The Mason-Stothers theorem for natural numbers

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Abstract

Based on the proof of Serge Lang given in his Book "Algebra" of the Mason-Stothers theorem, we give a proof of this theorem for natural numbers.

Introduction

This introduction was written with ChatGPT, after the notes were written by the author:

These notes offer a detailed examination of the Mason-Stothers theorem as it applies to natural numbers, drawing inspiration from Serge Lang's proof outlined in his seminal work "Algebra". This exploration into the arithmetic derivative, a concept introduced by José Mingot Shelly in 1911, underscores the intriguing parallels between polynomials and natural numbers, offering fresh insights into their interplay. With a rigorous mathematical framework, this paper delves into the theorem's implications, supported by definitions, propositions, and a comprehensive proof that sheds light on the underlying principles of arithmetic derivatives. Through the lens of this theorem, the paper endeavors to contribute to the ongoing discourse in mathematical circles, providing a foundational text for further academic inquiry.

The Mason-Stothers theorem for natural numbers

Let n be a natural number and let $n' := n \sum_{p|n} \frac{v_p(n)}{p}$ denote the arithmetic derivative, which according to Wikipedia was introduced by the Spanish mathematician Josè Mingot Shelly in 1911.

It satisfies the Leibniz Rule:

$$(mn)' = m'n + mn' \tag{1}$$

It is known that there is an analogy between polynomials and natural numbers. Using this analogy, we define the polynomials, given the natural number n:

$$f_n(x) = \prod_{p|n, p \text{ prime}} (x+p)^{v_p(n)} \tag{2}$$

It has the property, that when putting $x = 0$ we get the number n back and the derivatives for $x = 0$ coincide:

$$f_n(0) = n \tag{3}$$

$$f_n'(x)|_{x=0} = n' \tag{4}$$

Let $\mathrm{rad}(n) := \prod_{p|n} p$ denote the radical of the natural number n and let:

$$\mathrm{rad}(g(x)) := \prod_{\alpha \in \Omega_g} x - \alpha \tag{5}$$

where $\Omega_g :=$ is the set of distinct roots of g. It is known, that

$$\mathrm{rad}(g(x)) = \frac{g(x)}{\gcd(g(x), g'(x))} \tag{6}$$

and also if $g(x) = \prod_{i=1}^{s} g_i(x)^{r_i}$ is the decomposition of $g(x)$ into irreducible factors, then:

$$\mathrm{rad}(g(x)) = \prod_{i=1}^{s} g_i(x) \tag{7}$$

Proposition 0.1. *Let $f(x) \in \mathbb{N}[x]$ be a polynomial with $f(0) \neq 0$. Then the natural number $\mathrm{rad}(f(x))|_{x=0}$ is a divisor of $f(0)$.*

Proof. Let $f(x) = \prod_{i=1}^{s} f_i(x)^{r_i}$ be the decomposition of $f(x)$ into irreducible factors $f_i(x)$, some of which might $(r_i > 1)$ occur with repetition. Then $0 \neq f(0) = \prod_{i=1}^{s} f_i(0)^{r_i}$ and $\mathrm{rad}(f(x)) = \prod_{i=1}^{s} f_i(x)$. But then the natural number $\mathrm{rad}(f(x))|_{x=0} = \prod_{i=1}^{s} f_i(0)$ is a divisor of $\prod_{i=1}^{s} f_i(0)^{r_i} = f(0)$.

□

Proposition 0.2. *Let m, n be two natural numbers with $\gcd(m,n) = 1$, such that:*

$$W(m,n) := m'n - mn' \neq 0 \tag{8}$$

where $W(m,n)$ denotes the Wronskian of the arithmetic derivative. Let $g(x) := f_m(x) + f_n(x)$. Then we have the following inequality:

$$mn(m+n) \leq \mathrm{rad}(mn)\,|\mathrm{rad(g(x))}|_{\mathrm{x=0}}| \cdot |W(m,n)| \tag{9}$$

