - a > 0
    r = QQ(1) - alpha

    # Represent r as sum of four rational squares
    beta1, beta2, beta3, beta4 = four_squares_rational(r)

    # Form phi(n) by concatenating w plus these four betas
    phi_n = list(w) + [beta1, beta2, beta3, beta4]
    return phi_n

# ---------------------------------------------------------
# Example 1: k(a,b) = min(a,b)/max(a,b)
# ---------------------------------------------------------
def k1(a, b):
    a, b = Integer(a), Integer(b)
    return QQ(min(a,b), max(a,b))

# Compute phi(1), ..., phi(6) for k1
print("Embedding vectors for k(a,b) = min(a,b)/max(a,b):")
for i in range(1, 7):
    vec = phi(i, k1)
    print(f"phi({i}) = {vec}")

# ---------------------------------------------------------
# Example 2: k(a,b) = gcd(a,b)^2 / (a*b)
# ---------------------------------------------------------
def k2(a, b):
    a, b = Integer(a), Integer(b)
    return QQ(gcd(a,b)**2, a*b)

print("\nEmbedding vectors for k(a,b) = gcd(a,b)^2/(a*b):")
for i in range(1, 7):
    vec = phi(i, k2)
    print(f"phi({i}) = {vec}")

# ---------------------------------------------------------
# Example 3: k(a,b) = 2*gcd(a,b)/(a+b)
# ---------------------------------------------------------
```

```
def k3(a, b):
    a, b = Integer(a), Integer(b)
    return QQ(2*gcd(a,b), a+b)

