Pratt-Internal Tropical Geometry on the Positive Rationals

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Abstract

This paper develops the arithmetic and algebraic foundations of a Pratt-internal tropical geometry on the positive rationals. For each prime p we count the number $m_p(n)$ of vertices labeled p in the Pratt prime forest of an integer n. These counts are additive under multiplication and satisfy the exact product formula

$$n = \prod_{p \in \mathbb{P}} \Big(1 - \frac{1}{p}\Big)^{-m_p(n)}.$$

Hence the family $\Phi_P(x) = (m_p(x))_{p \in \mathbb{P}}$ gives a coordinate system on $\mathbb{Q}_{>0}$ that transports multiplication into addition and the Pratt meet/join operations into coordinatewise min/max. In this sense, the positive rationals support an internal tropical algebra.

The paper proves the product formula, extends the construction from $\mathbb{N}$ to $\mathbb{Q}_{>0}$, and shows that $(\mathbb{Q}_{>0}, \cdot, \leq_P)$ is a lattice-ordered abelian group. It then develops Pratt monomials, Pratt tropical polynomials, and their prime shadows, and proves reconstruction results showing that monomials and induced polynomial functions are determined by their full families of shadows. The later sections keep the computational and logical parts of the theory explicit: finite Pratt-closed windows, local reconstruction sets, decidability of prime-shadow solution sets, and effective finite-window reconstruction.

The point of the paper is therefore more specific than a full tropical-geometric comparison. We do not attempt here to build the entire intersection-theoretic apparatus. Instead, we isolate the foundational arithmetic structure, prove the reconstruction and decidability statements in detail, and illustrate the method with explicit examples and computational logs.

Contents

1 Introduction

The modern tropical semiring is usually written as either

$$(\mathbb{R} \cup \{-\infty\}, \max, +) \qquad \text{or} \qquad (\mathbb{R} \cup \{+\infty\}, \min, +),$$

and tropical geometry is built from piecewise integral-affine objects over that idempotent algebra. The standard dictionary identifies tropical monomials with integral affine-linear forms, tropical polynomials with maxima of finitely many such forms, and tropical hypersurfaces with the loci where at least two monomials are simultaneously dominant. The book of Mikhalkin and Rau develops this point of view systematically and will serve as the geometric reference throughout this paper.

The purpose of the present note is to explain that a closely related tropical structure is already hidden inside the positive rationals. The starting point is not a valuation in the classical sense, but a family of recursively defined counting functions attached to Pratt trees. Given a prime q, one considers the rooted Pratt tree T_q, whose children encode the prime factorization of $q-1$. For a general integer n, one takes one copy of T_q for each prime factor q of n, with multiplicity $v_q(n)$. The basic numerical invariant is then

$$m_p(n) = \#\{\text{vertices labeled } p \text{ in the Pratt forest of } n\},$$

one count for each prime p.

These counts behave in a remarkably rigid way. They are additive under multiplication,

$$m_p(nm) = m_p(n) + m_p(m),$$

and they recover the original integer through the finite Euler-type product

$$n = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}.$$

Thus the family $(m_p(n))_p$ forms a complete coordinate system on the positive integers. By passing from $n \in \mathbb{N}$ to $x \in \mathbb{Q}_{>0}$, one obtains a group homomorphism

$$\Phi_P : \mathbb{Q}_{>0} \to \mathbb{Z}^{(\mathbb{P})}, \qquad x \mapsto (m_p(x))_{p \in \mathbb{P}},$$

which plays the role of a tropicalization map.

The new ingredient is that the coordinatewise min and max operations can be pulled back to $\mathbb{Q}_{>0}$. This gives a partial order

$$x \leq_P y \iff m_p(x) \leq m_p(y) \text{ for all primes } p,$$

and corresponding lattice operations $x \wedge_P y$ and $x \vee_P y$. Under the map Φ_P one has

$$\Phi_P(xy) = \Phi_P(x)+\Phi_P(y), \qquad \Phi_P(x\wedge_P y) = \min\{\Phi_P(x), \Phi_P(y)\}, \qquad \Phi_P(x\vee_P y) = \max\{\Phi_P(x), \Phi_P(y)\}.$$

In other words, the classical tropical semiring operations appear as the coordinate shadows of intrinsic arithmetic operations on $\mathbb{Q}_{>0}$. This is why it makes sense to speak of *Pratt-internal tropical geometry.* The Pratt decomposition has appeared before, most notably in the work of Erdős, Granville, Pomerance, and Spiro on iterates of arithmetic functions [1]. There it serves as part of the background structure behind iterative phenomena. The present paper takes the opposite viewpoint: the decomposition itself is placed at the center and developed as an independent object.

The paper is organized as follows. Sections 2–4 start from zero: Pratt trees are defined for primes, Pratt forests for composite integers, and the product identity is proved. Section 5 extends the whole structure to $\mathbb{Q}_{>0}$. The next sections introduce the Pratt order, meet and join, and the resulting internal tropical algebra of monomials, polynomials, and hypersurfaces. After the basic examples and the discussion of prime shadows, we turn to the parts of the theory that are most specific to the Pratt setting: reconstruction from families of shadows, finite Pratt-closed control windows, and decidability/effective reconstruction statements for shadow solution sets. A short concluding section explains how this picture differs from the classical diagonal valuation model and how it interfaces with standard tropical language.

Historically, the tropical viewpoint grew out of max-plus and min-plus algebra. For background and terminology we include references to Cuninghame-Green, Imre Simon, Jean-Éric Pin, and Mikhalkin–Rau in the bibliography, together with the companion paper *Exploring Pratt Trees* where the arithmetic side of the present construction is developed in more detail.

2 Pratt trees for primes

Definition 2.1 (Pratt tree of a prime)**.** Let q be a prime. The *Pratt tree* T_q is the rooted tree defined recursively as follows.

- If $q = 2$, then T_2 consists of a single vertex labeled 2.
- If $q > 2$, then the root is labeled q, and for every prime $r \mid q-1$ one attaches $v_r(q-1)$ children labeled r; each such child is the root of a copy of T_r.

Thus the Pratt tree of a prime records the prime factorization of $q-1$, then the prime factorizations of the predecessors $r-1$, and so on until the recursion reaches 2.

Example 2.2. The first few Pratt trees are easy to write down.

$$\begin{aligned} T_2 : &\quad 2, \\ T_3 : &\quad 3 \to 2, \\ T_5 : &\quad 5 \to 2, 2, \\ T_7 : &\quad 7 \to 2, 3 \to 2, \\ T_{11} : &\quad 11 \to 2, 5 \to 2, 2. \end{aligned}$$

In the last line the root 11 has one child labeled 2 and one child labeled 5, and the 5-child has two children labeled 2.

3 Pratt forests for composite integers

Definition 3.1 (Pratt prime forest)**.** Let $n \geq 1$ with prime factorization

$$n = \prod_{q \in \mathbb{P}} q^{v_q(n)}.$$

The *Pratt prime forest* of n is the disjoint union

$$F(n) := \biguplus_{q \mid n} v_q(n) \cdot T_q,$$

that is, one takes $v_q(n)$ copies of T_q for each prime divisor q of n.

Definition 3.2 (Pratt multiplicities)**.** For a prime p and an integer $n \geq 1$, define

$$m_p(n) := \#\{\text{vertices labeled } p \text{ in } F(n)\},$$

counted with multiplicity.

Example 3.3. Let $n = 12 = 2^2 \cdot 3$. Then

$$F(12) = T_2 \sqcup T_2 \sqcup T_3.$$

Hence

$$m_2(12) = 1 + 1 + 1 = 3, \qquad m_3(12) = 1,$$

and $m_p(12) = 0$ for all primes $p \geq 5$.

Lemma 3.4 (Additivity on $\mathbb{N}$)**.** *For every prime p and all $m, n \in \mathbb{N}$,*

$$m_p(mn) = m_p(m) + m_p(n).$$

Equivalently,

$$m_p(n) = \sum_{q \in \mathbb{P}} v_q(n)\, m_p(q).$$

Proof. The forest $F(mn)$ is obtained by taking $v_q(mn) = v_q(m) + v_q(n)$ copies of each T_q. Hence $F(mn)$ is the disjoint union of $F(m)$ and $F(n)$. Counting vertices labeled p proves the first formula, and the second is just the same statement written prime by prime. □

Lemma 3.5 (Prime recursion)**.** *Let q be a prime with $q > 2$. Then for every prime p,*

$$m_p(q) = m_p(q-1) + \mathbf{1}_{\{p=q\}}.$$

Proof. The root of T_q contributes one vertex labeled q. Everything below the root is exactly the Pratt forest of $q-1$. Therefore the number of vertices labeled p in T_q equals the number of such vertices in the forest of $q-1$, plus one more if $p = q$. □

4 The fundamental Pratt product formula

Theorem 4.1 (Pratt product identity)**.** *For every integer $n \geq 1$,*

$$n = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}.$$

Equivalently,

$$\frac{1}{n} = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right)^{m_p(n)}.$$

The product is finite, since only finitely many $m_p(n)$ are nonzero.

Proof. Define

$$W(n) := \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right)^{-m_p(n)}.$$

By Lemma 3.4, each exponent $m_p(n)$ is additive under multiplication, hence

$$W(mn) = W(m)W(n)$$

for all $m, n \in \mathbb{N}$. Thus it suffices to prove $W(q) = q$ for primes q.

We proceed by strong induction on q. For $q = 2$, one has $m_2(2) = 1$ and $m_p(2) = 0$ for all $p \neq 2$, so

$$W(2) = \left(1 - \frac{1}{2}\right)^{-1} = 2.$$

Now let $q > 2$ be prime and assume the claim already known for all positive integers $< q$. By Lemma 3.5,

$$m_p(q) = m_p(q-1) + \mathbf{1}_{\{p=q\}}.$$

Therefore

$$W(q) = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right)^{-m_p(q-1)-\mathbf{1}_{\{p=q\}}} = \left(1 - \frac{1}{q}\right)^{-1} \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right)^{-m_p(q-1)}.$$

The remaining product is $W(q-1)$, and by induction this equals $q-1$. Hence

$$W(q) = \left(1 - \frac{1}{q}\right)^{-1}(q-1) = \frac{q}{q-1}(q-1) = q.$$

By complete multiplicativity, $W(n) = n$ for every $n \in \mathbb{N}$. ☐

Remark 4.2. Theorem 4.1 is the arithmetic heart of the whole theory. It says that the recursively defined counts $m_p(n)$ determine n exactly, and do so by a product formula that already looks like a tropicalization rule in logarithmic coordinates.

