The first hundert thousand numbers

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Abstract

We introduce a multiplicative representation of integers based on *Pratt prime forests*. To every $n \in \mathbb{N}$ we associate exponents $m_p(n) \in \mathbb{Z}_{\geq 0}$ counting how often a prime label p occurs as a vertex in the Pratt forest of n. This yields the finite product identity

$$n = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-m_p(n)},$$

and, after taking logarithms, a linear readout of $\log n$ from the feature vector $\phi(n) = (m_p(n))_p$ via an ℓ^2 inner product with a fixed weight vector $w = (\log(1 - 1/p))_p$. We use these sparse high-dimensional embeddings for the first 100,000 natural numbers and visualize their geometry by applying UMAP to obtain a two-dimensional layout. An animated incremental rendering highlights how arithmetic structure emerges as n grows. The implementation and outputs (paper, code, and video) are available online [1, 2, 3, 4]. The visualization approach is inspired by the prime-factor UMAP experiment of Williamson [5], while our codebase adapts and extends that idea to the Pratt-forest feature map, with substantial refactoring assisted by a large language model.

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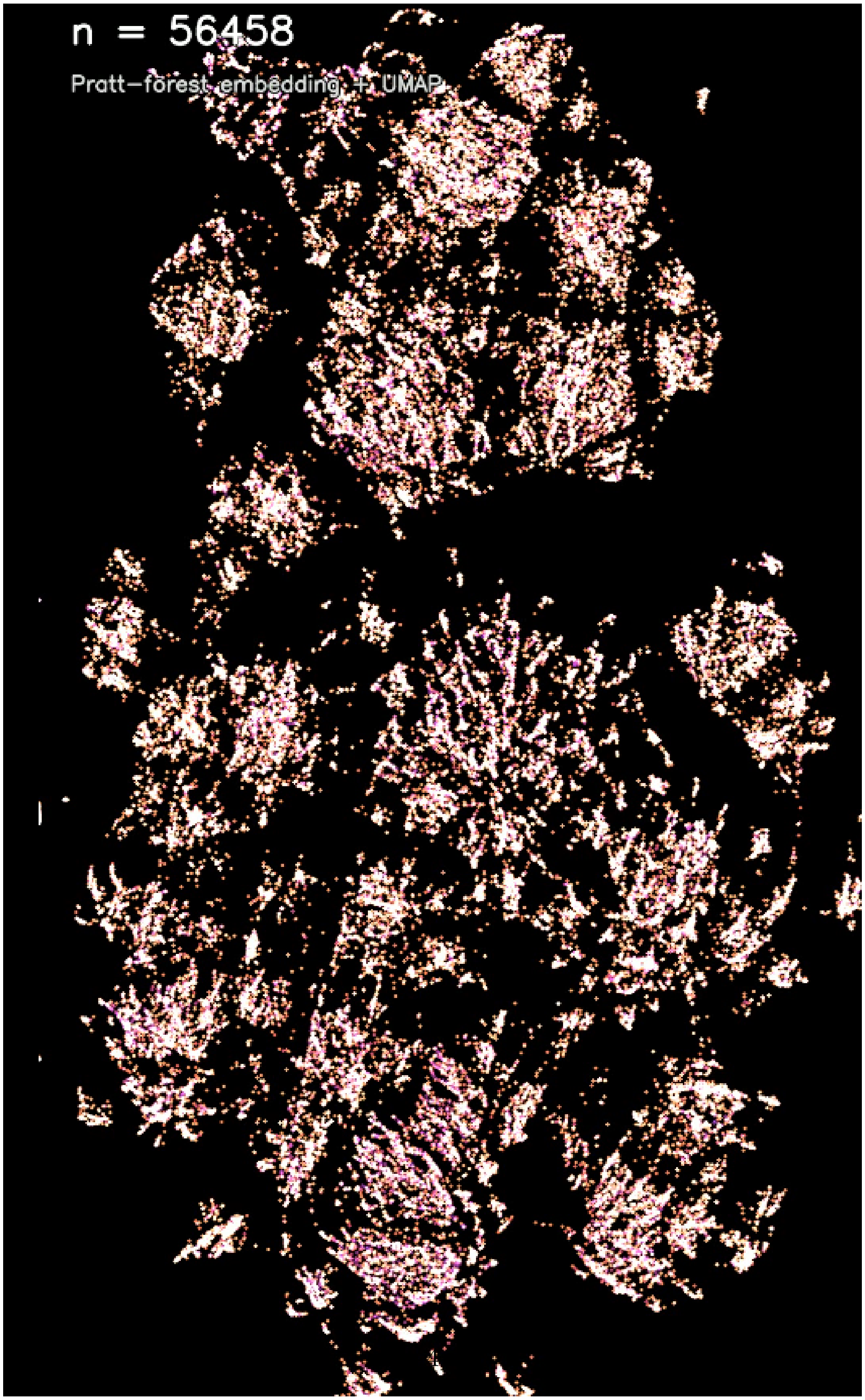
n = 56458
Pratt-forest embedding + UMAP

1 Introduction

Prime factorization provides a canonical description of integers, but it is not immediately geometric: it lives in a combinatorial space of exponents and does not come with a natural notion of distance or angle. A simple way to introduce geometry is to embed integers into a Hilbert space using prime-indexed coordinates and then compare the resulting vectors by inner products.

In this work we consider a refinement of the usual p-adic valuation feature map. Instead of the valuation $v_p(n)$, we use the *Pratt prime forest* of n. For each prime p, the Pratt tree T_p recursively decomposes p through the factorization of $p-1$ until reaching 2; for general n one takes the multiset union of $v_p(n)$ copies of T_p over all primes dividing n. We write $m_p(n)$ for the number of vertices labeled p in this forest. The resulting statistics are finitely supported in p for each fixed n, and they satisfy complete additivity in n (forest additivity).

A central consequence is the finite product identity

$$n \;=\; \prod_{p \text{ prime}} \Bigl(1-\frac{1}{p}\Bigr)^{-m_p(n)}. \tag{1}$$

Thus the integer n can be reconstructed from the feature vector $\phi(n) := (m_p(n))_p$ by a fixed multiplicative recipe. Taking logarithms turns (1) into a linear functional: if $w := (\log(1-1/p))_p$, then

$$\log n \;=\; -\langle \phi(n), w\rangle,$$

so $\log n$ is a *linear readout* and $n = \exp(-\langle \phi(n), w\rangle)$ is an *exponential readout*. In this sense, the arithmetic observable $\log n$ becomes an affine-linear coordinate on the embedded point cloud.

The practical motivation is visualization. We compute $\phi(n)$ for $1 \leq n \leq 100{,}000$ and apply UMAP to obtain a two-dimensional embedding that preserves local similarity under a cosine-type metric. We then render the points incrementally (adding the next batch of integers per frame) to produce a short animation, revealing large-scale clusters and filamentary structure that appear to correlate with the recursive prime ancestry encoded by Pratt trees. The resulting paper and artifacts are documented in [1, 4]. The overall visualization pipeline is inspired by Williamson's earlier experiment that applied UMAP to sparse prime-factor features [5], but here the input space is replaced by the Pratt-forest exponents $m_p(n)$, which encode more than just the top-level factorization of n.

2 A recursion for $f_n(x)$ and the induced "derivative" $n' = f_n'(2)$

2.1 Definition of f_n

We define polynomials $f_n(x) \in \mathbb{Z}[x]$ recursively by

$$\begin{aligned} f_1(x) &= 1, \\ f_2(x) &= x, \\ f_p(x) &= 1 + f_{p-1}(x) \qquad \text{for primes } p, \\ f_n(x) &= \prod_{p|n} f_p(x)^{v_p(n)} \qquad \text{for composite } n, \end{aligned}$$

where $v_p(n)$ is the p-adic valuation.

This is precisely the recursion encoded in `ff(n,x)` in the code snippet.

2.2 Basic evaluation at $x = 2$

Lemma 1. *For all $n \geq 1$ we have $f_n(2) = n$.*

Proof. We proceed by strong induction on n.

For $n = 1, 2$ the statement is immediate. Let $n \geq 3$.

If $n = p$ is prime, then

$$f_p(2) = 1 + f_{p-1}(2) = 1 + (p - 1) = p$$

using the induction hypothesis.

If n is composite, then by definition

$$f_n(2) = \prod_{p|n} f_p(2)^{v_p(n)} = \prod_{p|n} p^{v_p(n)} = n,$$

again by induction. □

2.3 Definition of n' and the correct recursion

Definition 1. *Define*

$$n' := f_n'(2).$$

The code proposes a recursive function `der(n)`. The prime case is correct, but for composites the code uses a *product*. The correct formula for n' is instead a *sum*. We now prove the right recursion.

Proposition 1. *The values* $n' = f_n'(2)$ *satisfy:*

$$\begin{aligned} 1' &= 0, \\ 2' &= 1, \\ p' &= (p-1)' \qquad \textit{for primes } p, \\ n' &= n \sum_{p|n} v_p(n) \frac{p'}{p} \qquad \textit{for composite } n. \end{aligned}$$

Proof. The initial values are immediate from $f_1(x) = 1$ and $f_2(x) = x$.