Proof. Since $\gcd(m,n) = 1$ and we know the complete factorization of $f_m(x)$, $f_n(x)$, we see that also $\gcd(f_m(x), f_n(x)) = 1$ and so, $f_m(x), f_n(x), g(x)$ are pairwise prime polynomials. We have:

$$g'(x) = f'_m(x) + f'_n(x) \tag{10}$$

hence:

$$g'(x)|_{x=0} = f'_m(x)|_{x=0} + f'_n(x)|_{x=0} = m' + n' \tag{11}$$

and also:

$$g(0) = f_m(0) + f_n(0) = m + n \tag{12}$$

Let $F_m(x) := \frac{f_m(x)}{g(x)}, F_n(x) := \frac{f_n(x)}{g(x)}$ so that:

$$F_m(x) + F_n(x) = 1 \tag{13}$$

Taking the derivative of this last equation we get:

$$F'_m(x) + F'_n(x) = 0 \tag{14}$$

which we write as in "Algebra" by Serge Lang as:

$$\frac{F'_m(x)}{F_m(x)} F_m(x) + \frac{F'_n(x)}{F_n(x)} F_n(x) = 0 \tag{15}$$

from which we deduce that:

$$\frac{\frac{F'_m(x)}{F_m(x)}}{\frac{F'_n(x)}{F_n(x)}} = -\frac{F_n(x)}{F_m(x)} = -\frac{f_n(x)}{f_m(x)} \tag{16}$$

By the quotient rule we have:

$$F'_m(x) = \frac{f'_m(x)g(x) - g'(x)f_m(x)}{g(x)^2} \tag{17}$$

and we deduce that:

$$\frac{F'_m(x)}{F_m(x)} = \frac{f'_m(x)g(x) - g'(x)f_m(x)}{f_m(x)g(x)} \tag{18}$$

Hence it follows that

$$\frac{F'_m(x)}{F_m(x)}|_{x=0} = \frac{m'(m+n) - (m'+n')m}{m(m+n)} = \frac{m'}{m} - \frac{m'+n'}{m+n} = \frac{W(m,n)}{m(m+n)} \tag{19}$$

which by assumption on the Wronskian is a non-zero rational number. If

$$g(x) = c \prod_k (x - \gamma_k)^{r_k} \tag{20}$$

is the factorization of $g(x)$ over the complex numbers, then by Exercise 11 in Serge Lang's book "Algebra", we have:

$$\frac{f_n(x)}{f_m(x)} = -\frac{\frac{F'_m(x)}{F_m(x)}}{\frac{F'_n(x)}{F_n(x)}} = -\frac{\sum_{p|m} \frac{v_p(m)}{x+p} - \sum_k \frac{r_k}{x-\gamma_k}}{\sum_{q|n} \frac{v_q(n)}{x+q} - \sum_k \frac{r_k}{x-\gamma_k}} \tag{21}$$

A common denominator of $F'_m(x)/F_m(x), F'_n(x)/F_n(x)$ is given by the product:

$$N_0(x) := \prod_{p|m}(x+p) \prod_{q|n}(x+q) \prod_k (x - \gamma_k) \tag{22}$$

We observe, that if we put $x = 0$ in the last equation, we get:

$$N_0(0) = \mathrm{rad}(m)\,\mathrm{rad}(n)\,\mathrm{rad}(\mathrm{g(x)})\,|_{x=0} \tag{23}$$

We observe also, that $N_0(x)F'_m(x)/F_m(x), N_0(x)F'_n(x)/F_n(x)$ are both polynomials in $\mathbb{C}[x]$ and we have:

$$\frac{f_n(x)}{f_m(x)} = -\frac{N_0(x)F'_m(x)/F_m(x)}{N_0(x)F'_n(x)/F_n(x)} \tag{24}$$

It is known that for any polynomial $f(x)$ we have:

$$\mathrm{rad}(f(x)) = \frac{f(x)}{\gcd(f(x), f'(x))} \tag{25}$$

Hence

$$\mathrm{rad}(g(x)) = \frac{g(x)}{\gcd(g(x), g'(x))} \tag{26}$$

from which we see, that:

$$|\mathrm{rad}(g(x))|_{x=0}| = |\prod_k \gamma_k| = \prod_k |\gamma_k| \tag{27}$$

is a natural number. So by equation (19) and the last equation we get:

$$\frac{n}{m} = \frac{f_n(0)}{f_m(0)} = | - \frac{\frac{F'_m(x)}{F_m(x)}}{\frac{F'_n(x)}{F_n(x)}}|_{x=0}| = | - \frac{\sum_{p|m} \frac{v_p(m)}{p} - \sum_k \frac{r_k}{-\gamma_k}}{\sum_{q|n} \frac{v_q(n)}{q} - \sum_k \frac{r_k}{-\gamma_k}}| \tag{28}$$

which after multiplying by $|N_0(0)|$ gets:

$$\frac{n}{m} = |\frac{\mathrm{rad(mn)}\,\mathrm{rad}(g(x))|_{x=0}(\sum_{p|m} \frac{v_p(m)}{p} - \sum_k \frac{r_k}{-\gamma_k})}{\mathrm{rad(mn)}\,\mathrm{rad}(g(x))|_{x=0}(\sum_{q|n} \frac{v_q(n)}{q} - \sum_k \frac{r_k}{-\gamma_k})}| \tag{29}$$

The left hand side of this last equation consists of a ratio n/m of two natural numbers n, m with $\gcd(n, m) = 1$. The right hand side of this last equation consists of a ratio of two natural numbers.