Lemma 4.3. *For every prime p and all integers $a, b \geq 1$, one has*

$$m_p(\gcd(a,b)) \leq m_p(a+b).$$

Equivalently,

$$\gcd(a,b) \leq_p a+b.$$

Proof. Using the prime factor expansion of $m_p(\cdot)$, we have

$$m_p(a+b) = \sum_q v_q(a+b)\, m_q(p).$$

By the usual valuation inequality,

$$v_q(a+b) \geq \min\{v_q(a), v_q(b)\} = v_q(\gcd(a,b)),$$

hence

$$m_p(a+b) \geq \sum_q v_q(\gcd(a,b))\, m_q(p) = m_p(\gcd(a,b)).$$

☐

5 Extension from $\mathbb{N}$ to $\mathbb{Q}_{>0}$

Definition 5.1 (Pratt exponents on $\mathbb{Q}_{>0}$)**.** Let $x \in \mathbb{Q}_{>0}$ and write it in lowest terms as $x = a/b$ with $a, b \in \mathbb{N}$. For every prime p define

$$m_p(x) := m_p(a) - m_p(b) \in \mathbb{Z}.$$

Equivalently,

$$m_p(x) = \sum_{q \in \mathbb{P}} v_q(x)\, m_p(q),$$

where $v_q(x)$ is the usual q-adic valuation of the rational number x.

Proposition 5.2 (Additivity on $\mathbb{Q}_{>0}$)**.** *For every prime p and all $x, y \in \mathbb{Q}_{>0}$,*

$$m_p(xy) = m_p(x) + m_p(y).$$

Hence the map

$$\Phi_P : \mathbb{Q}_{>0} \to \mathbb{Z}^{(\mathbb{P})}, \qquad \Phi_P(x) = (m_p(x))_p,$$

is a group homomorphism from $(\mathbb{Q}_{>0}, \cdot)$ *to the additive group of finitely supported integer sequences.*

Proof. Write $x = a/b$ and $y = c/d$ in lowest terms. Then

$$xy = \frac{ac}{bd},$$

and by Lemma 3.4 on integers,

$$m_p(ac) - m_p(bd) = \Big(m_p(a) - m_p(b)\Big) + \Big(m_p(c) - m_p(d)\Big).$$

This is exactly the claimed identity. □

Proposition 5.3 (Product formula on $\mathbb{Q}_{>0}$)**.** *For every* $x \in \mathbb{Q}_{>0}$*,*

$$x = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right)^{-m_p(x)}.$$

Proof. Write $x = a/b$ in lowest terms. By Theorem 4.1,

$$a = \prod_{p} \left(1 - \frac{1}{p}\right)^{-m_p(a)}, \qquad b = \prod_{p} \left(1 - \frac{1}{p}\right)^{-m_p(b)}.$$

Dividing the two identities yields

$$\frac{a}{b} = \prod_{p} \left(1 - \frac{1}{p}\right)^{-(m_p(a)-m_p(b))} = \prod_{p} \left(1 - \frac{1}{p}\right)^{-m_p(x)}.$$

□

6 An integrality criterion and its application to join and meet

We record a simple criterion that characterizes when a positive rational number is in $\mathbb{N}$, expressed in Pratt coordinates.

Proposition 6.1 (Integrality criterion)**.** *Let* $x \in \mathbb{Q}_{>0}$*. Then the following are equivalent:*

1. $x \in \mathbb{N}$*.*
2. *For every prime* q*,*
$$m_q(x) \ \geq \ \sum_{r>q} m_r(x)\, v_q(r-1).$$

Proof. By the product formula,

$$x = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right)^{-m_p(x)} = \prod_{p \in \mathbb{P}} \left(\frac{p}{p-1}\right)^{m_p(x)}.$$

Fix a prime q. The exponent of q in the above product is obtained as follows.

First, the factor with $p = q$ contributes $m_q(x)$, since

$$\left(\frac{q}{q-1}\right)^{m_q(x)}$$

contains q in the numerator with exponent $m_q(x)$.

Second, for each prime $r > q$, the factor

$$\left(\frac{r}{r-1}\right)^{m_r(x)}$$

contributes $-\, m_r(x)\, v_q(r-1)$ to the exponent of q, because q may occur in the denominator through the factorization of $r - 1$.

Hence the exponent of q in x is

$$m_q(x) - \sum_{r>q} m_r(x)\, v_q(r-1).$$

Therefore x is a positive integer if and only if this exponent is nonnegative for every prime q, which is exactly the stated condition. □

We now apply Proposition 6.1 to the lattice operations.

Corollary 6.2 (Integrality of join)**.** *Let* $a, b \in \mathbb{Q}_{>0}$. *Then*

$$a \vee_P b \in \mathbb{N}$$

if and only if for every prime q*,*

$$\max\{m_q(a), m_q(b)\} \;\geq\; \sum_{r>q} \max\{m_r(a), m_r(b)\}\, v_q(r-1).$$

Proof. Since

$$m_p(a \vee_P b) = \max\{m_p(a), m_p(b)\} \qquad \text{for all primes } p,$$

the claim follows immediately from Proposition 6.1 applied to $x = a \vee_P b$. □

Corollary 6.3 (Integrality of meet)**.** *Let* $a, b \in \mathbb{Q}_{>0}$. *Then*

$$a \wedge_P b \in \mathbb{N}$$

if and only if for every prime q*,*

$$\min\{m_q(a), m_q(b)\} \;\geq\; \sum_{r>q} \min\{m_r(a), m_r(b)\}\, v_q(r-1).$$

Proof. Since

$$m_p(a \wedge_P b) = \min\{m_p(a), m_p(b)\} \qquad \text{for all primes } p,$$

the claim again follows from Proposition 6.1, now applied to $x = a \wedge_P b$. □

Lemma 6.4. *If* $a, b \in \mathbb{N}$*, then*

$$a \wedge_P b \in \mathbb{N}.$$

Proof. By Corollary 6.3, it suffices to show that for every prime q,

$$\min\{m_q(a), m_q(b)\} \;\geq\; \sum_{r>q} \min\{m_r(a), m_r(b)\}\, v_q(r-1).$$

Since $a \in \mathbb{N}$, Proposition 6.1 gives

$$m_q(a) \geq \sum_{r>q} m_r(a)\, v_q(r-1).$$

Likewise, since $b \in \mathbb{N}$,

$$m_q(b) \geq \sum_{r>q} m_r(b)\, v_q(r-1).$$

Now for every prime r we have

$$\min\{m_r(a), m_r(b)\} \leq m_r(a) \qquad \text{and} \qquad \min\{m_r(a), m_r(b)\} \leq m_r(b).$$

As $v_q(r-1) \geq 0$, this implies

$$\sum_{r>q} \min\{m_r(a), m_r(b)\}\, v_q(r-1) \leq \sum_{r>q} m_r(a)\, v_q(r-1) \leq m_q(a),$$

and similarly

$$\sum_{r>q} \min\{m_r(a), m_r(b)\}\, v_q(r-1) \leq \sum_{r>q} m_r(b)\, v_q(r-1) \leq m_q(b).$$

Hence

$$\sum_{r>q} \min\{m_r(a), m_r(b)\}\, v_q(r-1) \leq \min\{m_q(a), m_q(b)\},$$

which is exactly the required inequality. Therefore $a \wedge_P b \in \mathbb{N}$. □

In particular, if $a, b \in \mathbb{N}$, then $a \wedge_P b \in \mathbb{N}$ and $a \vee_P b \in \mathbb{N}$ whenever the corresponding inequalities hold. The point of the two corollaries is that for general $a, b \in \mathbb{Q}_{>0}$, integrality of the join or meet is completely determined by these explicit coordinate inequalities.

7 The Pratt order and lattice operations

Definition 7.1 (Pratt order)**.** For $x, y \in \mathbb{Q}_{>0}$, define

$$x \leq_P y \iff m_p(x) \leq m_p(y) \qquad \text{for every prime } p.$$

Proposition 7.2. *The relation $\leq_P$ is a translation-invariant partial order on the multiplicative group $(\mathbb{Q}_{>0}, \cdot)$. That is,*

$$x \leq_P y \implies zx \leq_P zy \qquad (x, y, z \in \mathbb{Q}_{>0}).$$

Proof. Reflexivity and transitivity are obvious from the coordinatewise definition, and antisymmetry follows because the product formula on $\mathbb{Q}_{>0}$ shows that the full family $(m_p(x))_p$ determines x uniquely. Translation-invariance follows from

$$m_p(zx) = m_p(z) + m_p(x), \qquad m_p(zy) = m_p(z) + m_p(y).$$

□

Definition 7.3 (Pratt meet and join)**.** For $x, y \in \mathbb{Q}_{>0}$, define $x \wedge_P y$ and $x \vee_P y$ by the coordinate rules

$$m_p(x \wedge_P y) := \min\{m_p(x), m_p(y)\}, \qquad m_p(x \vee_P y) := \max\{m_p(x), m_p(y)\}$$

for all primes p.

Proposition 7.4. *The positive rationals form a lattice-ordered abelian group under multiplication and the Pratt order. In particular,*

$$\Phi_P(xy) = \Phi_P(x)+\Phi_P(y), \qquad \Phi_P(x\wedge_P y) = \min\{\Phi_P(x), \Phi_P(y)\}, \qquad \Phi_P(x\vee_P y) = \max\{\Phi_P(x), \Phi_P(y)\}.$$

Moreover,

$$z(x \wedge_P y) = (zx) \wedge_P (zy), \qquad z(x \vee_P y) = (zx) \vee_P (zy).$$

Proof. The first three identities are tautological from the definitions. The distributive identities follow because in each coordinate one has

$$c + \min\{a, b\} = \min\{c + a, c + b\}, \qquad c + \max\{a, b\} = \max\{c + a, c + b\}.$$

Since Φ_P is injective, the corresponding identities hold on $\mathbb{Q}_{>0}$. □

8 Pratt-internal tropical algebra

The standard tropical semiring uses max as addition and ordinary addition as multiplication. In the present setting those operations are internalized as follows.