If p is prime, $f_p(x) = 1 + f_{p-1}(x)$, hence

$$p' = f_p'(2) = f_{p-1}'(2) = (p-1)'.$$

If n is composite, write

$$f_n(x) = \prod_{p|n} f_p(x)^{v_p(n)}.$$

Taking a logarithmic derivative in x gives

$$\frac{f_n'(x)}{f_n(x)} = \sum_{p|n} v_p(n) \frac{f_p'(x)}{f_p(x)}.$$

Evaluating at $x = 2$ and using Lemma 1 yields

$$\frac{n'}{n} = \sum_{p|n} v_p(n) \frac{p'}{p},$$

which is equivalent to the stated formula. □

Remark 1. *Thus, if one wants* `der(n)` *to equal* $f_n'(2)$*, the composite branch in the code should be implemented as* a sum*:*

```
der(n) = n * sum( valuation(n,p) * der(p) / p for p in prime_divisors(n) ).
```

3 The squarefree Dirichlet series $z(x, s)$ and the ratio $c(s) = a(s)/b(s)$

3.1 Definition of $z(x, s)$

Definition 2. *Let* $\mathrm{rad}(n)$ *denote the radical of* n*. Define*

$$z(x, s) := \sum_{\mathrm{rad}(n)=n} \frac{1}{f_n(x)^s}.$$

For $\Re(s) > 1$ and x in a neighborhood of 2 this formal Dirichlet series admits the Euler product

$$z(x,s) = \prod_p \left(1 + f_p(x)^{-s}\right),$$

since the sum ranges over squarefree integers.

At $x = 2$, Lemma 1 implies

$$z(2,s) = \sum_{\mathrm{rad}(n)=n} \frac{1}{n^s} = \prod_p (1 + p^{-s}) = \frac{\zeta(s)}{\zeta(2s)}, \qquad \Re(s) > 1. \tag{2}$$

3.2 Logarithmic derivative in x

Differentiate the Euler product logarithmically with respect to x. Formally,

$$\begin{aligned} \frac{\partial_x z(x,s)}{z(x,s)} &= \sum_p \frac{\partial_x (1 + f_p(x)^{-s})}{1 + f_p(x)^{-s}} \\ &= \sum_p \frac{-s f_p'(x)\, f_p(x)^{-s-1}}{1 + f_p(x)^{-s}}. \end{aligned} \tag{3}$$

Evaluating at $x = 2$ gives

$$\frac{z_x'(2,s)}{z(2,s)} = -s \sum_p \frac{p'\, p^{-s-1}}{1 + p^{-s}}. \tag{4}$$

3.3 The series $a(s)$ and $b(s)$

Definition 3. *For $\Re(s) > 1$, define*

$$a(s) := \sum_p \frac{p'\, p^{-s-1}}{1 + p^{-s}},$$

$$b(s) := \sum_{\mathrm{rad}(n)=n} \frac{n'}{n^{s+1}}.$$

On the other hand, differentiating $z(x,s)$ termwise in the squarefree sum yields

$$z_x'(2,s) = \sum_{\mathrm{rad}(n)=n} \frac{-s\, n'}{n^{s+1}} = -s\, b(s).$$

Combining this with (4) gives, for $\Re(s) > 1$,

$$\frac{-s\, b(s)}{z(2,s)} = -s\, a(s) \quad \Longrightarrow \quad z(2,s) = \frac{b(s)}{a(s)}. \tag{5}$$

Definition 4. *Set*

$$c(s) := \frac{a(s)}{b(s)} \qquad (\Re(s) > 1).$$

Using (2) and (5), we obtain

$$c(s) = \frac{\zeta(2s)}{\zeta(s)}, \qquad \Re(s) > 1. \tag{6}$$

4 Meromorphic continuation of $c(s)$

Proposition 2. *The function $c(s)$ admits a meromorphic continuation to all of $\mathbb{C}$ by*

$$c(s) := \frac{\zeta(2s)}{\zeta(s)}. \tag{7}$$

This continuation agrees with the Dirichlet-series-defined ratio $a(s)/b(s)$ on $\Re(s) > 1$.

Proof. Equation (6) identifies the ratio $a(s)/b(s)$ with $\zeta(2s)/\zeta(s)$ on a nonempty domain. The right-hand side is meromorphic on $\mathbb{C}$, hence defines a meromorphic continuation. □

5 A recursive "log-free on the right" expansion for $\log f_n(x)$

5.1 The polynomials $f_n(x)$

We define polynomials $f_n(x) \in \mathbb{Z}[x]$ by the recursion

$$\begin{aligned}
f_1(x) &= 1, \\
f_2(x) &= x, \\
f_p(x) &= 1 + f_{p-1}(x) \qquad \text{for primes } p \geq 3, \\
f_n(x) &= \prod_{p \mid n} f_p(x)^{v_p(n)} \qquad \text{for composite } n,
\end{aligned}$$

where $v_p(n)$ denotes the p-adic valuation. (The prime step is intentionally stated only for $p \geq 3$, since the base value $f_2(x) = x$ does not match the rule $f_2 = 1 + f_1$.)

5.2 The power-series block

For any $k \geq 1$ we introduce the formal power series

$$\mathrm{PS}(k;x) := \log\Big(1 + \frac{1}{f_k(x)}\Big) = \sum_{m\geq 1} (-1)^{m+1} \frac{1}{m} \frac{1}{f_k(x)^m}. \tag{8}$$

Analytically, (8) holds whenever $|f_k(x)| > 1$, but we will use it as a formal expansion rule.

5.3 A recursive expansion operator

Define a formal expression $E(n)$ recursively by:

$$E(1) := 0, \tag{9}$$

$$E(2) := \log x, \tag{10}$$

$$E(p) := E(p-1) + \mathrm{PS}(p-1;x) \qquad (p \geq 3 \text{ prime}), \tag{11}$$

$$E(n) := \sum_{p|n} v_p(n)\, E(p) \qquad (n \text{ composite}). \tag{12}$$

Lemma 2. *For every $n \geq 1$ we have*

$$E(n) = \log f_n(x).$$

Proof. We proceed by induction on n.

The cases $n = 1, 2$ follow from (9)–(10). If $n = p \geq 3$ is prime, then by definition $f_p = 1 + f_{p-1}$, hence

$$\log f_p = \log f_{p-1} + \log\Big(1 + \frac{1}{f_{p-1}}\Big) = E(p-1) + \mathrm{PS}(p-1;x) = E(p).$$

If n is composite, then $f_n = \prod_{p|n} f_p^{v_p(n)}$, so

$$\log f_n = \sum_{p|n} v_p(n) \log f_p = \sum_{p|n} v_p(n) E(p) = E(n).$$

□

5.4 Coefficient tree and a "log-free" right-hand side

We now expand $E(n)$ in the basis $\{\log x\}$ and the blocks $\mathrm{PS}(k;x)$. Write formally

$$E(n) = \alpha(n)\, \log x + \sum_{k\geq 1} C_n(k)\, \mathrm{PS}(k;x), \tag{13}$$

with integer coefficients $\alpha(n), C_n(k)$.

From the recursion (9)–(12) one immediately reads:

$$\begin{aligned}
\alpha(1) &= 0, \qquad \alpha(2) = 1,\\
\alpha(p) &= \alpha(p-1) \qquad (p \geq 3 \text{ prime}),\\
\alpha(n) &= \sum_{p|n} v_p(n)\, \alpha(p) \qquad (n \text{ composite}),
\end{aligned}$$

and

$$\begin{aligned}
C_1(k) &= 0, \qquad C_2(k) = 0,\\
C_p(k) &= C_{p-1}(k) + \delta_{k,p-1} \qquad (p \geq 3 \text{ prime}),\\
C_n(k) &= \sum_{p|n} v_p(n)\, C_p(k) \qquad (n \text{ composite}).
\end{aligned}$$

Now substitute (8) into (13) and *factor out* the coefficients $(-1)^{m+1}\frac{1}{m}$ from the PS-sums. This yields the desired "log-free right-hand side" representation:

Proposition 3. *For every $n \geq 1$,*

$$\log f_n(x) = \alpha(n)\, \log x + \sum_{m\geq 1} (-1)^{m+1} \frac{1}{m} \sum_{k\geq 1} C_n(k)\, \frac{1}{f_k(x)^m}. \tag{14}$$

Proof. Combine Lemma 2 with (13) and (8), then exchange the finite sum over k with the formal series in m. □

5.5 Small examples

The first values of $f_n(x)$ are

$$\begin{aligned}
&f_1 = 1, \quad f_2 = x, \quad f_3 = x+1, \quad f_4 = x^2,\\
&f_5 = x^2+1, \quad f_6 = x(x+1), \quad f_7 = x^2+x+1, \quad f_8 = x^3.
\end{aligned}$$

Using the recursion for $E(n)$:

$$\log f_3 = \log x + \mathrm{PS}(2;x) = \log x + \sum_{m\geq 1} (-1)^{m+1} \frac{1}{m} \frac{1}{f_2(x)^m},$$

$$\log f_5 = 2\log x + \mathrm{PS}(4;x) = 2\log x + \sum_{m\geq 1} (-1)^{m+1} \frac{1}{m} \frac{1}{f_4(x)^m},$$

$$\begin{aligned}
\log f_7 &= 2\log x + \mathrm{PS}(2;x) + \mathrm{PS}(6;x)\\
&= 2\log x + \sum_{m\geq 1} (-1)^{m+1} \frac{1}{m} \left(\frac{1}{f_2(x)^m} + \frac{1}{f_6(x)^m} \right).
\end{aligned}$$

Two slightly larger prime examples illustrate the requested "factored" form.

