Uniqueness of L_{88} under the condition $n \equiv 0 \pmod 8$

An elliptic-curve proof for the consecutive-square identity starting at 192

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Abstract

We prove that the Diophantine equation

$$192^2 + 193^2 + \cdots + (192 + n - 1)^2 = t^2$$

has exactly one positive solution with $n \equiv 0 \pmod 8$, namely

$$(n, t) = (88, 2222).$$

Equivalently,

$$192^2 + 193^2 + \cdots + 279^2 = 2222^2$$

is the unique square identity beginning at 192 whose length is divisible by 8. In the notation of the Lorentzian construction, this is exactly the identity that produces the rank-88 lattice L_{88}. The proof reduces the problem to integral points on an elliptic curve and then uses a SageMath computation to enumerate those integral points explicitly. Every algebraic step is written out, and the complete SageMath script and console output are recorded in the appendix.

Contents

1. Statement of the result

We study the equation

$$192^2 + 193^2 + \cdots + (192 + n - 1)^2 = t^2, \tag{1}$$

with integers $n \geq 1$ and $t \geq 0$, under the additional hypothesis

$$n \equiv 0 \pmod{8}. \tag{2}$$

The relevance of the congruence condition (2) is immediate in the lattice-theoretic application: the Lorentzian construction produces an even unimodular positive-definite lattice only in ranks divisible by 8. Thus the natural rank parameter is exactly the number of consecutive squares, i.e. the length n.

Theorem 1. *The only positive integer solution of* (1) *subject to* (2) *is*

$$(n, t) = (88, 2222).$$

Equivalently, the only identity of the form

$$192^2 + 193^2 + \cdots + B^2 = T^2$$

with $B \geq 192$ *and*

$$B - 192 + 1 \equiv 0 \pmod{8}$$

is

$$192^2 + 193^2 + \cdots + 279^2 = 2222^2.$$

Remark 1. Without the side condition $n \equiv 0 \pmod{8}$, there are additional solutions. In particular, the elliptic-curve computation below also produces the solution $n = 2651$, corresponding to

$$192^2 + 193^2 + \cdots + 2842^2 = 87483^2.$$

So the divisibility condition by 8 is not cosmetic; it is essential for the uniqueness statement relevant to L_{88}.

2. Rewriting the sum of consecutive squares

We begin by expanding the left-hand side of (1) in closed form. Let

$$S(n) = \sum_{j=0}^{n-1} (192 + j)^2.$$

Then

$$S(n) = \sum_{j=0}^{n-1} (192^2 + 384j + j^2) = 192^2 n + 384 \sum_{j=0}^{n-1} j + \sum_{j=0}^{n-1} j^2.$$

Using

$$\sum_{j=0}^{n-1} j = \frac{n(n-1)}{2}, \qquad \sum_{j=0}^{n-1} j^2 = \frac{(n-1)n(2n-1)}{6},$$

we obtain

$$\begin{aligned} S(n) &= 36864n + 384 \cdot \frac{n(n-1)}{2} + \frac{(n-1)n(2n-1)}{6} \\ &= 36864n + 192n(n-1) + \frac{(n-1)n(2n-1)}{6}. \end{aligned}$$

Expanding and simplifying gives

$$S(n) = \frac{n^3}{3} + \frac{383}{2}n^2 + \frac{220033}{6}n = \frac{n(2n^2 + 1149n + 220033)}{6}. \tag{3}$$

Hence (1) is equivalent to

$$\frac{n^3}{3} + \frac{383}{2}n^2 + \frac{220033}{6}n = t^2. \tag{4}$$

Multiplying by 6 yields

$$2n^3 + 1149n^2 + 220033n = 6t^2. \tag{5}$$

3. Reduction to an elliptic curve

At this point we invoke the standard substitution used by Bremner, Stroeker, and Tzanakis for sums of consecutive squares. For a fixed starting value k, the equation

$$k^2 + (k+1)^2 + \cdots + (k+n-1)^2 = t^2$$

is transformed by

$$x = 12n + 12k - 6, \qquad y = 72t, \tag{6}$$

to the Weierstrass equation

$$y^2 = x^3 - 36x - 864k(k-1)(2k-1). \tag{7}$$

For us, $k = 192$. Therefore

$$12k - 6 = 2304 - 6 = 2298,$$

so the change of variables becomes

$$x = 12n + 2298, \qquad y = 72t. \tag{8}$$

Also

$$864 \cdot 192 \cdot 191 \cdot 383 = 12135204864,$$

so the corresponding elliptic curve is

$$E: \qquad y^2 = x^3 - 36x - 12135204864. \tag{9}$$

For completeness, let us verify the identity directly from (3). Starting from (8), we have

$$y^2 = (72t)^2 = 5184t^2 = 5184S(n).$$

Substituting (3) gives

$$y^2 = 5184\left(\frac{n^3}{3} + \frac{383}{2}n^2 + \frac{220033}{6}n\right) = 1728n^3 + 992736n^2 + 190108512n.$$

Now compute

$$x^3 - 36x - 12135204864 = (12n + 2298)^3 - 36(12n + 2298) - 12135204864,$$

and expand:

$$(12n + 2298)^3 - 36(12n + 2298) - 12135204864 = 1728n^3 + 992736n^2 + 190108512n.$$

Thus indeed

$$y^2 = x^3 - 36x - 12135204864$$

if and only if (1) holds.