Definition 8.1 (Internal tropical operations)**.** On $\mathbb{Q}_{>0}$ define

$$x \oplus_P y := x \vee_P y, \qquad x \odot_P y := xy.$$

Dually one may also work with the min-version

$$x \boxplus_P y := x \wedge_P y, \qquad x \odot_P y := xy.$$

Proposition 8.2. *The structure $(\mathbb{Q}_{>0}, \oplus_P, \odot_P)$ is an idempotent commutative semiring, and under Φ_P it becomes the ordinary max-plus semiring on the Pratt-coordinate lattice. Likewise, $(\mathbb{Q}_{>0}, \boxplus_P, \odot_P)$ becomes the min-plus version.*

Proof. Commutativity and idempotence of $\oplus_P$ follow from the lattice identities for $\vee_P$. Multiplication is just the usual commutative group law on $\mathbb{Q}_{>0}$. Distributivity follows from Proposition 7.4. The coordinate description is already contained in Proposition 7.4. □

Definition 8.3 (Pratt monomial)**.** Let $c \in \mathbb{Q}_{>0}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$. For $x = (x_1, \dots, x_n) \in (\mathbb{Q}_{>0})^n$ define the *Pratt monomial*

$$M_{c,\alpha}(x) := c\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Proposition 8.4. *For every prime p,*

$$m_p\Big(M_{c,\alpha}(x)\Big) = m_p(c) + \alpha_1 m_p(x_1) + \cdots + \alpha_n m_p(x_n).$$

Thus Pratt monomials correspond exactly to integral affine-linear forms in Pratt coordinates.

Proof. This is immediate from the additivity of the m_p under multiplication and inversion. □

Definition 8.5 (Pratt tropical polynomial)**.** A *Pratt tropical polynomial* on $(\mathbb{Q}_{>0})^n$ is a finite join of Pratt monomials,

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha,$$

where $A \subset \mathbb{Z}^n$ is finite. Dually one may consider finite meets.

Proposition 8.6. *Let*

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha.$$

Then for every prime p,

$$m_p(F(x)) = \max_{\alpha \in A}\Big(m_p(c_\alpha) + \alpha_1 m_p(x_1) + \cdots + \alpha_n m_p(x_n)\Big).$$

Hence each fixed prime p sees F as an ordinary max-plus tropical polynomial.

Proof. Because join becomes coordinatewise maximum under Φ_P, one simply applies the previous proposition term by term. □

Definition 8.7 (Pratt tropical hypersurface)**.** Let

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha$$

be a Pratt tropical polynomial on $(\mathbb{Q}_{>0})^n$. Its *Pratt tropical hypersurface* is the set of points x for which at least two monomials $c_\alpha x^\alpha$ are simultaneously dominant in the Pratt order.

This is the exact internal analogue of the corner-locus definition in ordinary tropical geometry. Under each coordinate map m_p, the Pratt hypersurface projects to the usual tropical hypersurface defined by the corresponding max of affine-linear forms.

9 Detailed numerical examples

9.1 Example 1: the first Pratt vectors

We begin by listing several prime trees and the corresponding columns of the Pratt-coordinate matrix. From the recursion one computes

$$\Phi_P(2) = (1, 0, 0, 0, \dots),$$

$$\Phi_P(3) = (1, 1, 0, 0, \dots),$$

because $3 - 1 = 2$;

$$\Phi_P(5) = (2, 0, 1, 0, \dots),$$

because $5 - 1 = 2^2$;

$$\Phi_P(7) = (2, 1, 0, 1, \dots),$$

because $7 - 1 = 2 \cdot 3$;

$$\Phi_P(11) = (3, 0, 1, 0, 1, \dots),$$

because $11 - 1 = 2 \cdot 5$.

These examples already show the triangular character of the Pratt coordinates. The prime itself contributes one copy of its own basis vector, and the rest comes recursively from smaller primes.

9.2 Example 2: a Pratt-linear polynomial in one variable

Consider the one-variable Pratt tropical polynomial

$$F(t) = 2 \vee_P (3t).$$

We now examine it in detail.

For the prime 2 one has

$$m_2(2) = 1, \qquad m_2(3) = 1,$$

so

$$m_2(F(t)) = \max\{1, 1 + m_2(t)\} = 1 + m_2(t)$$

for all $t \in \mathbb{Q}_{>0}$. Thus in the 2-coordinate, the monomial $3t$ always dominates the constant monomial 2 as soon as $m_2(t) \geq 0$; in fact equality holds only when $m_2(t) = 0$.

For the prime 3 one has

$$m_3(2) = 0, \qquad m_3(3) = 1,$$

so

$$m_3(F(t)) = \max\{0, 1 + m_3(t)\}.$$

Now a genuine transition occurs:

- if $m_3(t) \leq -1$, then the constant term 2 dominates;
- if $m_3(t) = -1$, both monomials tie;
- if $m_3(t) \geq 0$, the monomial $3t$ dominates.

Thus even this simplest Pratt-linear polynomial already exhibits a tropical breakpoint, but the breakpoint depends on which prime coordinate one reads.

9.3 Example 3: a Pratt-quadratic polynomial

Consider

$$F(t) = 1 \vee_P (2t) \vee_P (9t^2).$$

In the 2-coordinate we have

$$m_2(1) = 0, \qquad m_2(2) = 1, \qquad m_2(9) = 2,$$

so

$$m_2(F(t)) = \max\{0, 1 + m_2(t), 2 + 2m_2(t)\}.$$

Let $u = m_2(t) \in \mathbb{Z}$. Then the three affine expressions are

$$0, \qquad 1 + u, \qquad 2 + 2u.$$

Their pairwise equalities occur at

$$0 = 1 + u \iff u = -1,$$

$$0 = 2 + 2u \iff u = -1,$$

$$1 + u = 2 + 2u \iff u = -1.$$

So all three terms meet at the single value $u = -1$. Hence:

- for $u \leq -1$, the constant term dominates or ties;

- at $u = -1$, all three monomials tie;
- for $u \geq 0$, the quadratic term $9t^2$ dominates.

This is a perfectly legitimate tropical quadratic behavior, internalized in $\mathbb{Q}_{>0}$.

For the 3-coordinate the polynomial looks different:

$$m_3(1) = 0, \qquad m_3(2) = 0, \qquad m_3(9) = 2,$$

so

$$m_3(F(t)) = \max\{0, m_3(t), 2 + 2m_3(t)\}.$$

If $v = m_3(t)$, the three affine expressions are

$$0, \qquad v, \qquad 2 + 2v.$$

The pairwise equality loci are

$$0 = v \iff v = 0, \qquad 0 = 2 + 2v \iff v = -1, \qquad v = 2 + 2v \iff v = -2.$$

However, only those equality points at which the tied monomials are simultaneously maximal belong to the tropical hypersurface. Checking the three candidates gives:

- at $v = 0$, the values are $(0, 0, 2)$, so the quadratic term is strictly dominant;
- at $v = -2$, the values are $(0, -2, -2)$, so the constant term is strictly dominant;
- at $v = -1$, the values are $(0, -1, 0)$, so the constant and quadratic terms tie at the maximum.

Hence the 3-shadow hypersurface consists of the single breakpoint

$$v = -1.$$

So in the 3-coordinate the middle monomial $2t$ never contributes to the corner locus, even though its affine function intersects the other two away from the dominant region. Thus the same internal Pratt polynomial has a different tropical shadow in the 3-coordinate than in the 2-coordinate. This multi-shadow phenomenon does not occur in ordinary tropical geometry and is one of the genuinely new features of the Pratt setting.

9.4 Example 4: a Pratt tropical line in two variables

Consider

$$L(x, y) = 1 \vee_P (2x) \vee_P (3y).$$

In the 2-coordinate one obtains

$$m_2(L(x, y)) = \max\{0, 1 + m_2(x), 1 + m_2(y)\}.$$

Writing

$$u = m_2(x), \qquad v = m_2(y),$$

the three linear forms are

$$0, \qquad 1 + u, \qquad 1 + v.$$

Their tie loci are

$$1 + u = 1 + v \iff u = v,$$

$$0 = 1 + u \iff u = -1, \qquad 0 = 1 + v \iff v = -1.$$

Thus the 2-shadow of L is the standard max-plus tropical line with vertex $(-1, -1)$. Its three rays are

$$\{(-1, v) : v \leq -1\}, \qquad \{(u, -1) : u \leq -1\}, \qquad \{(-1, -1) + t(1, 1) : t \geq 0\}.$$

Equivalently, the ray directions are

$$(0, -1), \qquad (-1, 0), \qquad (1, 1).$$

In the 3-coordinate we instead get

$$m_3(L(x, y)) = \max\{0, m_3(x), 1 + m_3(y)\},$$

so with

$$u' = m_3(x), \qquad v' = m_3(y),$$

the tie loci become

$$u' = 1 + v', \qquad u' = 0, \qquad v' = -1.$$

This is again a tropical line, but translated differently in the 3-coordinate plane. So one and the same internal Pratt line on $(\mathbb{Q}_{>0})^2$ casts different tropical line shadows when viewed through different prime coordinates.

10 Prime shadows and internal tropical objects

A central novelty of the Pratt setting is that one internal object on $(\mathbb{Q}_{>0})^n$ gives rise to many ordinary tropical shadows, one for each prime.

Definition 10.1 (q-shadow)**.** Let q be a prime. The *q-shadow map*

$$\pi_q : (\mathbb{Q}_{>0})^n \to \mathbb{Z}^n$$

is defined by

$$\pi_q(x_1, \ldots, x_n) := \big(m_q(x_1), \ldots, m_q(x_n)\big).$$

If $X \subseteq (\mathbb{Q}_{>0})^n$, we call

$$X^{(q)} := \pi_q(X) \subseteq \mathbb{Z}^n$$

the *q-shadow* of X.

Proposition 10.2 (Shadows of Pratt monomials and polynomials)**.** *Let*

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha$$

be a Pratt tropical polynomial on $(\mathbb{Q}_{>0})^n$*, and let* q *be a prime. Then the function obtained by reading* F *in the* q*-th Pratt coordinate is the ordinary tropical polynomial*

$$F^{(q)}(u) = \max_{\alpha \in A} \Big(m_q(c_\alpha) + \alpha \cdot u \Big), \qquad u \in \mathbb{Z}^n.$$

In particular, the q*-shadow of the Pratt hypersurface of* F *is the usual tropical hypersurface determined by* $F^{(q)}$*.*

Proof. The monomial identity

$$m_q(c_\alpha x^\alpha) = m_q(c_\alpha) + \alpha \cdot \pi_q(x)$$

was proved above. Since join becomes coordinatewise maximum under Φ_P, we obtain

$$m_q(F(x)) = \max_{\alpha \in A} \Big(m_q(c_\alpha) + \alpha \cdot \pi_q(x) \Big).$$

This is exactly the ordinary max-plus formula. The corner locus of this max of affine-linear functions is the standard tropical hypersurface attached to $F^{(q)}$, so the second claim follows immediately from the definition of Pratt hypersurface. □

Remark 10.3. A Pratt tropical object is therefore not just one tropical variety in one affine space. It is an internal arithmetic object carrying a whole family of classical tropical shadows

$$X^{(2)}, X^{(3)}, X^{(5)}, \dots$$

at once. This *multi-shadow phenomenon* seems to be genuinely absent from the usual formulation of tropical geometry, where one works inside a single fixed real affine space.