Example $p = 11$. Since $10 = 2 \cdot 5$ and $f_{10} = f_2 f_5 = x(1 + x^2) = x + x^3$,

$$\log f_{11} = 3\log x + \mathrm{PS}(4;x) + \mathrm{PS}(10;x) = 3\log x + \sum_{m\geq 1}(-1)^{m+1}\frac{1}{m}\left(\frac{1}{f_4(x)^m} + \frac{1}{f_{10}(x)^m}\right).$$

Example $p = 13$. Since $12 = 2^2 \cdot 3$ and $f_{12} = f_2^2 f_3 = x^2(x+1) = x^3 + x^2$,

$$\log f_{13} = 3\log x + \mathrm{PS}(2;x) + \mathrm{PS}(12;x) = 3\log x + \sum_{m\geq 1}(-1)^{m+1}\frac{1}{m}\left(\frac{1}{f_2(x)^m} + \frac{1}{f_{12}(x)^m}\right).$$

Remark 2. *Formula (14) packages the entire "prime-step tree" into the integer multiplicities $C_n(k)$. In this sense, the right-hand side is "log-free": the only remaining logarithm is the base term $\alpha(n)\log x$, while all prime-step corrections appear as explicit series in $1/f_k(x)^m$ with the common scalar factor $(-1)^{m+1}\frac{1}{m}$.*

6 Examples

$$\log f_n(x) = \alpha(n)\log x + \sum_{m\geq 1}(-1)^{m+1}\frac{1}{m}\sum_k \frac{C_n(k)}{f_k(x)^m}.$$

Remark 3. *The function $\alpha(n)$ is completely additive by construction.*

7 Identification of $\alpha(n)$ with OEIS A064415

7.1 A recursion matching A064415

The OEIS entry A064415 lists several equivalent characterizations. In particular, it states that the function $a(n)$ is completely additive, with $a(1) = 0$, $a(2) = 1$, and for $n > 2$:

$$a(n) = \sum_{p|n} a(p-1), \tag{15}$$

where the sum is over primes dividing n with multiplicity. (Equivalently $a(n) = A003434(n) - (n \bmod 2)$, etc.) This recursion is consistent with the prime rule $a(p) = a(p-1)$ for odd primes.

n	$\alpha(n)$	$\sum_k \frac{C_n(k)}{f_k(x)^m}$
1	0	0
2	1	0
3	1	$\frac{1}{f_2(x)^m}$
4	2	0
5	2	$\frac{1}{f_4(x)^m}$
6	2	$\frac{1}{f_2(x)^m}$
7	2	$\frac{1}{f_2(x)^m} + \frac{1}{f_6(x)^m}$
8	3	0
9	2	$2\frac{1}{f_2(x)^m}$
10	3	$\frac{1}{f_4(x)^m}$
11	3	$\frac{1}{f_4(x)^m} + \frac{1}{f_{10}(x)^m}$
12	3	$\frac{1}{f_2(x)^m}$
13	3	$\frac{1}{f_2(x)^m} + \frac{1}{f_{12}(x)^m}$
14	3	$\frac{1}{f_2(x)^m} + \frac{1}{f_6(x)^m}$
15	3	$\frac{1}{f_2(x)^m} + \frac{1}{f_4(x)^m}$
16	4	0
17	4	$\frac{1}{f_{16}(x)^m}$
18	3	$2\frac{1}{f_2(x)^m}$
19	3	$2\frac{1}{f_2(x)^m} + \frac{1}{f_{18}(x)^m}$
20	4	$\frac{1}{f_4(x)^m}$

Table 1: Coefficients $\alpha(n)$ and inner sums $\sum_k C_n(k)/f_k(x)^m$ in the expansion $\log f_n(x) = \alpha(n)\log x + \sum_{m\geq 1}(-1)^{m+1}\frac{1}{m}\sum_k C_n(k)/f_k(x)^m$.

Proposition 4. *The function $\alpha(n)$ satisfies the OEIS recursion* (15).

Proof. Let $n > 2$. If n is composite, then

$$\alpha(n) = \sum_{p|n} v_p(n)\,\alpha(p).$$

For any odd prime p, we have $\alpha(p) = \alpha(p-1)$. Hence

$$\alpha(n) = \sum_{p|n} v_p(n)\,\alpha(p-1),$$

which is precisely (15) with multiplicity. □

n	$\alpha(n)$	$\sum_k \frac{C_n(k)}{f_k(x)^m}$
21	3	$2\frac{1}{f_2(x)^m} + \frac{1}{f_6(x)^m}$
22	4	$\frac{1}{f_4(x)^m} + \frac{1}{f_{10}(x)^m}$
23	4	$\frac{1}{f_4(x)^m} + \frac{1}{f_{10}(x)^m} + \frac{1}{f_{22}(x)^m}$
24	4	$\frac{1}{f_2(x)^m}$
25	4	$2\frac{1}{f_4(x)^m}$
26	4	$\frac{1}{f_2(x)^m} + \frac{1}{f_{12}(x)^m}$
27	3	$3\frac{1}{f_2(x)^m}$
28	4	$\frac{1}{f_2(x)^m} + \frac{1}{f_6(x)^m}$
29	4	$\frac{1}{f_2(x)^m} + \frac{1}{f_6(x)^m} + \frac{1}{f_{28}(x)^m}$
30	4	$\frac{1}{f_2(x)^m} + \frac{1}{f_4(x)^m}$
31	4	$\frac{1}{f_2(x)^m} + \frac{1}{f_4(x)^m} + \frac{1}{f_{30}(x)^m}$
32	5	0
33	4	$\frac{1}{f_2(x)^m} + \frac{1}{f_4(x)^m} + \frac{1}{f_{10}(x)^m}$
34	5	$\frac{1}{f_{16}(x)^m}$
35	4	$\frac{1}{f_2(x)^m} + \frac{1}{f_4(x)^m} + \frac{1}{f_6(x)^m}$
36	4	$2\frac{1}{f_2(x)^m}$
37	4	$2\frac{1}{f_2(x)^m} + \frac{1}{f_{36}(x)^m}$
38	4	$2\frac{1}{f_2(x)^m} + \frac{1}{f_{18}(x)^m}$
39	4	$2\frac{1}{f_2(x)^m} + \frac{1}{f_{12}(x)^m}$
40	5	$\frac{1}{f_4(x)^m}$

Table 2: Coefficients $\alpha(n)$ and inner sums $\sum_k C_n(k)/f_k(x)^m$ in the expansion $\log f_n(x) = \alpha(n)\log x + \sum_{m\geq 1}(-1)^{m+1}\frac{1}{m}\sum_k C_n(k)/f_k(x)^m$.

7.2 Equality of sequences

Theorem 1. *For all $n \geq 1$,*

$$\alpha(n) = A064415(n).$$

Proof. Both sequences are determined uniquely by $a(1) = 0$, $a(2) = 1$, complete additivity, and the recursion $a(p) = a(p-1)$ for odd primes (or equivalently the divisor-sum form (15)). These properties hold for $\alpha(n)$, and the OEIS entry states them for A064415. Thus the two functions coincide. □

Remark 4. *Independently, one may observe that* $\alpha(n) = \deg f_n(x)$ *because* $f_p(x) = 1 + f_{p-1}(x)$ *preserves degree for* $p \geq 3$ *and* $f_n(x)$ *is multiplicative in* n*. The MathOverflow discussion notes that these degrees match A064415.*

8 Exponentiation and the representation of $f_n(x)$

Corollary 1. *Formally (and analytically whenever the series converges),*

$$f_n(x) = x^{\alpha(n)} \exp\left(\sum_{m \geq 1} (-1)^{m+1} \frac{1}{m} \sum_{k \geq 1} C_n(k) \, \frac{1}{f_k(x)^m} \right). \tag{16}$$

Proof. Exponentiate the "log-free" equation for $f_n(x)$. □

9 Specialization at $x = 2$

Using Lemma 1, we have $f_k(2) = k$ for all k. Thus (16) yields:

Theorem 2. *For every* $n \geq 1$,

$$n = 2^{\alpha(n)} \exp\left(\sum_{m \geq 1} (-1)^{m+1} \frac{1}{m} \sum_{k \geq 1} C_n(k) \, \frac{1}{k^m} \right). \tag{17}$$

Proof. Evaluate (16) at $x = 2$ and use $f_n(2) = n$. □

Remark 5. *By Theorem 1, the exponent of* 2 *in* (17) *is precisely A064415*(n)*. Thus the representation* (17) *decomposes each integer* n *into a canonical power of* 2 *times an exponential correction encoded by the prime-step coefficient tree* $C_n(k)$*.*

10 Concluding comments

The recursion defining $f_n(x)$ produces two parallel structures:

- a completely additive "degree-like" invariant $\alpha(n)$,
- and a refinement encoded by the coefficients $C_n(k)$ measuring how often prime-step corrections $\mathrm{PS}(k; x)$ appear when expanding $\log f_n(x)$ down to the base index 2.