print("\nEmbedding vectors for k(a,b) = 2*gcd(a,b)/(a+b):")
for i in range(1, 7):
    vec = phi(i, k3)
    print(f"phi({i}) = {vec}")
```

4 Computed Examples

Below are the explicit embedding vectors $\phi(n)$ returned by the above SageMath code, for three different kernels.

4.1 1. Kernel

$$\begin{aligned}\phi(1) &= [\,1\,],\\ \phi(2) &= [\,\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\,],\\ \phi(3) &= [\,\tfrac{1}{3},\ 0,\ \tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3},\ 0,\ 0,\ \tfrac{1}{3},\ \tfrac{2}{3}\,],\end{aligned}$$

4.2 2. Kernel

$$\begin{aligned}\phi(1) &= [\,1\,],\\ \phi(2) &= [\,\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\,],\\ \phi(3) &= [\,\tfrac{1}{3},\ 0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{2}{9},\ \tfrac{2}{9},\ \tfrac{8}{9}\,],\\ \phi(4) &= [\,\tfrac{1}{4},\ 0,\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4},\ 0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\,],\end{aligned}$$

4.3 3. Kernel

$$\begin{aligned}\phi(1) &= [\,1\,],\\ \phi(2) &= [\,\tfrac{2}{3},\ 0,\ 0,\ \tfrac{1}{3},\ \tfrac{2}{3}\,],\\ \phi(3) &= [\,\tfrac{1}{2},\ 0,\ 0,\ \tfrac{1}{25},\ \tfrac{2}{25},\ \tfrac{1}{125},\ \tfrac{1}{50},\ \tfrac{11}{250},\ \tfrac{43}{50}\,],\end{aligned}$$

5 Extending to p.d. kernels k not necessarily with $k(n,n) =$ constant

The method can be extended to any p.d. kernels k over the natural numbers, taking rational values, by letting in the induction start:

$$\phi(1) = (a, b, c, d)$$

where the rational coordinates $a, \cdots, d$ satisfy $a^2+b^2+c^2+d^2 = k(1,1)$ by Lagranges theorem and in the induction step we chose a, b, c, d to match $a^2 + b^2 + c^2 + d^2 = k(n,n) - |w_{n-1}|^2 > 0$.

6 Equivalence between Integral Conditions and Integer-valued Embeddings

Let

$$k\colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}$$

be a positive-definite kernel. For each $n \in \mathbb{N}$, denote

$$G_n \;=\; \big[k(i,j)\big]_{1\le i,j\le n}$$

its $n \times n$ Gram matrix, and let

$$v_n = \big(k(1,n+1),\, k(2,n+1),\, \dots,\, k(n,n+1)\big)^T \;\in\; \mathbb{Q}^n.$$

We write $\operatorname{adj}(M)$ for the adjugate (classical adjoint) of a matrix M.

Theorem 6.1. The following two statements are equivalent:

I. Integral conditions.

1. $k(1,1) \in \mathbb{N}$.
2. For every n, $G_n^{-1} v_n \in \mathbb{Z}^n$.
3. $k(a,b) \in \mathbb{Z}$ for all $a, b \in \mathbb{N}$.

II. Integer-valued Hilbert embedding.

1. There exists a map $\phi\colon \mathbb{N} \to \ell^2(\mathbb{Z})$ such that

$$k(a,b) \;=\; \big\langle \phi(a),\, \phi(b)\big\rangle_{\ell^2} \quad \forall\, a,b \in \mathbb{N}.$$

2. For every n,

$$\det(G_n) \;\big|\; \big[\operatorname{adj}(G_n)\, v_n\big]_i \quad (1 \le i \le n),$$

i.e. $\det(G_n)$ divides each entry of $\operatorname{adj}(G_n)\, v_n$.

Before proving the equivalence, we recall:

Remark 6.2 (Sylvester's criterion)**.** A real symmetric matrix A is positive definite if and only if all its leading principal minors are positive. In particular, $\det(G_n) > 0$ for every n.

Proof of (II) $\Longrightarrow$ (I). 1. Since $\phi(1) \in \ell^2(\mathbb{Z})$ has finitely many nonzero integer coordinates a_i,

$$k(1,1) = \langle \phi(1), \phi(1)\rangle = \sum_i a_i^2 \;\in \mathbb{N},$$

and is nonzero by positive definiteness.

2. By Sylvester's criterion, $\det(G_n) > 0$ and

$$G_n^{-1} = \frac{\operatorname{adj}(G_n)}{\det(G_n)},$$

so II.2 gives

$$G_n^{-1} v_n = \frac{\operatorname{adj}(G_n)\, v_n}{\det(G_n)} \;\in\; \mathbb{Z}^n,$$

proving I.2.

3. Finally, for any $a, b \in \mathbb{N}$,

$$k(a,b) = \langle \phi(a), \phi(b)\rangle \;\in\; \mathbb{Z},$$

establishing I.3.

□

Proof of (I) $\Longrightarrow$ (II). We construct $\phi(1), \phi(2), \ldots$ inductively in $\ell^2(\mathbb{Z})$.

Base case ($n = 1$). By Lagrange's four-square theorem, write

$$k(1,1) = a^2 + b^2 + c^2 + d^2, \quad a, b, c, d \in \mathbb{Z}.$$

Set $\phi(1) = (a, b, c, d, 0, 0, \ldots)$ so that $\|\phi(1)\|^2 = k(1,1)$.

Inductive step. Assume $\phi(1), \ldots, \phi(n)$ satisfy

$$\big\langle \phi(i), \phi(j) \big\rangle = k(i,j) \quad (1 \le i, j \le n).$$

Let

$$G_n = [k(i,j)]_{1 \le i,j \le n}, \quad v_n = (k(1, n+1), \ldots, k(n, n+1))^T.$$

By I.2–I.3, $G_n \in \mathbb{Z}^{n \times n}$ is invertible over $\mathbb{Q}$ and

$$w_n \;=\; G_n^{-1} v_n \;\in\; \mathbb{Z}^n.$$

Then Sylvester's criterion gives

$$\det(G_{n+1}) = \det(G_n)\,\big(k(n+1, n+1) - v_n^T G_n^{-1} v_n\big) = \det(G_n)\,\big(k(n+1, n+1) - \|w_n\|^2\big) > 0,$$

so $\|w_n\|^2 < k(n+1, n+1)$, and the integer $k(n+1, n+1) - \|w_n\|^2 > 0$ is a sum of four squares,

$$k(n+1, n+1) - \|w_n\|^2 = \beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2, \quad \beta_i \in \mathbb{Z}.$$

Define

$$\phi(n+1) = \big(w_{n,1}, \ldots, w_{n,n}, \beta_1, \beta_2, \beta_3, \beta_4, 0, \ldots\big).$$

One checks

$$\langle \phi(i), \phi(n+1) \rangle = v_{n,i} = k(i, n+1), \quad \|\phi(n+1)\|^2 = \|w_n\|^2 + \sum_{j=1}^{4} \beta_j^2 = k(n+1, n+1),$$

and $\phi(n+1) \notin \operatorname{span}\{\phi(1), \ldots, \phi(n)\}$. This completes the induction and yields the desired ϕ. Finally, I.2 is equivalent to II.2 by the adjugate formula. $\square$