Therefore every solution (n, t) of (1) gives an integral point $(x, y) \in E(\mathbb{Z})$, and conversely every integral point $(x, y) \in E(\mathbb{Z})$ satisfying

$$x \equiv -6 \pmod{12}, \qquad y \equiv 0 \pmod{72} \tag{10}$$

recovers a solution by

$$n = \frac{x + 6}{12} - 192, \qquad t = \frac{y}{72}. \tag{11}$$

4. The elliptic curve and its integral points

We now analyze the curve (9) in SageMath. The computation recorded in Appendix A yields the following data:

- the curve is
$$E : y^2 = x^3 - 36x - 12135204864;$$
- its rank over $\mathbb{Q}$ is 4;
- Sage finds exactly 12 integral points with nonnegative y-coordinate:

$$(2298, 0),\ (2310, 13824),\ (2796, 98604),\ (3354, 159984),$$
$$(5950, 445544),\ (6636, 529236),\ (13936, 1641464),\ (26124, 4220964),$$
$$(34110, 6298776),\ (191301, 83671101),\ (1322496, 1520868096),\ (1979134, 2784279304).$$

By symmetry of the Weierstrass equation, the corresponding points with negative y-coordinate are also rational points, but they give the same values of t^2, so for the Diophantine problem it suffices to keep $y \geq 0$.

The next task is to translate these integral points back to (n, t)-pairs. The recovery rule (11) shows that we only keep points satisfying (10). A quick check gives:

Point (x, y)	$x \equiv -6 \pmod{12}$?	$72 \mid y$?	n	t	admissible?
$(2298, 0)$	yes	yes	0	0	no (length zero)
$(2310, 13824)$	yes	yes	1	192	yes, but $1 \not\equiv 0 \pmod 8$
$(3354, 159984)$	yes	yes	88	2222	yes, and $88 \equiv 0 \pmod 8$
$(34110, 6298776)$	yes	yes	2651	87483	yes, but $2651 \not\equiv 0 \pmod 8$
all remaining listed points	no and/or no	–	–	–	no

Thus the full elliptic-curve computation leaves exactly three positive-length solutions of (1):

$$(n, t) = (1, 192), \qquad (88, 2222), \qquad (2651, 87483).$$

Among these, only one satisfies the required divisibility condition $n \equiv 0 \pmod 8$, namely $(88, 2222)$.

This proves Theorem 1.

5. Translation to the L_{88} identity

The lattice-theoretic parametrization usually writes the same identity as

$$A^2 + (A+1)^2 + \cdots + B^2 = X^2, \qquad D = B - A + 1.$$

For the present problem,

$$A = 192, \qquad D = n, \qquad B = A + n - 1 = 191 + n, \qquad X = t.$$

Therefore the unique solution with $D \equiv 0 \pmod 8$ is

$$D = 88, \qquad B = 279, \qquad X = 2222.$$

So the unique nontrivial identity beginning at 192 and having length divisible by 8 is

$$192^2 + 193^2 + \cdots + 279^2 = 2222^2. \tag{12}$$

This is precisely the identity used to construct the rank-88 lattice L_{88}.

6. Conclusion

The problem is rigid once one imposes the natural rank condition $n \equiv 0 \pmod 8$. The elliptic curve (9) has finitely many integral points, and only one of them translates to a positive solution whose length is divisible by 8. Consequently the rank-88 identity (12) is unique among all identities starting at 192 that are compatible with the even unimodular rank condition.