The identity (17) at $x = 2$ makes this structure explicit at the level of integers.

11 A canonical feature embedding via the coefficients $C_n(k)$

11.1 Setup

Let $f_n(x)$ be defined by

$$\begin{aligned}
&f_1(x) = 1, \qquad f_2(x) = x, \\
&f_p(x) = 1 + f_{p-1}(x) \qquad (p \geq 3 \text{ prime}), \\
&f_n(x) = \prod_{p|n} f_p(x)^{v_p(n)} \qquad (n \text{ composite}).
\end{aligned}$$

Define

$$\mathrm{PS}(k; x) := \log\Big(1 + \frac{1}{f_k(x)}\Big) = \sum_{m \geq 1} (-1)^{m+1} \frac{1}{m} \frac{1}{f_k(x)^m}.$$

As before, define the expansion operator $E(n)$ by

$$\begin{aligned}
&E(1) = 0, \qquad E(2) = \log x, \\
&E(p) = E(p-1) + \mathrm{PS}(p-1; x) \qquad (p \geq 3 \text{ prime}), \\
&E(n) = \sum_{p|n} v_p(n)\, E(p) \qquad (n \text{ composite}).
\end{aligned}$$

Then one has $E(n) = \log f_n(x)$ for all $n \geq 1$.

We write uniquely

$$\log f_n(x) = \alpha(n) \log x + \sum_{k \geq 1} C_n(k)\, \mathrm{PS}(k; x), \tag{18}$$

where $\alpha(n) \in \mathbb{Z}_{\geq 0}$ and only finitely many $C_n(k)$ are nonzero.

11.2 Evaluation at $x = 2$

A key structural property is

$$f_n(2) = n \qquad (n \geq 1),$$

hence

$$\mathrm{PS}(k; 2) = \log\Big(1 + \frac{1}{k}\Big).$$

Evaluating (18) at $x = 2$ yields the identity

$$\log n = \alpha(n) \log 2 + \sum_{k \geq 1} C_n(k)\, \log\Big(1 + \frac{1}{k}\Big). \tag{19}$$

Equivalently, exponentiating,

$$n = 2^{\alpha(n)} \prod_{k \geq 1} \Big(1 + \frac{1}{k}\Big)^{C_n(k)}. \tag{20}$$

Since only finitely many $C_n(k) \neq 0$, the product is finite.

11.3 Injectivity of the feature map

Theorem 3. *The assignment*

$$n \longmapsto \big(\alpha(n), (C_n(k))_{k \geq 1}\big)$$

is injective on $\mathbb{N}$.

Proof. Assume that two integers $n, m \geq 1$ satisfy

$$\alpha(n) = \alpha(m) \quad \text{and} \quad C_n(k) = C_m(k) \text{ for all } k \geq 1.$$

Then the right-hand sides of (18) coincide as formal expressions, hence $\log f_n(x) = \log f_m(x)$. Evaluating at $x = 2$ gives $\log n = \log m$, so $n = m$. □

Remark 6. *The theorem shows that the finite-support vector* $(C_n(k))_{k \geq 1}$*, together with the scalar* $\alpha(n)$*, forms a complete invariant for* n *within this framework.*

11.4 A Hilbert space viewpoint

Let

$$\ell^2_{\text{fin}} := \{(c_k)_{k \geq 1} :\ c_k \in \mathbb{R},\ c_k = 0 \text{ for all but finitely many } k\} \subset \ell^2.$$

Define the Hilbert space

$$\mathcal{H} := \mathbb{R} \oplus \ell^2, \qquad \langle (a, c), (a', c') \rangle = aa' + \sum_{k \geq 1} c_k c'_k.$$

Then for each n we may define the feature embedding

$$\Phi(n) := \big(\alpha(n), (C_n(k))_{k \geq 1}\big) \in \mathcal{H}. \tag{21}$$

Because $(C_n(k))$ has finite support, $\Phi(n) \in \mathbb{R} \oplus \ell^2_{\text{fin}}$.

By Theorem 3, Φ is an injective (canonical) embedding of $\mathbb{N}$ into the Hilbert space $\mathcal{H}$. Identity (19) shows that the linear functional

$$L(a,(c_k)) := a\log 2 + \sum_{k\geq 1} c_k \log\Big(1+\frac{1}{k}\Big)$$

recovers $\log n$ from the feature vector:

$$L(\Phi(n)) = \log n.$$

Remark 7. *This provides a concrete "feature geometry" on the integers induced by the prime-step recursion of $f_n(x)$. One may study distances $\|\Phi(n)-\Phi(m)\|$ or angles in $\mathcal{H}$ as quantitative measures of similarity of prime-step structure.*

12 A concrete feature geometry on the integers

12.1 The ambient Hilbert space

Recall the coefficient expansion

$$\log f_n(x) = \alpha(n)\log x + \sum_{k\geq 1} C_n(k)\,\mathrm{PS}(k;x), \qquad \mathrm{PS}(k;x) = \log\Big(1+\frac{1}{f_k(x)}\Big),$$

with $\alpha(n)\in\mathbb{Z}_{\geq 0}$ and only finitely many $C_n(k)\neq 0$. We define the feature map

$$\Phi(n) := \big(\alpha(n),(C_n(k))_{k\geq 1}\big).$$

Let

$$\ell^2_{\mathrm{fin}} = \{(c_k)_{k\geq 1} : \ c_k = 0 \text{ for all but finitely many } k\} \subset \ell^2,$$

and consider the Hilbert space

$$\mathcal{H} := \mathbb{R}\oplus\ell^2, \qquad \langle (a,c),(a',c')\rangle_{\mathcal{H}} = aa' + \sum_{k\geq 1} c_k c'_k.$$

Since each C_n has finite support, we have $\Phi(n)\in\mathbb{R}\oplus\ell^2_{\mathrm{fin}}$.

12.2 Distance and angle

The induced norm and distance are

$$\|\Phi(n)\|^2 = \alpha(n)^2 + \sum_{k\geq 1} C_n(k)^2, \qquad d(n,m) := \|\Phi(n) - \Phi(m)\|.$$

Explicitly,

$$d(n,m)^2 = \big(\alpha(n) - \alpha(m)\big)^2 + \sum_{k\geq 1} \big(C_n(k) - C_m(k)\big)^2.$$

Thus the distance decomposes into two contributions:

- a *base discrepancy* measured by $\alpha(n) - \alpha(m)$,
- a *prime-step discrepancy* measured coefficientwise by $C_n(k) - C_m(k)$.

Because all C-vectors have finite support, the sum is finite and $d(n,m)$ is well-defined.

Whenever $\Phi(n)$ and $\Phi(m)$ are nonzero, we may also form the angle

$$\cos\theta(n,m) = \frac{\langle \Phi(n), \Phi(m)\rangle_{\mathcal{H}}}{\|\Phi(n)\|\,\|\Phi(m)\|}.$$

A small angle (equivalently large cosine) indicates that n and m share a similar pattern of prime-step corrections, in the sense that their coefficients $C_n(k)$ and $C_m(k)$ point in nearly the same direction in ℓ^2.

12.3 Interpretation via the $x = 2$ identity

Evaluating at $x = 2$ yields

$$\log n = \alpha(n)\log 2 + \sum_{k\geq 1} C_n(k)\ \log\!\Big(1 + \frac{1}{k}\Big).$$

Define the linear functional

$$L : \mathcal{H} \to \mathbb{R}, \qquad L\big(a, (c_k)\big) := a\log 2 + \sum_{k\geq 1} c_k \log\!\Big(1 + \frac{1}{k}\Big).$$

Then

$$L(\Phi(n)) = \log n.$$

Hence the feature embedding is not merely geometric: it is *arithmetically complete* in the sense that the scalar observable $\log n$ is a linear readout of the feature vector.

12.4 Nearest-neighbor intuition

The geometry suggests a natural "nearest-neighbor" heuristic: integers n and m with small $d(n,m)$ should share

1. similar values of $\alpha(\cdot)$ (i.e., similar A064415 values),
2. and similar collections of prime-step ancestors, encoded by the support and multiplicities of $C_n(k)$ and $C_m(k)$.

In particular, if n and m share many of the same prime-step indices k with comparable multiplicities, then the ℓ^2 part of $d(n,m)$ will be small.

12.5 Summary

The map

$$\Phi : \mathbb{N} \to \mathcal{H}, \qquad \Phi(n) = \big(\alpha(n), (C_n(k))_{k\geq 1}\big),$$

provides a canonical embedding of the integers into a Hilbert space. The induced distance and angle offer quantitative measures of similarity between the prime-step structures underlying $f_n(x)$, while the identity at $x = 2$ shows that standard arithmetic data ($\log n$) can be recovered as a linear functional of these features.