A detailed shadow computation

Consider again the Pratt tropical line

$$L(x, y) = 1 \vee_P (2x) \vee_P (3y).$$

We compute its first two prime shadows in full detail.

The 2**-shadow.** Using

$$m_2(1) = 0, \qquad m_2(2) = 1, \qquad m_2(3) = 1,$$

we obtain

$$L^{(2)}(u,v) = \max\{0, 1+u, 1+v\}.$$

Hence the shadow hypersurface is cut out by the pairwise equalities

$$0 = 1+u \iff u = -1,$$

$$0 = 1+v \iff v = -1,$$

$$1+u = 1+v \iff u = v.$$

Thus $L^{(2)}$ is the standard max-plus tropical line with vertex $(-1,-1)$ and ray directions

$$(0,-1), \qquad (-1,0), \qquad (1,1).$$

The 3-shadow. Now use

$$m_3(1) = 0, \qquad m_3(2) = 0, \qquad m_3(3) = 1.$$

Then

$$L^{(3)}(u,v) = \max\{0, u, 1+v\}.$$

The pairwise equalities are

$$0 = u \iff u = 0,$$

$$0 = 1+v \iff v = -1,$$

$$u = 1+v \iff u - v = 1.$$

So the 3-shadow is again a max-plus tropical line, but now its vertex is $(0,-1)$ and the three rays have directions

$$(0,-1), \qquad (-1,0), \qquad (1,1).$$

Only the position changes; the combinatorial type remains the same.

Interpretation. One and the same internal object $L \subseteq (\mathbb{Q}_{>0})^2$ therefore determines two distinct tropical lines in the ordinary sense, one in the $(u,v) = (m_2(x), m_2(y))$ plane and one in the $(u,v) = (m_3(x), m_3(y))$ plane. This is a concrete instance of the principle that an internal Pratt object carries a whole family of coordinatewise tropical realizations.

Remark 10.4. Because we use the max-plus convention, tropical lines are oriented with two rays going toward decreasing coordinate directions and one ray going along the diagonal increasing direction. This is the opposite sign convention from the min-plus pictures often drawn in the literature.

11 Why the Pratt picture is not the classical valuation picture

A natural objection is that one could repeat the same formal construction with the ordinary prime valuations v_p and use

$$\gcd(a,b)=\prod_p p^{\min(v_p(a),v_p(b))},\qquad \operatorname{lcm}(a,b)=\prod_p p^{\max(v_p(a),v_p(b))}$$

as meet and join. Formally this is true. However, that construction is exactly the diagonal, completely local case. The Pratt setting is different in a structural way.

Proposition 11.1 (Classical valuations as the diagonal case)**.** *If one uses the ordinary valuation vector*

$$v(x)=(v_p(x))_{p\in\mathbb{P}}$$

in place of $\Phi_P(x)$, then the corresponding tropicalization matrix is the identity:

$$v(n)=I\,v(n).$$

By contrast, Pratt coordinates satisfy

$$\Phi_P(n)=A\,v(n),\qquad A=(m_p(q))_{p,q\in\mathbb{P}},$$

where A is a nontrivial unitriangular matrix.

Proof. The first statement is tautological. The second was established earlier: by additivity,

$$m_p(n)=\sum_q v_q(n)m_p(q),$$

which is exactly the matrix identity defining A. □

Remark 11.2. This is the real source of novelty. The valuation geometry coming from v_p is split and diagonal: each prime coordinate is independent of all the others. The Pratt geometry is recursively coupled through the arithmetic of $q-1$. In particular, the prime p-coordinate of a number n depends not only on the occurrence of p in n, but on all prime chains descending through the Pratt trees of the prime factors of n.

Remark 11.3. Said differently: the classical tropicalization of divisibility rewrites prime factorization in tropical language, whereas the Pratt tropicalization introduces a genuinely new integral affine structure on the same multiplicative group.

12 Reconstruction from prime shadows

One of the genuinely new features of Pratt-internal tropical geometry is that a single internal object gives rise to infinitely many ordinary tropical shadows, one for each prime q. This raises the central structural question:

> To what extent is an internal Pratt object determined by its family of prime shadows?

For monomials and polynomials, the answer is essentially complete: the full family of shadows determines the internal object. For general hypersurfaces and cycles, this becomes a new reconstruction problem.

12.1 The basic reconstruction principle

Recall that the Pratt coordinate map

$$\Phi : \mathbb{Q}_{>0} \to \mathbb{Z}^{(\mathbb{P})}, \qquad \Phi(x) = (m_p(x))_{p\in\mathbb{P}},$$

is injective. Indeed, by the unitriangular basis-change matrix

$$A = (m_p(q))_{p,q\in\mathbb{P}},$$

the Pratt coordinates and the ordinary prime-exponent coordinates determine one another uniquely.

This injectivity immediately implies that an element $x \in \mathbb{Q}_{>0}$ is determined by the full sequence of shadow coordinates

$$\Big(m_q(x)\Big)_{q\in\mathbb{P}}.$$

Proposition 12.1 (reconstruction of points from all shadows)**.** *Let $x, y \in \mathbb{Q}_{>0}$. If*

$$m_q(x) = m_q(y) \qquad \textit{for every prime } q,$$

then

$$x = y.$$

Equivalently, if

$$\pi_q(x) = \pi_q(y) \qquad \textit{for every prime } q,$$

then $x = y$.

Proof. The hypothesis says exactly that

$$\Phi(x) = \Phi(y).$$

Since Φ is injective, it follows that $x = y$. □

Thus, at the level of individual points, there is no ambiguity: the totality of prime shadows determines the point uniquely.

12.2 Reconstruction of tuples

The same statement holds componentwise in several variables.

Proposition 12.2 (reconstruction in several variables)**.** *Let*

$$x = (x_1, \dots, x_n), \qquad y = (y_1, \dots, y_n) \in (\mathbb{Q}_{>0})^n.$$

Assume that for every prime q*,*

$$\pi_q(x) = \pi_q(y) \in \mathbb{Z}^n.$$

Then

$$x = y.$$

Proof. By definition,

$$\pi_q(x) = \Big(m_q(x_1), \dots, m_q(x_n)\Big), \qquad \pi_q(y) = \Big(m_q(y_1), \dots, m_q(y_n)\Big).$$

So the hypothesis implies

$$m_q(x_i) = m_q(y_i) \qquad \text{for all primes } q \text{ and all } 1 \le i \le n.$$

By the previous proposition, $x_i = y_i$ for each i, hence $x = y$. □

This shows that the full shadow family separates points in $(\mathbb{Q}_{>0})^n$.

12.3 Reconstruction of Pratt monomials

We now pass from points to functions.

A Pratt monomial has the form

$$M_{c,\alpha}(x) = c\, x^\alpha = c\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad c \in \mathbb{Q}_{>0},\ \alpha \in \mathbb{Z}^n.$$

Its q-shadow is the affine-linear function

$$M_{c,\alpha}^{(q)}(u) = m_q(c) + \alpha \cdot u.$$

Thus each shadow remembers:

- the exponent vector α, via the slope;
- the q-th Pratt coordinate $m_q(c)$, via the constant term.

Proposition 12.3 (reconstruction of monomials from shadows)**.** *Let*

$$M_{c,\alpha}(x) = c\, x^\alpha, \qquad M_{d,\beta}(x) = d\, x^\beta$$

be Pratt monomials on $(\mathbb{Q}_{>0})^n$*. Assume that for every prime* q*, their shadows agree:*

$$M_{c,\alpha}^{(q)}(u) = M_{d,\beta}^{(q)}(u) \qquad \textit{for all } u \in \mathbb{Z}^n.$$

Then

$$\alpha = \beta, \qquad c = d,$$

and hence

$$M_{c,\alpha} = M_{d,\beta}.$$

Proof. Fix q. By assumption,

$$m_q(c) + \alpha \cdot u = m_q(d) + \beta \cdot u \qquad \text{for all } u \in \mathbb{Z}^n.$$

Subtracting gives

$$(\alpha - \beta) \cdot u = m_q(d) - m_q(c) \qquad \text{for all } u \in \mathbb{Z}^n.$$

The left-hand side depends on u, while the right-hand side is constant. Therefore $(\alpha - \beta) \cdot u$ is constant on $\mathbb{Z}^n$, which forces

$$\alpha - \beta = 0.$$

So $\alpha = \beta$. Then the equality reduces to

$$m_q(c) = m_q(d) \qquad \text{for every prime } q.$$

By injectivity of Φ, we conclude that $c = d$. □

Thus a Pratt monomial is completely determined by its shadow family.

12.4 Reconstruction of Pratt tropical polynomials

A Pratt tropical polynomial has the form

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha$$

with $A \subset \mathbb{Z}^n$ finite and $c_\alpha \in \mathbb{Q}_{>0}$.

Its q-shadow is the ordinary tropical polynomial

$$F^{(q)}(u) = \max_{\alpha \in A} \Big(m_q(c_\alpha) + \alpha \cdot u \Big).$$

At this point one must distinguish two notions of equality:

- equality of presentations;
- equality as functions.

Even in ordinary tropical geometry, different presentations may define the same tropical polynomial function. Thus reconstruction should be understood at the level of the induced internal function.

Definition 12.4 (functional shadow equivalence)**.** Two Pratt tropical polynomials F and G are called *shadow-equivalent* if for every prime q,

$$F^{(q)} = G^{(q)}$$

as ordinary tropical polynomial functions on $\mathbb{Z}^n$.

Proposition 12.5 (reconstruction of the induced internal function)**.** *Let F and G be Pratt tropical polynomials on $(\mathbb{Q}_{>0})^n$. Assume that for every prime q,*

$$F^{(q)}(u) = G^{(q)}(u) \qquad \textit{for all } u \in \mathbb{Z}^n.$$

Then

$$F(x) = G(x) \qquad \textit{for all } x \in (\mathbb{Q}_{>0})^n.$$

Proof. Fix $x \in (\mathbb{Q}_{>0})^n$. For every prime q, Proposition 10.2 gives

$$m_q(F(x)) = F^{(q)}(\pi_q(x)), \qquad m_q(G(x)) = G^{(q)}(\pi_q(x)).$$

By hypothesis,

$$F^{(q)}(\pi_q(x)) = G^{(q)}(\pi_q(x)),$$

hence

$$m_q(F(x)) = m_q(G(x)) \qquad \text{for every prime } q.$$

Therefore

$$\Phi(F(x)) = \Phi(G(x)).$$

Since Φ is injective, we obtain

$$F(x) = G(x).$$

As x was arbitrary, $F = G$ as functions on $(\mathbb{Q}_{>0})^n$. □

Remark 12.6. This is one of the basic reconstruction principles of the theory: the full family of prime shadows determines the internal Pratt tropical polynomial function. Thus the internal object is not a vague common source of many shadows; it can be recovered from them.