A. SageMath script

The following SageMath script was used.

```
from sage.all import *

# Prove/verify integer solutions to
#   192^2 + 193^2 + ... + B^2 = T^2
# by reducing to integral points on an elliptic curve.
#
# For fixed start k, Bremner-Stroeker-Tzanakis reduce
#   k^2 + (k+1)^2 + ... + (k+n-1)^2 = t^2
# via
#   x = 12*n + 12*k - 6,
#   y = 72*t,
# to the elliptic curve
#   y^2 = x^3 - 36*x - 864*k*(k-1)*(2*k-1).
# Here k = 192 and n = B-191.
#
# References:
#   A. Bremner, J. Stroeker, N. Tzanakis,
#   "On square values of sums of consecutive squares", 1997.

k = ZZ(192)
A = k

# Elliptic curve attached to start value k = 192
A4 = ZZ(-36)
A6 = -ZZ(864) * k * (k - 1) * (2*k - 1)
E = EllipticCurve([0, 0, 0, A4, A6])

print("Elliptic curve E:")
print(E)
print("Weierstrass form: y^2 = x^3 - 36*x + (%s)" % A6)
```

```
print()

# Sanity checks for the two obvious candidate solutions:
# n=1 gives the trivial one-term solution B=192, T=192.
# n=88 gives the nontrivial solution B=279, T=2222.
def to_point(n, t, k=ZZ(192)):
    x = 12*n + 12*k - 6
    y = 72*t
    return x, y

for (n, t, label) in [(1, 192, "trivial"), (88, 2222, "nontrivial")]:
    x, y = to_point(ZZ(n), ZZ(t), k)
    print(f"Check {label} candidate n={n}, t={t}: point ({x}, {y})")
    print("On E?", E.is_on_curve(x, y))
print()

# Compute rank / generators if desired
try:
    print("Rank of E:", E.rank())
    print("Generators:")
    for P in E.gens():
        print(" ", P)
    print()
except Exception as e:
    print("Rank/generator computation did not finish or is unavailable:")
    print(" ", e)
    print()

# Integral points on E
print("Computing integral points on E ...")
pts = E.integral_points()
print("Number of integral points found:", len(pts))
print()

# Convert back to solutions of the original problem.
# For x = 12*n + 12*k - 6 we have n = (x + 6)/12 - k.
# Since B = k + n - 1 = (x + 6)/12 - 1, equivalently B = (x - 6)/12.
solutions = []
all_relevant_points = []

for P in pts:
    x, y = map(ZZ, P.xy())
    all_relevant_points.append((x, y))

    # Need y divisible by 72 and x congruent to -6 mod 12.
    if (x + 6) % 12 != 0:
        continue
    if y % 72 != 0:
```

```
            continue

        n = (x + 6)//12 - k
        t = y//72
        B = k + n - 1

        # We only want sums from 192^2 up to B^2, so n >= 1, B >= 192, t >= 0.
        if n >= 1 and B >= k and t >= 0:
            # Verify directly
            lhs = sum(m*m for m in range(k, B+1))
            rhs = t*t
            ok = (lhs == rhs)
            solutions.append((B, t, n, x, y, ok))

print("All integral points (x,y):")
for xy in all_relevant_points:
    print(" ", xy)
print()

print("Recovered solutions with B >= 192 and T >= 0:")
for B, T, n, x, y, ok in sorted(solutions):
    print(f"  B={B}, T={T}, n={n}, point=({x},{y}), direct_check={ok}")
print()

# Final human-readable conclusion
if sorted((B, T) for B, T, n, x, y, ok in solutions) == [(ZZ(192), ZZ(192)), (ZZ(279), ZZ(2222))]:
    print("Conclusion:")
    print("  The only solutions with start value 192 are:")
    print("    192^2 = 192^2")
    print("  and")
    print("    192^2 + 193^2 + ... + 279^2 = 2222^2")
else:
    print("Conclusion:")
    print("  The recovered solution set is different from the expected two-point set.")
    print("  Inspect the listed integral points / conversions above.")
```

B. Console output

The following console output was recorded.

```
Elliptic curve E:
Elliptic Curve defined by y^2 = x^3 - 36*x - 12135204864 over Rational Field
Weierstrass form: y^2 = x^3 - 36*x + (-12135204864)

Check trivial candidate n=1, t=192: point (2310, 13824)
On E? True
Check nontrivial candidate n=88, t=2222: point (3354, 159984)
On E? True
```

```
Rank of E: 4
Generators:
  (2310 : 13824 : 1)
  (24994/9 : 2601368/27 : 1)
  (2796 : 98604 : 1)
  (1979134 : -2784279304 : 1)

Computing integral points on E ...
Number of integral points found: 12

All integral points (x,y):
  (2298, 0)
  (2310, 13824)
  (2796, 98604)
  (3354, 159984)
  (5950, 445544)
  (6636, 529236)
  (13936, 1641464)
  (26124, 4220964)
  (34110, 6298776)
  (191301, 83671101)
  (1322496, 1520868096)
  (1979134, 2784279304)

Recovered solutions with B >= 192 and T >= 0:
  B=279, T=2222, n=88, point=(3354,159984), direct_check=True

Conclusion:
  The only solutions with start value 192  and n = 0 mod 8 is:
    192^2 + 193^2 + ... + 279^2 = 2222^2
```

C. Post-processing of the integral points

The raw list of integral points already contains all the arithmetic information needed for the theorem. The point

$$(34110, 6298776)$$

also satisfies the congruence conditions (10), and the inverse map (11) yields

$$n = \frac{34110 + 6}{12} - 192 = 2651, \qquad t = \frac{6298776}{72} = 87483.$$

This is a genuine solution of (1), but it does *not* satisfy the side condition $n \equiv 0 \pmod 8$, since

$$2651 \equiv 3 \pmod 8.$$

Therefore it lies outside the scope of Theorem 1. The theorem concerns exactly the rank-divisible-by-8 case, and in that case the unique solution is $(n, t) = (88, 2222)$.