13 The rank of the Gram matrix and the prime counting function

13.1 Feature vectors and Gram matrices

Recall the coefficient expansion

$$\log f_n(x) = \alpha(n) \log x + \sum_{k\geq 1} C_n(k)\, \mathrm{PS}(k;x), \qquad \mathrm{PS}(k;x) = \log\Big(1 + \frac{1}{f_k(x)}\Big), \tag{22}$$

where $\alpha(n) \in \mathbb{Z}_{\geq 0}$ and only finitely many $C_n(k)$ are nonzero. We define the feature vector

$$\Phi(n) := \big(\alpha(n), (C_n(k))_{k\geq 1}\big) \in \mathbb{R} \oplus \ell^2.$$

We equip $\mathcal{H} := \mathbb{R} \oplus \ell^2$ with the standard inner product

$$\langle (a,c), (a',c')\rangle_{\mathcal{H}} = aa' + \sum_{k\geq 1} c_k c'_k.$$

For $n \geq 1$ let

$$\mathcal{V}_n := \operatorname{span}\{\Phi(1), \ldots, \Phi(n)\} \subset \mathcal{H}.$$

The $n \times n$ Gram matrix G_n is defined by

$$(G_n)_{ij} := \langle \Phi(i), \Phi(j) \rangle_{\mathcal{H}} \qquad (1 \leq i, j \leq n).$$

It is a standard fact that

$$\operatorname{rank}(G_n) = \dim(\mathcal{V}_n). \tag{23}$$

13.2 Recursive structure of the coefficients

The coefficients $(\alpha(n), C_n(k))$ satisfy the following recursions:

$$\alpha(1) = 0, \qquad \alpha(2) = 1, \tag{24}$$

$$\alpha(p) = \alpha(p-1) \qquad (p \geq 3 \text{ prime}), \tag{25}$$

$$\alpha(n) = \sum_{p \mid n} v_p(n)\, \alpha(p) \qquad (n \text{ composite}), \tag{26}$$

and

$$C_1(k) = 0, \qquad C_2(k) = 0, \tag{27}$$

$$C_p(k) = C_{p-1}(k) + \delta_{k,p-1} \qquad (p \geq 3 \text{ prime}), \tag{28}$$

$$C_n(k) = \sum_{p \mid n} v_p(n)\, C_p(k) \qquad (n \text{ composite}). \tag{29}$$

13.3 Composites do not create new directions

Lemma 3. *If n is composite, then*

$$\Phi(n) = \sum_{p \mid n} v_p(n)\, \Phi(p).$$

Proof. The first component follows directly from (26). For the second component, fix $k \geq 1$ and use (29):

$$C_n(k) = \sum_{p \mid n} v_p(n)\, C_p(k).$$

Thus both components of $\Phi(n)$ equal the corresponding linear combination of the components of $\Phi(p)$ for primes $p \mid n$. □

Corollary 2. *For every $n \geq 1$,*

$$\mathcal{V}_n = \operatorname{span}\{\Phi(p) : p \leq n,\ p \text{ prime}\}.$$

Proof. The inclusion "$\supset$" is obvious. For "$\subset$", repeatedly apply Lemma 3 to replace each composite $\Phi(m)$ with a linear combination of prime feature vectors with indices $\leq m \leq n$. □

13.4 A uniqueness coordinate for each odd prime

The key observation is that each odd prime p introduces a genuinely new coordinate in the C-part of $\Phi(p)$.

Lemma 4. *Let $p \geq 3$ be prime. Then*

$$C_p(p-1) = 1, \qquad C_m(p-1) = 0 \text{ for all } 1 \leq m < p.$$

In particular, for any other prime $q \neq p$ with $q \leq n$,

$$C_q(p-1) = 0.$$

Proof. The identity $C_p(p-1) = 1$ is immediate from (28):

$$C_p(p-1) = C_{p-1}(p-1) + \delta_{p-1,p-1} = C_{p-1}(p-1) + 1.$$

It therefore suffices to show that $C_{p-1}(p-1) = 0$. We prove the stronger statement that $C_m(p-1) = 0$ for all $m < p$ by induction on m.

For $m = 1, 2$, this follows from (27). Assume $3 \leq m < p$, and that $C_r(p-1) = 0$ for all $r < m$.

Case 1: m is prime. Since $m < p$, we have $m - 1 \neq p - 1$, hence $\delta_{p-1,m-1} = 0$. Thus (28) gives

$$C_m(p-1) = C_{m-1}(p-1) + \delta_{p-1,m-1} = C_{m-1}(p-1) = 0$$

by the induction hypothesis.

Case 2: m is composite. Then (29) yields

$$C_m(p-1) = \sum_{q \mid m} v_q(m)\, C_q(p-1).$$

All primes $q \mid m$ satisfy $q \leq m < p$, so $C_q(p-1) = 0$ by the induction hypothesis. Hence $C_m(p-1) = 0$.

This completes the induction and proves the claim. □

13.5 Linear independence of the prime feature vectors

Lemma 5. *The set of vectors* $\{\Phi(p) : p \leq n,\ p \text{ prime}\}$ *is linearly independent.*

Proof. We separate the prime 2 from the odd primes.

First note that $\Phi(2) = (\alpha(2), (0, 0, \dots)) = (1, 0)$ is nonzero, hence provides at least one independent direction.

Now consider a linear relation among prime feature vectors:

$$\lambda_2 \Phi(2) + \sum_{\substack{p \leq n \\ p \geq 3 \text{ prime}}} \lambda_p \Phi(p) = 0.$$

Fix an odd prime $p_0 \leq n$. Look at the coordinate $k = p_0 - 1$ in the ℓ^2–component. By Lemma 4,

$$C_{p_0}(p_0 - 1) = 1, \quad C_p(p_0 - 1) = 0 \ (p \neq p_0), \quad C_2(p_0 - 1) = 0.$$

Therefore the $k = p_0 - 1$ coordinate of the above linear combination equals λ_{p_0}, which must be zero. Since this holds for every odd prime $p_0 \leq n$, all $\lambda_p = 0$ for $p \geq 3$.

The remaining relation is $\lambda_2 \Phi(2) = 0$, so $\lambda_2 = 0$. Thus the prime feature vectors are linearly independent. □

13.6 Rank equals the prime counting function

Theorem 4. *For every* $n \geq 1$*,*

$$\dim(\mathcal{V}_n) = \pi(n),$$

where $\pi(n)$ *denotes the number of primes* $\leq n$*. Consequently,*

$$\operatorname{rank}(G_n) = \pi(n).$$

Proof. By Corollary 2,

$$\mathcal{V}_n = \operatorname{span}\{\Phi(p) : p \leq n,\ p \text{ prime}\}.$$

By Lemma 5, this spanning set is linearly independent. Hence it is a basis of $\mathcal{V}_n$. Therefore

$$\dim(\mathcal{V}_n) = \#\{p \leq n : \ p \text{ prime}\} = \pi(n).$$

Finally, (23) yields $\operatorname{rank}(G_n) = \pi(n)$. □

Corollary 3. *For $n \geq 2$,*

$$\operatorname{rank}(G_n) - \operatorname{rank}(G_{n-1}) = \begin{cases} 1, & n \text{ prime}, \\ 0, & n \text{ composite}. \end{cases}$$

Proof. This is immediate from Theorem 4 and the identity $\pi(n) - \pi(n-1) = 1$ if and only if n is prime. □

13.7 Interpretation

Theorem 4 shows that the first n feature vectors $\Phi(1), \ldots, \Phi(n)$ generate exactly $\pi(n)$ independent directions. The C–coordinates $k = p - 1$ act as "signature features" for odd primes p, while the prime 2 is detected by the α–component. In this sense, the Gram-rank profile of the feature embedding is rigidly controlled by the distribution of primes.

14 Linear functional and continuity of L

We briefly recall the relevant notions and then check that the functional

$$L(a, (c_k)_{k\geq 1}) := a \log 2 + \sum_{k\geq 1} \frac{1}{k}\, c_k \, \log\Big(1 + \frac{1}{k}\Big)$$

is continuous in the natural Hilbert-space topology used for the feature vectors.

14.1 What is a linear functional?

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A *linear functional* on V is a linear map

$$L : V \to \mathbb{R} \quad (\text{or } \mathbb{C}),$$

i.e.

$$L(x + y) = L(x) + L(y), \qquad L(\lambda x) = \lambda L(x).$$

In this setting one works with a feature space of the form

$$H := \mathbb{R} \oplus \ell^2,$$

with inner product

$$\langle (a, c), (b, d) \rangle := ab + \sum_{k\geq 1} c_k d_k,$$

and norm $\|(a, c)\|_H^2 = a^2 + \sum_{k\geq 1} c_k^2$.

14.2 Continuity in which sense?

On a normed space $(V, \|\cdot\|)$, a linear functional L is called *continuous* (or *bounded*) if there exists $C > 0$ such that

$$|L(x)| \le C\|x\| \quad \text{for all } x \in V.$$

For Hilbert spaces this is the standard notion of continuity.

14.3 Continuity of L on $H = \mathbb{R} \oplus \ell^2$

Define the coefficient vector

$$w := \Big(\log 2,\ \big(w_k\big)_{k\ge 1}\Big), \qquad w_k := \frac{1}{k}\log\Big(1 + \frac{1}{k}\Big).$$

Then L can be written as the inner product

$$L(a, c) = \langle (a, c), w \rangle_H.$$

Hence L is continuous if and only if $w \in H$.