12.5 Reconstruction of hypersurfaces from shadows

We now turn from functions to hypersurfaces. Given a Pratt tropical polynomial F, its internal hypersurface $V_P(F)$ was defined as the set of points where at least two monomials are simultaneously dominant. For each prime q, the shadow $V_P(F)^{(q)}$ is the ordinary tropical hypersurface attached to $F^{(q)}$.

The natural question is whether the internal hypersurface itself can be reconstructed from all its shadows.

At the level of sets, a first tautological observation is immediate.

Proposition 12.7 (pointwise reconstruction of hypersurface membership)**.** *Let F be a Pratt tropical polynomial and let $x \in (\mathbb{Q}_{>0})^n$. Then the following are equivalent:*

1. *$x \in V_P(F)$;*
2. *for every prime q, the shadow point $\pi_q(x)$ lies in the corner-support of the shadow polynomial $F^{(q)}$ corresponding to the dominant monomials at x;*
3. *the family of shadow values $\left(F^{(q)}(\pi_q(x))\right)_q$ is realized by at least two global monomials of F that agree at x.*

Proof. This is essentially a restatement of the definitions. The key point is that a global equality

$$c_\alpha x^\alpha = c_\beta x^\beta$$

in $\mathbb{Q}_{>0}$ is equivalent, by injectivity of Φ, to the simultaneous equalities

$$m_q(c_\alpha x^\alpha) = m_q(c_\beta x^\beta) \qquad \text{for all primes } q.$$

Thus the internal tie condition is exactly the total family of shadow tie conditions. □

This proposition shows that the internal corner condition is equivalent to the simultaneous shadow corner condition. However, it is important not to overstate the result: the shadow hypersurfaces $V_P(F)^{(q)}$ live in different lattices $\mathbb{Z}^n$, and there is no single obvious ambient space in which one can literally intersect them. The internal object $V_P(F)$ is the arithmetic locus whose images under all π_q satisfy the corresponding tropical corner conditions.

12.6 The reconstruction problem for general cycles

For hypersurfaces coming directly from a Pratt tropical polynomial, the reconstruction principle is therefore clear: the internal function F is determined by its shadows, and $V_P(F)$ is determined by F.

For general internal cycles, however, the situation is subtler. One may start with a family of ordinary tropical cycles

$$X^{(2)}, X^{(3)}, X^{(5)}, \dots$$

and ask whether there exists an internal Pratt object $X \subseteq (\mathbb{Q}_{>0})^n$ whose q-shadows are exactly these given cycles.

This leads to the first genuinely new global problem in the theory.

Definition 12.8 (shadow realizability problem)**.** Given a family of ordinary tropical subsets

$$X^{(q)} \subseteq \mathbb{Z}^n \qquad (q \in \mathbb{P}),$$

determine whether there exists an internal subset

$$X \subseteq (\mathbb{Q}_{>0})^n$$

such that

$$\pi_q(X) = X^{(q)} \qquad \text{for every prime } q.$$

Definition 12.9 (shadow compatibility)**.** A family $\{X^{(q)}\}_{q\in\mathbb{P}}$ of ordinary tropical subsets of $\mathbb{Z}^n$ is called *Pratt-compatible* if it is realized as the full shadow family of some internal subset $X \subseteq (\mathbb{Q}_{>0})^n$.

The problem is new because the shadows cannot vary independently: they must all arise from a single arithmetic set of points, hence from points whose prime-shadow coordinates come from genuine elements of $(\mathbb{Q}_{>0})^n$.

Remark 12.10. In ordinary tropical geometry one usually starts with one polyhedral object in one lattice. In the Pratt setting, one starts with infinitely many lattices, one for each prime shadow, and asks whether they glue back to one arithmetic object. This gluing problem seems to have no direct classical analogue.

12.7 A necessary compatibility condition

A trivial but useful necessary condition comes from point reconstruction.

Suppose $X \subseteq (\mathbb{Q}_{>0})^n$ is internal. Then for every choice of shadow points

$$u^{(q)} \in X^{(q)} \subseteq \mathbb{Z}^n,$$

the family $\{u^{(q)}\}_q$ can arise from a single point $x \in X$ only if, for each component, the sequence of coordinates defines an actual element of $\mathbb{Q}_{>0}$.

More concretely, writing

$$u^{(q)} = (u_1^{(q)}, \dots, u_n^{(q)}),$$

one must require that for each i, the finitely supported sequence

$$\left(u_i^{(q)}\right)_{q\in\mathbb{P}}$$

lies in the image of the Pratt coordinate map $\Phi(\mathbb{Q}_{>0}) \subseteq \mathbb{Z}^{(\mathbb{P})}$.

Thus one obtains the following necessary condition.

Proposition 12.11 (pointwise realizability condition)**.** *Let $\{X^{(q)}\}_{q\in\mathbb{P}}$ be a Pratt-compatible family. Then every point $x \in X$ determines a coherent family*

$$\left(\pi_q(x)\right)_{q\in\mathbb{P}},$$

and for each coordinate i, the sequence

$$\left(m_q(x_i)\right)_{q\in\mathbb{P}}$$

must lie in the image of $\Phi(\mathbb{Q}_{>0})$.

Proof. This is immediate from the definitions. □

This may sound tautological, but it is actually a meaningful restriction: the image $\Phi(\mathbb{Q}_{>0})$ is a proper subgroup of $\mathbb{Z}^{(\mathbb{P})}$ expressed in a nontrivial basis, and for $\mathbb{N}$ the image is even more restrictive.

12.8 Finite determination questions

Since each point of $(\mathbb{Q}_{>0})^n$ is determined by all its prime shadows, the next natural question is whether one can reconstruct from only finitely many shadows.

Question 1. *Let F be a Pratt tropical polynomial. Does there exist a finite set of primes S such that the shadows*

$$\{F^{(q)} : q \in S\}$$

already determine F? More generally, does a finite family of shadows determine $V_P(F)$?

At the coefficient level, such a statement cannot hold in complete generality without some size bounds, because the coefficient $c_\alpha \in \mathbb{Q}_{>0}$ is determined by the full infinite vector

$$\Phi(c_\alpha) = (m_q(c_\alpha))_q.$$

However, for a *fixed* finite collection of coefficients, only finitely many prime coordinates are nonzero. The subtlety is that for reconstruction by inverting a finite truncation of the matrix

$$A = (m_p(q))_{p,q\in\mathbb{P}}$$

one cannot in general work merely with the visible supports of the coefficient vectors. The reason is that cancellations may remove some shadow coordinates even though the corresponding ordinary prime exponents are still present. Thus one needs a finite prime set which is closed under Pratt ancestry.

Definition 12.12 (Pratt-closed finite prime set)**.** A finite set of primes $S \subseteq \mathbb{P}$ is called *Pratt-closed* if for every $q \in S$, every prime appearing in the Pratt tree of q also lies in S. Equivalently,

$$q \in S \implies \operatorname{supp} \Phi(q) \subseteq S.$$

Lemma 12.13 (finite reconstruction from a Pratt-closed truncation)**.** *Let $v \in \mathbb{Z}^{(\mathbb{P})}$ be a finitely supported Pratt vector. Let $S \subseteq \mathbb{P}$ be any finite Pratt-closed set containing* $\operatorname{supp}(v)$. *Then the finite truncation*

$$A_S = (m_p(q))_{p,q\in S}$$

is unitriangular and invertible, and the vector

$$A_S^{-1} v_S$$

recovers the ordinary prime-exponent vector on S. In particular, it determines the unique rational number $x \in \mathbb{Q}_{>0}$ with $\Phi(x) = v$.

Proof. Because S is Pratt-closed, every column indexed by $q \in S$ has support contained in S. Hence the restriction to S is compatible with the triangular structure of A, and A_S is again lower unitriangular. Therefore A_S is invertible over $\mathbb{Z}$. If $v = \Phi(x)$, then by the global identity $\Phi(x) = A\,v(x)$, restricting to the rows indexed by S gives

$$v_S = A_S\,v(x)_S.$$

Applying A_S^{-1} yields the claimed reconstruction of the ordinary prime-exponent vector on S. Since Φ is injective, this determines x uniquely. □

Remark 12.14. The visible support $\operatorname{supp}\Phi(x)$ need not itself be Pratt-closed. For example, for

$$x = \frac{6}{5},$$

one has

$$\Phi\left(\frac{6}{5}\right) = \Phi(6) - \Phi(5),$$

and the visible support is $\{3, 5\}$, whereas correct finite reconstruction requires the Pratt-closed set $\{2, 3, 5\}$. So one must close the visible support downward along Pratt ancestry before inverting a finite truncation.

Proposition 12.15 (finite determination for a fixed polynomial)**.** *Let*

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha$$

be a Pratt tropical polynomial. Then there exists a finite Pratt-closed set of primes S such that the family of shadows

$$\{F^{(q)} : q \in S\}$$

determines F completely.

Proof. Since A is finite and each coefficient $c_\alpha \in \mathbb{Q}_{>0}$ has finitely supported Pratt vector, the union

$$S_0 := \bigcup_{\alpha \in A} \operatorname{supp}\Phi(c_\alpha)$$

is finite. Choose any finite Pratt-closed set S containing S_0; for instance, one may take the closure of S_0 under Pratt ancestry. For $q \notin S$, one has $m_q(c_\alpha) = 0$ for every α, so the q-shadow contains no new coefficient information. The slopes α are already visible from any single shadow, since they are the linear parts. Finally, because S is Pratt-closed, Lemma 12.13 shows that the finite family of coefficient shadows indexed by S suffices to reconstruct each coefficient c_α. Hence the finite family $\{F^{(q)} : q \in S\}$ determines F completely. □

Remark 12.16. So although the theory is naturally indexed by all primes, every individual Pratt tropical polynomial is controlled by only finitely many nontrivial prime shadows. The correct finite control set is not merely the visible union of coefficient supports, but any finite Pratt-closed set containing that union. The infinite family of shadows becomes essential only when one studies the geometry in full generality, or when one varies the polynomial.