Hence we must have:

$$n \leq |N_0(0)F'_m(0)/F_m(0)| \tag{30}$$

and also:

$$m \leq |N_0(0)F'_n(0)/F_n(0)| \tag{31}$$

But we have determined the value of the natural number:

$$|N_0(0)F'_m(0)/F_m(0)| = \mathrm{rad}(mn)\,|\mathrm{rad}(g(x))|_{x=0}|\,\frac{|W(m,n)|}{m(m+n)} \tag{32}$$

and by the last inequality about n, we get:

$$n \leq \mathrm{rad}(mn)\,|\mathrm{rad}(g(x))|_{x=0}|\,\frac{|W(m,n)|}{m(m+n)} \tag{33}$$

and after multiplying with $m(m + n)$ we get the desired inequality.

□

Proposition 0.3. *Let $f(x), g(x) \in \mathbb{N}[x]$ be two polynomials with $\gcd(f(x), g(x)) = 1$, $f(0) \neq 0, g(0) \neq 0$ and let $F(x) := \frac{f(x)}{f(x)+g(x)}, G(x) := \frac{g(x)}{f(x)+g(x)}$ and suppose that $F'(0) \neq 0, G'(0) \neq 0$. Then*

$$\frac{g(0)}{\gcd(f(0), g(0))} \leq \operatorname{rad}(f(x)g(x)(f(x)+g(x)))|_{x=0}|F'(0)/F(0)| \tag{34}$$

Proof. The proof is analogous to the proof given of the proposition before. □

Proposition 0.4. *Let m, n be two natural numbers with $\gcd(m,n) = 1, W(m,n) \neq 0, mn > |W(m,n)|, \operatorname{rad}(f_m(x) + f_n(x))|_{x=0} \leq \operatorname{rad}(m+n)$. Then:*

$$m + n < \operatorname{rad}(mn(m+n)) \tag{35}$$

Proof. Consider in the last proposition $f(x) := f_{m^2n}(x), g(x) := f_{mn^2}(x)$ with

$$\gcd(f(0), g(0)) = \gcd(m^2n, mn^2) = mn$$

. Then

$$f(x)g(x) = f_{m^2n}(x)f_{mn^2}(x) = f_{m^3n^3}(x) = f_{mn}(x)^3 \tag{36}$$

and

$$f(x) + g(x) = f_{m^2n}(x) + f_{mn^2}(x) = f_{mn}(x)(f_m(x) + f_n(x)) \tag{37}$$

From this we get:

$$\operatorname{rad}(f(x)g(x)(f(x)+g(x)))|_{x=0} = \operatorname{rad}(f_{mn}(x)^4(f_m(x)+f_n(x)))|_{x=0} = \ldots \tag{38}$$

$$\ldots = \operatorname{rad}(mn) \cdot \operatorname{rad}(f_m(x) + f_n(x))|_{x=0} \tag{39}$$

It follows by the last proposition and since $|F'(0)/F(0)| = \frac{|W(m,n)|}{m(m+n)}$:

$$n = \frac{n^2m}{mn} = \frac{g(0)}{\gcd(f(0), g(0))} \leq \operatorname{rad}(mn)\operatorname{rad}(f(x)+g(x))|_{x=0} \cdot |F'(0)/F(0)| \tag{40}$$

which leads to :

$$n \leq \operatorname{rad}(mn)\operatorname{rad}(f(x)+g(x))|_{x=0} \cdot \frac{|W(m,n)|}{m(m+n)} \tag{41}$$

hence by multiplying with $m(m+n)$ we get:

$$mn(m+n) \leq \operatorname{rad}(mn)\operatorname{rad}(f(x)+g(x))|_{x=0}|W(m,n)| \tag{42}$$

which by assumptions on is

$$mn(m+n) \leq \operatorname{rad}(mn)\operatorname{rad}(f(x)+g(x))|_{x=0}|W(m,n)| < mn\operatorname{rad}(mn(m+n)) \tag{43}$$

and the inequality follows when dividing by mn.

□

References

[1] Algebra — Serge Lang .