We verify $w \in H$: for large k,

$$\log\Big(1 + \frac{1}{k}\Big) = \frac{1}{k} + O\Big(\frac{1}{k^2}\Big),$$

so

$$w_k = \frac{1}{k}\log\Big(1 + \frac{1}{k}\Big) = \frac{1}{k^2} + O\Big(\frac{1}{k^3}\Big).$$

Thus $w_k^2 = O(k^{-4})$, and therefore

$$\sum_{k\ge 1} w_k^2 < \infty.$$

Consequently $w \in \mathbb{R} \oplus \ell^2$, and by Cauchy–Schwarz,

$$|L(a, c)| = |\langle (a, c), w \rangle| \le \|(a, c)\|_H \, \|w\|_H.$$

So L is a bounded (hence continuous) linear functional on H.

14.4 Interpretation via Riesz representation

By the Riesz representation theorem, every continuous linear functional on a Hilbert space is given by inner product with a unique vector. Here that representing vector is exactly w, and the operator norm is

$$\|L\| = \|w\|_H.$$

14.5 Remark

If one changed the ambient topology (e.g. replaced ℓ^2 by ℓ^1 or by the product topology), the continuity question would have a different answer. The above statement is the natural one for the Gram-kernel setup, since the Gram matrix arises from the ℓ^2-type inner product on features.

N	$r = \pi(N)$	$\det G_{\mathcal{P}(N)}$	Vol $= \sqrt{\det}$ (num)
2	1	1	1
3	2	1	1
5	3	1	1
7	4	1	1
11	5	1	1
13	6	1	1
17	7	1	1
19	8	1	1
23	9	1	1
29	10	1	1
31	11	1	1
37	12	1	1
41	13	1	1
43	14	1	1
47	15	1	1

Table 3: Gram determinants and volumes of the prime-spanned parallelotopes.

15 Unimodularity of the Prime Gram Determinant

We work in the Hilbert space

$$H \;=\; \mathbb{R}e_\alpha \;\oplus\; \ell^2(\mathbb{N}),$$

equipped with the orthonormal basis

$$\{e_\alpha\} \cup \{e_k\}_{k\geq 1}.$$

For each $n \geq 1$ define the feature vector

$$\Phi(n) = \alpha(n)e_\alpha + \sum_{k\geq 1} C_n(k)\, e_k,$$

where $\alpha(n) \in \mathbb{Z}$ and $C_n(k) \in \mathbb{Z}$ are given by the recursions:

- $\alpha(1) = 0$, $\alpha(2) = 1$;
- if $p \geq 3$ is prime, then $\alpha(p) = \alpha(p-1)$;
- if n is composite, then

$$\alpha(n) = \sum_{p|n} v_p(n)\, \alpha(p);$$

- $C_1 = 0$, $C_2 = 0$;
- if $p \geq 3$ is prime, then

$$C_p = C_{p-1} + \delta_{p-1};$$

- if n is composite, then

$$C_n = \sum_{p|n} v_p(n)\, C_p.$$

The inner product on H is

$$\langle \Phi(i), \Phi(j) \rangle = \alpha(i)\alpha(j) + \sum_{k \geq 1} C_i(k) C_j(k).$$

For $N \geq 2$ let

$$\mathcal{P}(N) = \{p_1 < \cdots < p_r\}$$

be the set of primes $\leq N$ with $p_1 = 2$, and let

$$G_{\mathcal{P}(N)} = \big(\langle \Phi(p_i), \Phi(p_j) \rangle\big)_{i,j=1}^{r}$$

be the Gram matrix of the prime feature vectors.

15.1 Support and new-coordinate lemmas

Lemma 6 (Prime-support lemma)**.** *For every $n \geq 1$,*

$$\operatorname{supp}(C_n) \subseteq \{\, q - 1 : \ q \leq n \ \textit{prime}, \ q \geq 3 \,\}.$$

Proof. We argue by induction on n. For $n = 1, 2$ the claim is immediate since $C_1 = C_2 = 0$. If $n = p \geq 3$ is prime, then

$$C_p = C_{p-1} + \delta_{p-1},$$

so the support of C_p is contained in the union of the support of C_{p-1} and the singleton $\{p-1\}$. If n is composite, then

$$C_n = \sum_{p|n} v_p(n) C_p,$$

so $\mathrm{supp}(C_n)$ is contained in a union of prime supports with primes $p \leq n$. □

Lemma 7 (New coordinate per prime). *Let $p \geq 3$ be prime. Then*

$$C_p(p-1) = 1 \quad \textit{and} \quad C_m(p-1) = 0 \textit{ for all } m < p.$$

Proof. By the prime-support lemma, for any $m < p$ the support of C_m is contained in $\{q-1 : \ q \leq m \text{ prime}\}$, hence cannot contain $p-1$. Thus $C_{p-1}(p-1) = 0$. Using

$$C_p = C_{p-1} + \delta_{p-1}$$

gives $C_p(p-1) = 1$. □

15.2 Determinant computation

Define the r-element orthonormal set

$$B_N := \{e_\alpha\} \cup \{e_{p_i-1} : \ i = 2, \dots, r\}.$$

By the prime-support lemma, every $\Phi(p_i)$ with $p_i \leq N$ lies in $\mathrm{span}(B_N)$. Let $A_N \in \mathbb{Z}^{r\times r}$ be the coordinate matrix of $\{\Phi(p_i)\}_{i=1}^r$ with respect to B_N (rows are the coordinates).

Lemma 8. *The matrix A_N is lower triangular with diagonal entries equal to* 1. *Hence* $\det(A_N) = 1$.

Proof. For $p_1 = 2$, we have $\Phi(2) = e_\alpha$, so the first row of A_N is $(1, 0, \dots, 0)$. For $i \geq 2$, the column corresponding to e_{p_i-1} records the coefficient $C_{p_j}(p_i - 1)$ in row j. By the new-coordinate lemma, this coefficient equals 1 when $j = i$ and equals 0 for all $j < i$. Therefore all entries above the diagonal in these columns vanish and the diagonal entries are 1. □

Proposition 5 (Prime Gram determinant)**.** *For every $N \geq 2$,*

$$\det\bigl(G_{\mathcal{P}(N)}\bigr) = 1.$$

Consequently the Euclidean volume of the parallelotope spanned by $\{\Phi(p) : p \leq N\}$ equals 1*.*

Proof. Since B_N is orthonormal and A_N records the coordinates of the prime vectors in that basis, we have

$$G_{\mathcal{P}(N)} = A_N A_N^T.$$

Thus

$$\det(G_{\mathcal{P}(N)}) = \det(A_N)^2 = 1.$$

The volume statement follows from the standard identity $\mathrm{Vol}^2 = \det(\mathrm{Gram})$. □

Remark 8. *The result reflects a strong "unimodularity" property of the prime feature family: each new prime $p \geq 3$ introduces a genuinely new orthonormal coordinate e_{p-1} with coefficient* 1 *that cannot occur earlier. This makes the coordinate matrix of the prime vectors triangular with unit diagonal, forcing the Gram determinant to be identically* 1*.*

16 Pratt prime forests and the $(\deg(f_n), C_n)$-representation

16.1 The Pratt prime forest and vertex counts

For a prime p, let T_p denote the *Pratt tree* of p: its root is labeled by p, its children are the primes r dividing $p-1$ with multiplicity $v_r(p-1)$, and the construction continues recursively, terminating at 2. (For $p = 2$ we take T_2 to be the single vertex labeled 2.)

Definition 5 (Pratt prime forest)**.** *Let $n \geq 1$ with prime factorization $n = \prod_p p^{v_p(n)}$. The* Pratt prime forest *of n is the multiset*

$$F(n) \;:=\; \biguplus_{p \mid n} v_p(n) \cdot T_p,$$

i.e. the disjoint union of $v_p(n)$ copies of T_p for each prime divisor p of n.

Definition 6 (Vertex counts)**.** *For a prime q, define*

$$m_q(n) \;:=\; \#\{\textit{vertices labeled } q \textit{ in } F(n)\},$$

counted with multiplicity (each copy of a tree contributes its own vertices).

Basic additivity. By construction of the forest as a disjoint union,

$$m_q(n) \;=\; \sum_{p\mid n} v_p(n)\, m_q(p) \qquad \text{for every prime } q. \tag{30}$$

In particular m_q is completely determined by its values on primes.

16.2 The recursion for $(\alpha(n), C_n)$

We work with the recursively defined invariants $\alpha(n)$ and $C_n(k)$ used in the computations:

- $\alpha(1) = 0$, $\alpha(2) = 1$;
- for primes $p \geq 3$: $\alpha(p) = \alpha(p-1)$;
- for composite n: $\alpha(n) = \sum_{p\mid n} v_p(n)\, \alpha(p)$.

Similarly, C_n is a finitely supported function $k \mapsto C_n(k) \in \mathbb{Z}$ with

- $C_1 = 0$, $C_2 = 0$;
- for primes $p \geq 3$: $C_p = C_{p-1} + \delta_{k,p-1}$ (i.e. add 1 to the coordinate $k = p-1$);
- for composite n: $C_n = \sum_{p\mid n} v_p(n)\, C_p$ (coordinatewise sum).