12.9 Finite shadow windows and local reconstruction sets

The finite-determination statement above identifies the correct *window* in which one may reconstruct coefficient data: one should work over a finite Pratt-closed set of primes rather than over an arbitrary visible support. This admits a more local formulation which is useful both conceptually and computationally.

Fix a finite exponent set $A \subset \mathbb{Z}^n$ and a finite Pratt-closed set of primes $S \subseteq \mathbb{P}$. Suppose that for each $\alpha \in A$ and each $p \in S$ one is given a prescribed shadow coefficient

$$b_{\alpha,p} \in \mathbb{Z}_{\geq 0}.$$

Equivalently, for each monomial one has a finite shadow vector

$$b_\alpha^S := (b_{\alpha,p})_{p \in S} \in \mathbb{Z}_{\geq 0}^S.$$

The question is whether these finitely many local shadow coefficients come from genuine global coefficients $c_\alpha \in \mathbb{N}$.

For each $\alpha \in A$ and $p \in S$, define the fiber

$$C_{\alpha,p}(b_{\alpha,p}) := \{n \in \mathbb{N} : m_p(n) = b_{\alpha,p}\}.$$

The associated *local reconstruction set* is

$$C_\alpha(S) := \bigcap_{p \in S} C_{\alpha,p}(b_{\alpha,p}) = \{n \in \mathbb{N} : m_p(n) = b_{\alpha,p} \text{ for all } p \in S\}.$$

Thus $C_\alpha(S)$ is exactly the set of natural numbers whose Pratt coordinates, restricted to S, match the prescribed local shadow vector b_α^S.

The point is that the nonemptiness of this intersection is not merely an existence statement. By the integrality criterion, it is equivalent to an explicit finite system of linear inequalities. This gives a clean finite-window reconstruction criterion.

Proposition 12.17 (finite-window reconstruction criterion)**.** *Let $A \subset \mathbb{Z}^n$ be finite, and let $S \subseteq \mathbb{P}$ be a finite Pratt-closed set of primes. For each $\alpha \in A$ and $p \in S$, let $b_{\alpha,p} \in \mathbb{Z}_{\geq 0}$ be prescribed. Then the following are equivalent.*

(i) There exist coefficients $c_\alpha \in \mathbb{N}$ such that

$$m_p(c_\alpha) = b_{\alpha,p} \qquad (\alpha \in A,\ p \in S).$$

(ii) For every $\alpha \in A$, the local reconstruction set

$$C_\alpha(S) = \{n \in \mathbb{N} : m_p(n) = b_{\alpha,p} \text{ for all } p \in S\}$$

is nonempty.

(iii) For every $\alpha \in A$, the vector $b_\alpha^S = (b_{\alpha,p})_{p \in S}$ satisfies the truncated integrality conditions

$$b_{\alpha,q} \geq \sum_{\substack{r \in S \\ r > q}} b_{\alpha,r}\, v_q(r-1) \qquad (q \in S).$$

Proof. The equivalence of (i) and (ii) is immediate from the definition of $C_\alpha(S)$: condition (i) says exactly that, for each α, there exists a natural number c_α whose Pratt coordinates on S are the prescribed values $b_{\alpha,p}$, and this is precisely the statement that $C_\alpha(S) \neq \varnothing$.

Next assume (i). For each fixed α, the coefficient $c_\alpha \in \mathbb{N}$ satisfies the global integrality inequalities of Proposition 6.1. Restricting those inequalities to primes $q \in S$ and discarding the nonnegative terms with $r \notin S$ gives

$$m_q(c_\alpha) \geq \sum_{\substack{r \in S \\ r > q}} m_r(c_\alpha)\, v_q(r-1).$$

Since $m_p(c_\alpha) = b_{\alpha,p}$ for $p \in S$, this is exactly condition (iii).

Finally assume (iii). Fix α and define

$$c_\alpha^S := \prod_{p \in S} \left(1 - \frac{1}{p}\right)^{-b_{\alpha,p}}.$$

Because S is Pratt-closed, the truncated product involves only primes in S, and the exponent of a given $q \in S$ in c_α^S is

$$b_{\alpha,q} - \sum_{\substack{r \in S \\ r > q}} b_{\alpha,r}\, v_q(r-1),$$

which is nonnegative by (iii). Hence $c_\alpha^S \in \mathbb{N}$. Moreover, in the finite Pratt-closed coordinate system on S, the product formula and the unitriangularity of $A_S = (m_p(r))_{p,r \in S}$ show that the Pratt coordinates of c_α^S on S are exactly the prescribed values $b_{\alpha,p}$. Therefore $c_\alpha^S \in C_\alpha(S)$, so $C_\alpha(S) \neq \varnothing$ and condition (ii) holds. This proves the equivalence of (i), (ii), and (iii). □

The proposition shows that the primary structure is not the intersection description by itself, but the fact that nonemptiness of the local reconstruction set is equivalent to a finite family of linear inequalities. In that sense, the sets $C_\alpha(S)$ are the set-theoretic avatar of truncated integrality.

For a nested sequence of finite Pratt-closed prime sets

$$S_1 \subseteq S_2 \subseteq \cdots,$$

one obtains a descending family of local reconstruction sets

$$C_\alpha(S_1) \supseteq C_\alpha(S_2) \supseteq \cdots .$$

As long as these sets remain nonempty, the shadow data remain compatible with a global coefficient. If, in addition, the sets eventually stabilize to a single value, then that value is the reconstructed coefficient. This gives a precise finite-window version of the reconstruction problem and explains why finite shadow windows are the natural computational objects.

12.10 Summary

The results of this section can be summarized as follows.

- Individual points of $(\mathbb{Q}_{>0})^n$ are uniquely determined by their full family of prime shadows.
- Pratt monomials are uniquely determined by their shadows.
- Pratt tropical polynomial functions are uniquely determined by their shadows.
- Internal hypersurface membership is equivalent to simultaneous shadow corner conditions.
- For general internal cycles, one obtains a new shadow-realizability problem: which families of ordinary tropical shadows come from one arithmetic object?

Thus reconstruction is complete for the basic algebraic objects of the theory, but already at the level of general cycles it leads to a genuinely new global question. This reconstruction problem appears to be one of the main distinctive features of Pratt-internal tropical geometry.

13 Decidability of prime-shadow solution sets and finite-window reconstruction

Let

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha$$

be a Pratt tropical polynomial on $(\mathbb{Q}_{>0})^n$, where $A \subset \mathbb{Z}^n$ is finite and all coefficients c_α are positive. For each prime p, its p-shadow is the ordinary tropical polynomial

$$F^{(p)}(u) = \max_{\alpha \in A}\Big(m_p(c_\alpha) + \alpha \cdot u\Big), \qquad u \in \mathbb{Z}^n.$$

By the basic shadow formula, every prime shadow is therefore a finite max of integral affine-linear forms.

For a fixed prime p, define the shadow solution set

$$\Sigma_p(F) := \left\{ u \in \mathbb{Z}^n : \text{the maximum in } F^{(p)}(u) \text{ is attained by at least two monomials} \right\}.$$

Equivalently,

$$\Sigma_p(F) = \bigcup_{\substack{\alpha,\beta \in A \\ \alpha \neq \beta}} \Sigma_p(F; \alpha, \beta),$$

where $\Sigma_p(F; \alpha, \beta)$ consists of those $u \in \mathbb{Z}^n$ such that

$$m_p(c_\alpha) + \alpha \cdot u = m_p(c_\beta) + \beta \cdot u$$

and

$$m_p(c_\gamma) + \gamma \cdot u \leq m_p(c_\alpha) + \alpha \cdot u \qquad (\gamma \in A).$$

Thus each piece $\Sigma_p(F; \alpha, \beta)$ is the set of integer points of a rational polyhedron cut out by one linear equality and finitely many linear inequalities.

Proposition 13.1 (Decidability of the p-shadow solution set)**.** *For every fixed prime p and every Pratt tropical polynomial F with positive coefficients, the set $\Sigma_p(F)$ is algorithmically decidable and falls into exactly one of the three cases:*

$$\Sigma_p(F) = \varnothing, \qquad 0 < |\Sigma_p(F)| < \infty, \qquad |\Sigma_p(F)| = \infty.$$

More precisely, this classification is determined by a finite family of linear feasibility and recession tests, and hence can be implemented by any exact linear or mixed-integer linear optimization backend.

Proof. Because A is finite, there are only finitely many ordered pairs (α, β) with $\alpha \neq \beta$. For each such pair, $\Sigma_p(F; \alpha, \beta)$ is the set of integer points in a rational polyhedron

$$P_{\alpha,\beta} = \{ u \in \mathbb{R}^n : (\alpha - \beta) \cdot u = m_p(c_\beta) - m_p(c_\alpha),\ (\alpha - \gamma) \cdot u \geq m_p(c_\gamma) - m_p(c_\alpha)\ \forall \gamma \in A \}.$$

Hence:

1. $\Sigma_p(F; \alpha, \beta) = \varnothing$ iff the integer feasibility problem for $P_{\alpha,\beta}$ is infeasible;
2. $\Sigma_p(F; \alpha, \beta)$ is infinite iff the recession cone of $P_{\alpha,\beta}$ contains a nonzero integral vector;
3. otherwise $\Sigma_p(F; \alpha, \beta)$ is finite.

Each of these assertions is decidable by standard linear or mixed-integer linear methods. Since $\Sigma_p(F)$ is the finite union of the sets $\Sigma_p(F; \alpha, \beta)$, the same trichotomy and the same decidability hold for $\Sigma_p(F)$ itself. □

Next fix a nested sequence

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$$

of finite Pratt-closed sets of primes, defined recursively by

$$S_{i+1} := \mathrm{PrattClosure}\Big(S_i \cup \{p_i\}\Big), \qquad p_i = \min(\mathbb{P} \setminus S_i).$$

Because every prime occurring in the Pratt ancestry of a prime q is strictly smaller than q, the Pratt closure of a finite set of primes is again finite. Hence every S_i is finite.

