Short Report on Part II (Work in Progress)

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March 10, 2026

Summary

Part II continues the program initiated in Part I and focuses on the summatory function

$$G(N) = \sum_{n \leq N} g(n),$$

where $g(n)$ denotes the positive Möbius weights associated with the factorization-poset meet kernel. The manuscript begins by reformulating $G(N)$ in reproducing-kernel and Gram-matrix terms, deriving the exact identity

$$G(N) = v_N^T G_N^{-1} v_N, \qquad v_N = (1, 2, \ldots, N)^T.$$

This interprets the arithmetic sum $G(N)$ as the squared norm of a canonical cutoff vector in the Hilbert space generated by the kernel.

The work then studies upper and lower bounds via Hilbert-space geometry and positive covering arguments on the truncated factorization poset. The decisive conceptual step, however, is the recognition that this positive covering approach is asymptotically too coarse. In response, the manuscript introduces an exact signed boundary-layer expansion based on the coefficient vector

$$c^{(N)} := G_N^{-1} v_N.$$

These coefficients are identified as truncated upper Möbius sums and subsequently reinterpreted in combinatorial and topological terms. This turns the problem from one of positive domination into one of cancellation along the maximal frontier of the truncated poset.

Main Mathematical Contributions

The first major contribution is the exact RKHS/Gram-matrix identity for $G(N)$. This is a natural but important extension of the kernel geometry developed in Part I, and it provides a clean energy interpretation of the summatory function.

The second major contribution is methodological: the manuscript shows that positive coverings, although natural, are unlikely to yield the conjecturally sharp order of growth. This is a valuable negative insight, because it prevents the project from remaining trapped in an overly rigid monotone framework.

The third and strongest contribution is the identification of the exact coefficient vector

$$c^{(N)} = (E_N^{-1})^T \mathbf{1},$$

whose entries satisfy

$$c_n^{(N)} = \sum_{k \leq N,\ n \preccurlyeq k} \mu_P(n, k).$$

Thus the coefficients are not merely matrix-inversion artifacts: they are truncated upper Möbius sums on the factorization poset and therefore true boundary quantities.

The fourth major contribution is the combinatorial-topological reinterpretation of these coefficients. They are expressed as inclusion-exclusion weights of maximal elements above a given node, and then as reduced Euler characteristics of naturally associated simplicial complexes:

$$c_n^{(N)} = -\widetilde{\chi}(\Delta_N(n)).$$

This gives the project a new structural layer. The asymptotic study of $G(N)$ is thereby recast as a problem of frontier topology and signed cancellation.

Strengths

The manuscript has a clear conceptual advance over the first stage of the project. It does not merely add estimates; it identifies a new mechanism. The move from positive covering geometry to signed boundary theory is mathematically convincing and well motivated.

The topological interpretation is particularly strong. It situates the new coefficients within a classical Möbius-inversion paradigm — namely, the relation between inclusion-exclusion and Euler characteristic — while applying it in an original arithmetic setting. This is the part of the manuscript that seems most likely to generate further rigorous progress.

Another strength is the internal honesty of the text: it recognizes the limitations of the initial positive method and uses that failure constructively to motivate a sharper framework.

Limitations and Open Points

The manuscript is explicitly a work in progress, and this status is mathematically visible. The new framework is structurally promising, but the decisive asymptotic consequences are not yet established.

In particular, the following kinds of statements still appear to be open:

- uniform control of the signed coefficients $c_n^{(N)}$,
- a precise description of their support and cancellation pattern,
- energy estimates strong enough to imply bounds of order $N(\log N)^2$,
- and ultimately the conjectural asymptotic behavior of $G(N)$.

Thus, Part II should not yet be read as a completed asymptotic theory. Rather, it is a structural reorientation of the project that isolates what may be the correct mechanism for future progress.

Overall Assessment

I regard this manuscript as mathematically interesting and substantially more than an informal note. Its principal achievement is conceptual: it shows that the right object is not a positive majorant but a signed boundary expansion, and it gives that expansion an exact Möbius-theoretic and topological meaning.

This does not yet amount to a final theorem of the desired asymptotic strength. However, as a second part of an ongoing project, it is both credible and promising. In fact, the shift to boundary topology may turn out to be the crucial innovation of the whole program.

Recommendation

My recommendation would be to present Part II explicitly as a work in progress or research note rather than as a finished article with definitive asymptotic claims. In that form, it is a valuable and original continuation of Part I. The main new idea — signed frontier cancellation encoded by reduced Euler characteristics — is strong enough to warrant serious attention.

Reference to Part I

https://www.orges-leka.de/lindstroem_bhat_matrices_and_prime_factorization_of_integers.pdf