Write $\deg(f_n)$ for the degree of the polynomial $f_n(x)$ defined by $f_1 = 1$, $f_2 = x$, $f_p = 1 + f_{p-1}$ for primes $p \geq 3$, and $f_n = \prod_{p\mid n} f_p^{v_p(n)}$ for composite n.

16.3 Goal and proof strategy

Define the forest-derived product

$$W(n) \;:=\; 2^{m_2(n)} \prod_{\substack{q \text{ prime} \\ q\geq 3}} \left(\frac{q}{q-1}\right)^{m_q(n)}. \tag{31}$$

The computations in Table 6 suggest the following identification.

Conjecture 1 (Pratt–$(\deg, C)$ dictionary)**.** *For every $n \geq 1$,*

$$\deg(f_n) \;=\; m_2(n), \qquad \textit{and} \qquad C_n(q-1) \;=\; m_q(n) \;\; \textit{for every prime } q \geq 3.$$

Equivalently, the right-hand side in Table 6 can be written as

$$2^{\deg(f_n)} \prod_{k\geq 1} \left(1 + \frac{1}{k}\right)^{C_n(k)} \;=\; 2^{m_2(n)} \prod_{q\geq 3 \text{ prime}} \left(\frac{q}{q-1}\right)^{m_q(n)} \;=\; W(n).$$

A natural proof approach is an induction on n, reduced to a prime induction by additivity.

16.4 Step 1: reduction to primes

Lemma 9 (Additivity reduces the dictionary to primes)**.** *Assume that for every prime p one has*

$$\alpha(p) = m_2(p) \quad \text{and} \quad C_p(q-1) = m_q(p) \quad \text{for all primes } q \geq 3.$$

Then the same identities hold for all $n \geq 1$.

Proof idea. For the m_q this is exactly (30). For α and C_n the composite-step definitions are coordinatewise additive:

$$\alpha(n) = \sum_{p|n} v_p(n)\alpha(p), \qquad C_n = \sum_{p|n} v_p(n)C_p.$$

Thus, once the equalities are known for primes, they extend to all n by the same additivity. □

16.5 Step 2: the prime recursion matches the Pratt recursion

The key structural fact about Pratt trees is that the subtree multiset below the root of T_p is precisely the Pratt prime forest of $p-1$.

Lemma 10 (Pratt recursion for vertex counts)**.** *Let $p \geq 3$ be prime. For every prime q,*

$$m_q(p) \;=\; m_q(p-1) + \mathbf{1}_{\{q=p\}}.$$

Proof idea. The root contributes one vertex labeled p. All remaining vertices come from the children subtrees, which are $v_r(p-1)$ copies of T_r for each prime $r \mid (p-1)$. By definition this is exactly the forest $F(p-1)$. Counting vertices labeled q yields the claimed recursion. □

Now compare this with the defining prime-step recursion for C_p:

$$C_p(k) = C_{p-1}(k) + \mathbf{1}_{\{k=p-1\}}.$$

Setting $k = q-1$ (so $q = k+1$) gives, for primes $q \geq 3$,

$$C_p(q-1) = C_{p-1}(q-1) + \mathbf{1}_{\{q=p\}}. \tag{32}$$

This is formally identical to Lemma 10. Hence:

Proposition 6 (Prime case: C counts Pratt vertices)**.** *For every prime p and every prime $q \geq 3$,*

$$C_p(q-1) = m_q(p).$$

Proof sketch. Induct on p in increasing order. The base cases are $p = 2$ (both sides 0 for $q \geq 3$) and $p = 3$ (both sides equal 1 only when $q = 3$). For the induction step $p \mapsto p'$ with p' prime, use the identical recursions (32) and Lemma 10. □

16.6 Step 3: $\alpha(n) = \deg(f_n)$ and the count of 2-vertices

The same philosophy applies to α and $\deg(f_n)$:

- At prime steps $p \geq 3$, $f_p = 1 + f_{p-1}$ implies $\deg(f_p) = \deg(f_{p-1})$, and $\alpha(p) = \alpha(p-1)$ matches this.
- At composite steps, $f_n = \prod_{p|n} f_p^{v_p(n)}$ gives $\deg(f_n) = \sum_{p|n} v_p(n)\deg(f_p)$, matching the additive definition of $\alpha(n)$.

Thus one proves $\alpha(n) = \deg(f_n)$ by induction on n.

On the Pratt side, Lemma 10 also applies to $q = 2$ (with the convention that T_2 is a single 2-vertex), giving $m_2(p) = m_2(p-1)$ for primes $p \geq 3$, and additivity (30) for composite n. This matches the recursion for α.

Proposition 7 (Degree equals number of 2-vertices)**.** *For every $n \geq 1$,*

$$\deg(f_n) = \alpha(n) = m_2(n).$$

16.7 Consequence: the combinatorial product formula

So we have proved

$$2^{\deg(f_n)} \prod_{k\geq 1} \left(1 + \frac{1}{k}\right)^{C_n(k)} = 2^{m_2(n)} \prod_{q\geq 3 \text{ prime}} \left(\frac{q}{q-1}\right)^{m_q(n)} = W(n).$$

In particular, the verified identity $n = 2^{\deg(f_n)} \prod_k (1 + \frac{1}{k})^{C_n(k)}$ can be reinterpreted as the purely combinatorial statement

$$n = \prod_{q \text{ prime}} \left(\frac{q}{q-1}\right)^{m_q(n)} = \prod_{p \text{ prime}} \left(\frac{1}{1-\frac{1}{p}}\right)^{m_p(n)} \tag{33}$$

i.e. n is obtained by multiplying the local factor $q/(q-1)$ once for each occurrence of a q-vertex in the Pratt prime forest, and multiplying by 2 once for each 2-vertex. We prove below that the coefficients $m_p(n)$ are unique.

17 Uniqueness of Pratt exponents

Lemma 11 (Uniqueness of exponents with finite support)**.** *Let* $(a_p)_p$ *and* $(b_p)_p$ *be families of integers (or rationals) indexed by the primes. Assume* finite support, *i.e.* $a_p = b_p = 0$ *for all but finitely many primes* p*. If*

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-a_p} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-b_p},$$

then $a_p = b_p$ *for every prime* p*. In particular, whenever* n *admits a representation*

$$n = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}$$

with $m_p(n) = 0$ *for all but finitely many primes, the coefficients* $m_p(n)$ *are uniquely determined.*

Proof. Rearranging gives

$$\prod_{p} \left(\frac{p}{p-1}\right)^{a_p - b_p} = 1.$$

Set $c_p := a_p - b_p$. By the finite-support assumption, only finitely many c_p are nonzero. Suppose for contradiction that some $c_p \neq 0$, and let P be the *largest* prime with $c_P \neq 0$.

Apply the P-adic valuation v_P to the left-hand side. The factor

$$\left(\frac{P}{P-1}\right)^{c_P}$$

contributes $v_P\big(P^{c_P}\big) = c_P$ because $P \nmid (P-1)$. For any prime $p < P$ we have $v_P(p) = 0$ and also $v_P(p-1) = 0$ since $p - 1 < P$, hence every factor with $p < P$ contributes zero to v_P. There are no factors with $p > P$ by maximality of P. Therefore,

$$v_P\left(\prod_{p} \left(\frac{p}{p-1}\right)^{c_p}\right) = c_P.$$

But the right-hand side equals 1, so $v_P(1) = 0$, forcing $c_P = 0$, a contradiction. Hence no such P exists and all $c_p = 0$, i.e. $a_p = b_p$ for every prime p. □

Remark 9. *The finite-support hypothesis is essential for treating the product as an honest finite product in $\mathbb{Q}^\times$. Without it, one must specify a convergence notion (absolute convergence, Euler product interpretation, ordering issues, etc.), and uniqueness can fail without additional structure.*

18 A Hilbert-space embedding and a linear read-out of $\log n$

Let p_k denote the k-th prime and let $H := \ell^2(\mathbb{N})$ be the real Hilbert space with standard orthonormal basis $(e_k)_{k\geq 1}$. For each $n \in \mathbb{N}$ we consider the Pratt exponents $m_p(n) \in \mathbb{Z}_{\geq 0}$ and define the (finitely supported) feature vector

$$\phi(n) \;:=\; \sum_{k=1}^{\infty} m_{p_k}(n)\, e_k \;\in\; H. \tag{34}$$

(The sum is finite because $m_p(n) = 0$ for all but finitely many primes p.)

Next define the weight vector

$$w \;:=\; \sum_{k=1}^{\infty} \log\Big(1 - \frac{1}{p_k}\Big)\, e_k. \tag{35}$$

Since $\log(1 - \frac{1}{p}) = -\frac{1}{p} + O(\frac{1}{p^2})$ and $\sum_p \frac{1}{p^2} < \infty$, we have $w \in \ell^2(\mathbb{N}) = H$.