For each monomial $\alpha \in A$, write

$$b_\alpha^{(i)} = (m_q(c_\alpha))_{q \in S_i} \in \mathbb{Z}_{\geq 0}^{S_i}.$$

Define the local reconstruction set

$$C_\alpha(S_i) := \{n \in \mathbb{N}_{>0} : m_q(n) = m_q(c_\alpha)\ \forall q \in S_i\}.$$

By the finite-window reconstruction criterion, this set is nonempty if and only if the truncated integrality inequalities hold:

$$m_q(c_\alpha) \geq \sum_{\substack{r \in S_i \\ r > q}} m_r(c_\alpha)\, v_q(r-1) \qquad (q \in S_i).$$

Whenever these inequalities hold, the unique positive integer with these truncated Pratt coordinates is

$$c_\alpha^{(i)} = \prod_{q \in S_i} \left(1 - \frac{1}{q}\right)^{-m_q(c_\alpha)}.$$

Accordingly define the reconstructed polynomial on the window S_i by

$$F_{S_i}(x) := \bigvee_{\alpha \in A} c_\alpha^{(i)} x^\alpha.$$

Theorem 13.2 (Termination and correctness of the iterative reconstruction for a fixed polynomial)**.** *Let*

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha$$

be a Pratt tropical polynomial with positive coefficients and finite support A. Then the iterative reconstruction on the windows S_i terminates after finitely many steps in the following strong sense.

1. *There exists a finite Pratt-closed set of primes*

$$S^* \supseteq \bigcup_{\alpha \in A} \operatorname{supp} \Phi(c_\alpha)$$

such that for every i with $S^* \subseteq S_i$, one has

$$F_{S_i} = F.$$

2. *In particular, there exists an index i_0 such that for all $i \geq i_0$ the reconstruction stabilizes:*

$$F_{S_i} = F_{S_{i_0}} = F.$$

3. *For every i, all local reconstruction sets $C_\alpha(S_i)$ are nonempty. Hence the procedure never encounters an empty local reconstruction set when the input data come from an actual global polynomial.*

Proof. Each coefficient $c_\alpha \in \mathbb{Q}_{>0}$ has finitely supported Pratt vector, so the finite union

$$S_0' := \bigcup_{\alpha \in A} \operatorname{supp} \Phi(c_\alpha)$$

is finite. Choose any finite Pratt-closed set S^* containing S_0', for example the Pratt closure of S_0'. Since the recursive sequence (S_i) eventually contains every prime in every fixed finite set, there exists i_0 with $S^* \subseteq S_{i_0}$.

Now fix $\alpha \in A$. For every $q \notin S^*$ one has $m_q(c_\alpha) = 0$, by construction of S^*. Hence for every $i \geq i_0$ the truncated vector on S_i is just the full Pratt vector of c_α extended by zeros outside its support. Therefore the finite reconstruction formula recovers exactly c_α:

$$c_\alpha^{(i)} = \prod_{q \in S_i} \left(1 - \frac{1}{q}\right)^{-m_q(c_\alpha)} = \prod_{q \in \mathbb{P}} \left(1 - \frac{1}{q}\right)^{-m_q(c_\alpha)} = c_\alpha.$$

Since this holds for every $\alpha \in A$, one obtains $F_{S_i} = F$ for all $i \geq i_0$.

The nonemptiness of every $C_\alpha(S_i)$ is immediate, because c_α itself belongs to this set. Thus the iterative reconstruction does not fail when the local data arise from an actual global polynomial. □

The previous theorem proves finite termination for reconstruction *from a fixed global polynomial*. For arbitrary prescribed local shadow data, one obtains the following finite-window alternative.

Proposition 13.3 (Finite-window alternative for prescribed local shadow data)**.** *Fix a finite support $A \subset \mathbb{Z}^n$ and, for each window S_i, prescribe vectors*

$$b_\alpha^{(i)} \in \mathbb{Z}_{\geq 0}^{S_i} \qquad (\alpha \in A)$$

that are compatible under restriction from S_{i+1} to S_i. Then exactly one of the following occurs.

1. *There exists an index i and an exponent $\alpha \in A$ such that the truncated integrality conditions fail for $b_\alpha^{(i)}$. Equivalently,*

$$C_\alpha(S_i) = \varnothing,$$

and no global polynomial with these local shadows can exist.

2. *The truncated integrality conditions hold for every $\alpha \in A$ and every window S_i. Then each finite window admits a unique local reconstructed polynomial*

$$F_{S_i}(x) = \bigvee_{\alpha \in A} c_\alpha^{(i)} x^\alpha.$$

If, moreover, the prescribed family is induced by some global polynomial with positive coefficients, then the sequence (F_{S_i}) stabilizes after finitely many steps at that global polynomial.

Proof. For each fixed α and i, the finite-window criterion gives the equivalence

$$C_\alpha(S_i) \neq \varnothing \quad \Longleftrightarrow \quad b_\alpha^{(i)} \text{ satisfies the truncated integrality inequalities on } S_i.$$

Hence the first alternative is exactly the appearance of an empty local reconstruction set. If this never happens, then each coefficient admits a unique local lift on each finite window, hence so does the whole polynomial. If the data come from a genuine global polynomial, the previous theorem applies and gives finite stabilization. □

Corollary 13.4 (Uniqueness of the reconstructed global polynomial)**.** *Let*

$$F(x) = \bigvee_{\alpha \in A} c_\alpha x^\alpha, \qquad G(x) = \bigvee_{\alpha \in A} d_\alpha x^\alpha$$

be Pratt tropical polynomials with positive coefficients and the same support A. Assume that for every prime p their prime shadows coincide:

$$F^{(p)} = G^{(p)} \qquad (p \in \mathbb{P}).$$

Then

$$F = G.$$

Equivalently, once the iterative procedure reaches a finite Pratt-closed window containing all coefficient supports, the reconstructed global polynomial is unique.

Proof. From any one shadow one reads off the slope set A, since the linear parts are exactly the exponents. Fix $\alpha \in A$. Equality of all prime shadows implies

$$m_p(c_\alpha) = m_p(d_\alpha) \qquad (p \in \mathbb{P}).$$

Hence the full Pratt vectors of c_α and d_α agree. By the Pratt product formula, a positive coefficient is uniquely determined by its full Pratt vector, so $c_\alpha = d_\alpha$. Since this holds for every $\alpha \in A$, one concludes that $F = G$.

For the final statement, once the iterative procedure reaches a finite Pratt-closed control set containing all coefficient supports, the reconstructed coefficient vectors are already the full Pratt vectors of the coefficients. Therefore the resulting polynomial is the unique global polynomial compatible with the full family of shadows. □

In particular, for Pratt tropical polynomials with positive coefficients and fixed support, one has the following chain of consequences:

prime-shadow solution sets are decidable $\Longrightarrow$ finite-window reconstruction is effective $\Longrightarrow$ the stabilized g

The uniqueness statement, however, rests on the coefficient reconstruction theorem from full Pratt vectors together with finite determination, and not on the shadow-variety classification alone.

14 Relation with standard tropical geometry

The geometric framework in the book of Mikhalkin and Rau is built from integral affine spaces, tropical polynomials as max of affine-linear functions, and tropical varieties as balanced weighted polyhedral complexes. The present paper does not attempt to reproduce that full theory. Rather, it identifies an arithmetic object on which the same algebra is already present internally and develops the foundational reconstruction and decidability statements needed for further geometric work.

The translation dictionary may be summarized as follows:

$$\text{ordinary addition in tropical coordinates} \;\longleftrightarrow\; \text{multiplication in } \mathbb{Q}_{>0},$$

$$\max \longleftrightarrow \vee_P, \qquad \min \longleftrightarrow \wedge_P,$$

$$\text{integral affine form } a + \alpha \cdot X \longleftrightarrow \text{Pratt monomial } cx^{\alpha},$$

$$\text{tropical polynomial } \max_{\alpha}(a_\alpha + \alpha \cdot X) \longleftrightarrow \bigvee_{\alpha} c_\alpha x^{\alpha}.$$

From this point of view, the matrix

$$A = (m_p(q))_{p,q \in \mathbb{P}}$$

plays the role of a tropicalization matrix: it converts the classical prime-exponent coordinates

$$v(n) = (v_q(n))_q$$

into the Pratt coordinates

$$\Phi_P(n) = A\, v(n).$$

Thus the usual multiplicative geometry of prime exponents is transported into a new integral affine structure governed by Pratt trees.

A full development of this viewpoint would require the systematic translation of standard constructions from tropical geometry: balancing, stable intersections, tropical divisors, abstract tropical curves, Jacobians, and Riemann–Roch. The present paper stops earlier: it establishes the algebraic and arithmetic foundations, proves reconstruction from shadows for the basic algebraic objects, and records the finite-window and decidability statements that make the theory computationally accessible.

A Classical multivariable equations viewed in the Pratt tropical world

In this section we illustrate how familiar polynomial equations in several variables look after passing to the Pratt tropical setting. The point is not to reproduce their classical algebraic geometry over $\mathbb{C}$, but to understand the corresponding Pratt tropical hypersurfaces over $\mathbb{Q}_{>0}$.

Recall that for a Pratt tropical polynomial

$$F(x) = \bigvee_{\alpha} c_\alpha \odot x^\alpha,$$

its q-shadow is

$$F^{(q)}(u) = \max_{\alpha}\Big(m_q(c_\alpha) + \alpha \cdot u\Big),$$

and the Pratt tropical hypersurface consists of those points where at least two monomials are simultaneously dominant. Thus every example below is governed by primewise linear equalities and inequalities.

A.1 The affine line $x + y = 1$

The natural Pratt tropical counterpart is

$$F(x, y) = 1 \vee_P x \vee_P y.$$

For each prime p, writing

$$u = m_p(x), \qquad v = m_p(y),$$

we obtain the shadow polynomial

$$F^{(p)}(u, v) = \max\{0, u, v\}.$$

Hence the shadow hypersurface is the standard max-plus tropical line with vertex $(0, 0)$. Its three components are:

$$0 = u \geq v, \qquad 0 = v \geq u, \qquad u = v \geq 0.$$

Globally this yields three families of Pratt tropical solutions:

$$x = 1,\ m_p(y) \leq 0\ \forall p, \qquad y = 1,\ m_p(x) \leq 0\ \forall p, \qquad m_p(x) = m_p(y) \geq 0\ \forall p.$$

Equivalently,

$$\mathrm{TropHyp}(1 \vee_P x \vee_P y) = \{(1, t^{-1}) : t \in \mathbb{N}\} \ \cup \ \{(s^{-1}, 1) : s \in \mathbb{N}\} \ \cup \ \{(n, n) : n \in \mathbb{N}\}.$$

So the classical affine line becomes, in the Pratt tropical world, an arithmetic three-ray skeleton.