We know the (finite-support) product expansion

$$n \;=\; \prod_{p\ \text{prime}} \Big(1 - \frac{1}{p}\Big)^{-m_p(n)}. \tag{36}$$

Taking logarithms and rewriting in terms of the enumeration (p_k) yields

$$\begin{aligned} \log n &= \sum_{p\ \text{prime}} -m_p(n)\, \log\Big(1 - \frac{1}{p}\Big) \\ &= \sum_{k=1}^{\infty} -m_{p_k}(n)\, \log\Big(1 - \frac{1}{p_k}\Big) = \langle -\phi(n), w\rangle_H. \end{aligned} \tag{37}$$

In particular, the integer n can be recovered from $\phi(n)$ by an exponential readout:

$$n \;=\; \exp\big(-\langle \phi(n), w\rangle_H\big). \tag{38}$$

Equivalently, define the linear functional

$$L : H \to \mathbb{R}, \qquad L(x) := \langle x, w\rangle_H. \tag{39}$$

Then (37) reads $L(\phi(n)) = \log n$, and (38) becomes

$$n = \exp\bigl(-L(\phi(n))\bigr).$$

Remark 10. *The map $\phi : \mathbb{N} \to H$ is multiplicative in the sense that $\phi(nm) = \phi(n) + \phi(m)$, since each coordinate $m_p(\cdot)$ is completely additive. Moreover, L is a bounded linear functional on H by Cauchy–Schwarz, because $w \in H$.*

19 Conclusion

We presented a Hilbert-space feature geometry on $\mathbb{N}$ derived from Pratt prime forests. The key identity

$$n \;=\; \prod_{p \text{ prime}} \Bigl(1 - \frac{1}{p}\Bigr)^{-m_p(n)}$$

shows that the integer n is recovered from the forest vertex counts $m_p(n)$ via a finite Euler-type product. Equivalently, $\log n$ becomes a bounded linear functional of the embedding vector $\phi(n)$, providing a clean bridge between multiplicative arithmetic and linear geometry.

Beyond the formal identity, the main empirical outcome is visual: when the first 100,000 integers are embedded by $\phi(n)$ and reduced to two dimensions with UMAP, the point cloud exhibits striking structure, and an incremental animation makes the emergence of that structure easy to perceive. This supports the viewpoint that the recursive prime ancestry contained in Pratt trees induces a meaningful similarity notion among integers that differs from (and refines) the standard valuation-based encoding.

Several directions remain open. One may compare quantitatively the Pratt-forest embedding against classical encodings (e.g. valuation vectors, divisor functions, or additive arithmetic features), study the spectral properties of the induced Gram matrices, or investigate whether the geometry captures analytic information relevant to Euler products and Dirichlet series. All code and rendered artifacts referenced here are available online [1, 2, 3, 4].

References

[1] Orges Leka. The first hundert thousand numbers (pdf). `https://www.orges-leka.de/the_first_hundert_thousand_numbers.pdf`,

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[4] Orges Leka. Umap visualization of the first 100,000 numbers (video). https://www.youtube.com/watch?v=FA1KEr2zXz0, 2025. Rendered animation (YouTube).

[5] John H. Williamson. Umap of prime factorizations. https://johnhw.github.io/umap_primes/index.md.html, 2018. Original visualization idea; our pipeline adapts this approach to Pratt-forest features.

n	$\deg(f_n)$	$2^{\deg(f_n)} \cdot \prod_{C_n(k)\neq 0} \left(1+\frac{1}{k}\right)^{C_n(k)}$
1	0	2^0
2	1	2^1
3	1	$2^1 \cdot \left(1+\frac{1}{2}\right)^1$
4	2	2^2
5	2	$2^2 \cdot \left(1+\frac{1}{4}\right)^1$
6	2	$2^2 \cdot \left(1+\frac{1}{2}\right)^1$
7	2	$2^2 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1$
8	3	2^3
9	2	$2^2 \cdot \left(1+\frac{1}{2}\right)^2$
10	3	$2^3 \cdot \left(1+\frac{1}{4}\right)^1$
11	3	$2^3 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1$
12	3	$2^3 \cdot \left(1+\frac{1}{2}\right)^1$
13	3	$2^3 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{12}\right)^1$
14	3	$2^3 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1$
15	3	$2^3 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1$
16	4	2^4
17	4	$2^4 \cdot \left(1+\frac{1}{16}\right)^1$
18	3	$2^3 \cdot \left(1+\frac{1}{2}\right)^2$
19	3	$2^3 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{18}\right)^1$
20	4	$2^4 \cdot \left(1+\frac{1}{4}\right)^1$
21	3	$2^3 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^1$
22	4	$2^4 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1$
23	4	$2^4 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1 \cdot \left(1+\frac{1}{22}\right)^1$
24	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^1$
25	4	$2^4 \cdot \left(1+\frac{1}{4}\right)^2$
26	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{12}\right)^1$
27	3	$2^3 \cdot \left(1+\frac{1}{2}\right)^3$
28	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1$
29	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{28}\right)^1$
30	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1$
31	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{30}\right)^1$
32	5	2^5
33	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1$
34	5	$2^5 \cdot \left(1+\frac{1}{16}\right)^1$
35	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1$

Table 4: Table for the identity $n = 2^{\deg(f_n)} \prod_k (1+\frac{1}{k})^{C_n(k)}$ for $1 \leq n \leq 35$.

n	$\deg(f_n)$	$2^{\deg(f_n)} \cdot \prod_{C_n(k)\neq 0} \left(1+\frac{1}{k}\right)^{C_n(k)}$
36	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^2$
37	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{36}\right)^1$
38	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{18}\right)^1$
39	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{12}\right)^1$
40	5	$2^5 \cdot \left(1+\frac{1}{4}\right)^1$
41	5	$2^5 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{40}\right)^1$
42	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^1$
43	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{42}\right)^1$
44	5	$2^5 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1$
45	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{4}\right)^1$
46	5	$2^5 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1 \cdot \left(1+\frac{1}{22}\right)^1$
47	5	$2^5 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1 \cdot \left(1+\frac{1}{22}\right)^1 \cdot \left(1+\frac{1}{46}\right)^1$
48	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1$
49	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^2$
50	5	$2^5 \cdot \left(1+\frac{1}{4}\right)^2$
51	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{16}\right)^1$
52	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{12}\right)^1$
53	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{12}\right)^1 \cdot \left(1+\frac{1}{52}\right)^1$
54	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^3$
55	5	$2^5 \cdot \left(1+\frac{1}{4}\right)^2 \cdot \left(1+\frac{1}{10}\right)^1$
56	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1$
57	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^3 \cdot \left(1+\frac{1}{18}\right)^1$
58	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{28}\right)^1$
59	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{28}\right)^1 \cdot \left(1+\frac{1}{58}\right)^1$
60	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1$
61	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{60}\right)^1$
62	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{30}\right)^1$
63	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^3 \cdot \left(1+\frac{1}{6}\right)^1$
64	6	2^6

Table 5: Table for the identity $n = 2^{\deg(f_n)} \prod_k (1+\frac{1}{k})^{C_n(k)}$ for $36 \leq n \leq 64$.

n	$\deg(f_n)$	$2^{\deg(f_n)} \cdot \prod_{C_n(k)\neq 0} \left(1+\frac{1}{k}\right)^{C_n(k)}$
65	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{12}\right)^1$
66	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1$
67	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1 \cdot \left(1+\frac{1}{66}\right)^1$
68	6	$2^6 \cdot \left(1+\frac{1}{16}\right)^1$
69	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1 \cdot \left(1+\frac{1}{22}\right)^1$
70	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1$
71	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{70}\right)^1$
72	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2$
73	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{72}\right)^1$
74	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{36}\right)^1$
75	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^2$
76	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{18}\right)^1$
77	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1$
78	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{12}\right)^1$
79	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{12}\right)^1 \cdot \left(1+\frac{1}{78}\right)^1$
80	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^1$
81	4	$2^4 \cdot \left(1+\frac{1}{2}\right)^4$
82	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{40}\right)^1$
83	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{40}\right)^1 \cdot \left(1+\frac{1}{82}\right)^1$
84	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^1$
85	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{16}\right)^1$
86	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{42}\right)^1$
87	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{28}\right)^1$
88	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1$
89	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1 \cdot \left(1+\frac{1}{88}\right)^1$
90	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{4}\right)^1$
91	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^1 \cdot \left(1+\frac{1}{12}\right)^1$
92	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1 \cdot \left(1+\frac{1}{22}\right)^1$
93	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{30}\right)^1$
94	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1 \cdot \left(1+\frac{1}{22}\right)^1 \cdot \left(1+\frac{1}{46}\right)^1$
95	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{18}\right)^1$
96	6	$2^6 \cdot \left(1+\frac{1}{2}\right)^1$
97	6	$2^6 \cdot \left(1+\frac{1}{2}\right)^1 \cdot \left(1+\frac{1}{96}\right)^1$
98	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{6}\right)^2$
99	5	$2^5 \cdot \left(1+\frac{1}{2}\right)^2 \cdot \left(1+\frac{1}{4}\right)^1 \cdot \left(1+\frac{1}{10}\right)^1$
100	6	$2^6 \cdot \left(1+\frac{1}{4}\right)^2$

Table 6: Table for the identity $n = 2^{\deg(f_n)} \prod_k (1+\frac{1}{k})^{C_n(k)}$ for $65 \leq n \leq 100$.