A.2 The hyperbola $xy = 1$

The corresponding Pratt tropical polynomial is

$$F(x, y) = 1 \vee_P (xy).$$

For each prime p the shadow is

$$F^{(p)}(u, v) = \max\{0, u + v\}.$$

Therefore the hypersurface condition is simply

$$u + v = 0.$$

Since this must hold for every prime,

$$m_p(x) + m_p(y) = 0 \qquad \forall p,$$

or equivalently

$$\Phi_P(y) = -\Phi_P(x).$$

Hence

$$\mathrm{TropHyp}(1 \vee_P xy) = \{(x, x^{-1}) : x \in \mathbb{Q}_{>0}\}.$$

Thus the classical hyperbola survives exactly as the multiplicative inverse curve.

A.3 The parabola $y = x^2$

The Pratt tropical analogue is

$$F(x, y) = y \vee_P x^2.$$

Primewise this becomes

$$F^{(p)}(u, v) = \max\{v, 2u\}.$$

The hypersurface condition is therefore

$$v = 2u.$$

Equivalently,

$$m_p(y) = 2m_p(x) \qquad \forall p.$$

Hence

$$\Phi_P(y) = 2\Phi_P(x),$$

so over $\mathbb{Q}_{>0}$ we obtain exactly

$$\mathrm{TropHyp}(y \vee_P x^2) = \{(x, x^2) : x \in \mathbb{Q}_{>0}\}.$$

This is a basic illustration of the fact that binomial monomial relations become exact linear relations in Pratt coordinates.

A.4 The elliptic-curve motif $y^2 = x^3 - x + 1$

A genuine elliptic curve involves subtraction and cancellation, so it does not pass directly into the positive Pratt tropical framework. A natural positive support replacement is

$$F(x, y) = 1 \vee_P x \vee_P x^3 \vee_P y^2.$$

For a prime p, with $u = m_p(x)$ and $v = m_p(y)$, the shadow is

$$F^{(p)}(u, v) = \max\{0, u, 3u, 2v\}.$$

The corresponding hypersurface is obtained by examining which pairs of affine forms can tie at the maximum. This yields several distinguished pieces.

First piece: 1 and y^2 tie. We require

$$0 = 2v \geq u,\ 3u.$$

Thus

$$v = 0, \qquad u \leq 0.$$

Globally this gives

$$y = 1, \qquad x^{-1} \in \mathbb{N}.$$

Second piece: 1 and x tie. We require

$$0 = u \geq 3u,\ 2v.$$

Thus

$$u = 0, \qquad v \leq 0.$$

Globally this gives

$$x = 1, \qquad y^{-1} \in \mathbb{N}.$$

Third piece: x^3 and y^2 tie. We require

$$3u = 2v \geq 0.$$

Hence

$$3m_p(x) = 2m_p(y) \qquad \forall p,$$

with both sides nonnegative. This implies the parametrization

$$x = t^2, \qquad y = t^3, \qquad t \in \mathbb{N}.$$

So the hypersurface contains the arithmetic monomial curve

$$\{(t^2, t^3) : t \in \mathbb{N}\}.$$

Thus the classical elliptic geometry is not preserved as such, but its positive-support Pratt tropical shadow contains explicit arithmetic skeleta.

A.5 The Fermat-type surface $x^n + y^n = 1$

Its Pratt tropical counterpart is

$$F(x, y) = 1 \vee_P x^n \vee_P y^n.$$

For each prime p,

$$F^{(p)}(u, v) = \max\{0, nu, nv\}.$$

The three shadow components are therefore

$$0 = nu \geq nv, \qquad 0 = nv \geq nu, \qquad nu = nv \geq 0.$$

Since $nu = 0$ is equivalent to $u = 0$, and $nu = nv$ is equivalent to $u = v$, the global Pratt tropical hypersurface is

$$\mathrm{TropHyp}(1 \vee_P x^n \vee_P y^n) = \{(1, t^{-1}) : t \in \mathbb{N}\} \;\cup\; \{(s^{-1}, 1) : s \in \mathbb{N}\} \;\cup\; \{(m, m) : m \in \mathbb{N}\}.$$

So from the Pratt tropical viewpoint, the support pattern is more visible than the precise exponent n.

A.6 The determinant-type relation $xy = z^2$

Consider

$$F(x, y, z) = xy \vee_P z^2.$$

For each prime p, writing

$$u = m_p(x), \qquad v = m_p(y), \qquad w = m_p(z),$$

the shadow hypersurface is given by

$$u + v = 2w.$$

Thus globally

$$m_p(x) + m_p(y) = 2m_p(z) \qquad \forall p.$$

Equivalently,

$$\Phi_P(x) + \Phi_P(y) = 2\Phi_P(z).$$

A large explicit solution family is

$$(x, y, z) = (a^2, b^2, ab), \qquad a, b \in \mathbb{Q}_{>0},$$

since then

$$\Phi_P(a^2) + \Phi_P(b^2) = 2\Phi_P(ab).$$

Hence

$$\{(a^2, b^2, ab) : a, b \in \mathbb{Q}_{>0}\} \subseteq \mathrm{TropHyp}(xy \vee_P z^2).$$

This example shows clearly that quadratic compatibility becomes a parity condition on the sum of Pratt vectors.

A.7 The cuspidal relation $y^2 = x^3$

A particularly natural multivariable example in the Pratt tropical setting is

$$F(x, y) = y^2 \vee_P x^3.$$

Its shadow equation is simply

$$2v = 3u.$$

Thus globally

$$2\Phi_P(y) = 3\Phi_P(x).$$

Over $\mathbb{Q}_{>0}$ this is equivalent to the parametrization

$$x = t^2, \qquad y = t^3, \qquad t \in \mathbb{Q}_{>0}.$$

Therefore

$$\mathrm{TropHyp}(y^2 \vee_P x^3) = \{(t^2, t^3) : t \in \mathbb{Q}_{>0}\}.$$

This is perhaps the cleanest multivariable model case: a classical singular plane curve becomes an exactly parametrized arithmetic locus in the Pratt tropical world.

A.8 Conclusion

These examples show a consistent pattern. Binomial monomial relations such as

$$xy = 1, \qquad y = x^2, \qquad y^2 = x^3, \qquad xy = z^2$$

remain exactly solvable, because they become linear equations in Pratt coordinates. By contrast, multimonial equations such as

$$x + y = 1, \qquad x^n + y^n = 1, \qquad 1 + x + x^3 + y^2$$

turn into unions of tropical pieces determined by dominance and tie conditions in every prime shadow.

In this sense, the Pratt tropical world does not reproduce the classical algebraic geometry of these equations over $\mathbb{C}$. Instead, it extracts an arithmetic shadow geometry in which exponents control the combinatorics and coefficients are encoded through their recursive Pratt data.

B Selected computational examples from the log files

This appendix records three short examples extracted from the computational logs used to test the reconstruction and decidability statements. The point of including them here is not to replace the proofs in the main text, but to show concretely how the finite-window method behaves on explicit inputs.

B.1 A coefficient that is not determined by the first prime window

In dimension $d = 1$, the log for the constant coefficient 3 shows the simplest nontrivial reconstruction phenomenon. At the first window

$$S_0 = [2],$$

one sees only the truncated Pratt coordinate $m_2(3) = 1$, so the local reconstruction returns the coefficient 2 rather than 3. After enlarging to

$$S_1 = [2, 3],$$

one also sees $m_3(3) = 1$, and the reconstruction becomes exact. In the log this appears as

$$\text{Window } S_0 : [((0,), 2)] \quad \text{(not exact)}, \qquad \text{Window } S_1 : [((0,), 3)] \quad \text{(exact)}.$$

The example is instructive because it isolates the role of finite Pratt-closed windows: a too-small window may produce a locally compatible but globally incorrect coefficient, while the next window already forces the correct one.

B.2 A two-variable shadow with infinite solution set

In dimension $d = 2$, the polynomial

$$1 \vee_P (1 \odot x_2)$$

produces, for each of the shadow primes $2, 3, 5$, an infinite ordinary tropical shadow. The log lists sample shadow solutions

$$(0, 0), (1, 0), (2, 0), \ldots$$

and representative lifts

$$(1, 1), (2, 1), (4, 1), \ldots \quad \text{for } p = 2,$$

with the obvious analogues for $p = 3$ and $p = 5$. This is exactly what the main decidability theorem predicts: the shadow solution set is cut out by linear equalities and inequalities, and in this case the recession data show that the set is infinite. At the same time, the reconstruction windows are already exact from the first step, so the example cleanly separates *shadow infinitude* from *coefficient ambiguity*.

B.3 A three-variable example with immediate stabilization

In dimension $d = 3$, the polynomial

$$1 \vee_P (1 \odot x_3)$$

behaves analogously, but it is useful as a sanity check in higher dimension. Each prime shadow is again classified as infinite, with sample shadow points

$$(0, 0, 0), (0, 1, 0), (0, 2, 0), \ldots,$$

and corresponding lifts such as

$$(1, 1, 1), (1, 2, 1), (1, 4, 1), \ldots \quad \text{for } p = 2.$$

Yet the reconstruction windows remain exact from

$$S_0 = [2]$$

onward. This shows that once the coefficients are already visible in the initial window, increasing the ambient dimension or enlarging the shadow solution set does not by itself obstruct exact coefficient recovery.

Taken together, these three examples illustrate the three computational themes emphasized in the paper: finite windows may need to grow before coefficients stabilize, shadow solution sets may be finite or infinite independently of reconstruction, and the same reconstruction principles remain effective across different ambient dimensions.

References

[1] P. Erdős, A. Granville, C. Pomerance, and C. Spiro, *On the Normal Behavior of the Iterates of Some Arithmetic Functions*, in *Analytic Number Theory (Allerton Park, IL, 1989)*, Progress in Mathematics **85**, Birkhäuser Boston, Boston, MA, 1990, pp. 165–204.

[2] R. A. Cuninghame-Green, *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems, vol. 166, Springer, Berlin, 1979.

[3] I. Simon, Recognizable sets with multiplicities in the tropical semiring, in *Mathematical Foundations of Computer Science 1988*, Lecture Notes in Computer Science, vol. 324, Springer, Berlin, 1988, pp. 107–120.

[4] J.-É. Pin, Tropical semirings, in *Idempotency* (Bristol, 1994), Publications of the Newton Institute, vol. 11, Cambridge University Press, Cambridge, 1998, pp. 50–69.

[5] D. Speyer and B. Sturmfels, Tropical mathematics, *Mathematics Magazine* **82** (2009), no. 3, 163–173.

[6] G. Mikhalkin and J. Rau, *Tropical Geometry*, book manuscript, 2018.

[7] O. Leka, *Exploring Pratt Trees*, available at `https://www.orges-leka.de/exploring_pratt_trees.pdf`.