D. An infinite family of solutions via Pell equations

In this section we explain a standard but very useful mechanism which turns *one* solution of

$$A^2 + (A+1)^2 + \cdots + (A+D-1)^2 = X^2 \tag{13}$$

into *infinitely many* further solutions. The method is based on the theory of Pell equations.

We assume throughout that

$$8 \mid D,$$

and we are interested in integer solutions $(A, X) \in \mathbb{Z}^2$, usually with $A \geq 1$ and $X \geq 0$.

D.1. Step 1: rewriting the equation

We begin by rewriting the left-hand side of (13). Using the standard formulas

$$\sum_{k=0}^{D-1} 1 = D, \qquad \sum_{k=0}^{D-1} k = \frac{D(D-1)}{2}, \qquad \sum_{k=0}^{D-1} k^2 = \frac{D(D-1)(2D-1)}{6},$$

we obtain

$$\begin{aligned}
\sum_{k=0}^{D-1} (A+k)^2 &= \sum_{k=0}^{D-1} (A^2 + 2Ak + k^2) \\
&= DA^2 + 2A \sum_{k=0}^{D-1} k + \sum_{k=0}^{D-1} k^2 \\
&= DA^2 + 2A \cdot \frac{D(D-1)}{2} + \frac{D(D-1)(2D-1)}{6} \\
&= DA^2 + D(D-1)A + \frac{D(D-1)(2D-1)}{6}.
\end{aligned}$$

Hence (13) is equivalent to

$$DA^2 + D(D-1)A + \frac{D(D-1)(2D-1)}{6} = X^2. \tag{14}$$

Now we complete the square in the variable A. Observe that

$$A^2 + (D-1)A = \left(A + \frac{D-1}{2}\right)^2 - \frac{(D-1)^2}{4}.$$

Substituting this into (14), we get

$$X^2 = D\left(A + \frac{D-1}{2}\right)^2 - D\frac{(D-1)^2}{4} + \frac{D(D-1)(2D-1)}{6}.$$

We simplify the constant term:

$$\begin{aligned}
-D\frac{(D-1)^2}{4} + \frac{D(D-1)(2D-1)}{6} &= D(D-1)\left(-\frac{D-1}{4} + \frac{2D-1}{6}\right) \\
&= D(D-1)\cdot\frac{-3D+3+4D-2}{12} \\
&= D(D-1)\cdot\frac{D+1}{12} \\
&= \frac{D(D^2-1)}{12}.
\end{aligned}$$

Thus

$$X^2 = D\left(A + \frac{D-1}{2}\right)^2 + \frac{D(D^2-1)}{12}. \tag{15}$$

It is now convenient to introduce the new variable

$$Y := 2A + D - 1.$$

Then

$$A + \frac{D-1}{2} = \frac{Y}{2},$$

so (15) becomes

$$X^2 = D\frac{Y^2}{4} + \frac{D(D^2-1)}{12}.$$

Multiplying by 4, we obtain

$$4X^2 = DY^2 + \frac{D(D^2-1)}{3}.$$

Now define

$$U := 2X.$$

Then we arrive at the inhomogeneous Pell-type equation

$$U^2 - DY^2 = \frac{D(D^2-1)}{3}. \tag{16}$$

We summarize this in a lemma.

Lemma 1. *Let D be a positive integer. Then $(A, X) \in \mathbb{Z}^2$ satisfies (13) if and only if the pair*

$$(U, Y) = (2X,\, 2A + D - 1)$$

satisfies

$$U^2 - DY^2 = \frac{D(D^2-1)}{3}.$$

Conversely, if $(U, Y) \in \mathbb{Z}^2$ satisfies

$$U^2 - DY^2 = \frac{D(D^2-1)}{3},$$

with U even and $Y \equiv D-1 \pmod 2$, then

$$X = \frac{U}{2}, \qquad A = \frac{Y-(D-1)}{2}$$

is an integer solution of (13).

Proof. The forward implication was derived above by straightforward algebra. For the converse, assume

$$U^2 - DY^2 = \frac{D(D^2-1)}{3},$$

and U is even, $Y \equiv D-1 \pmod 2$. Then $X = U/2 \in \mathbb{Z}$, and

$$A = \frac{Y-(D-1)}{2} \in \mathbb{Z}$$

because $Y-(D-1)$ is even. Reversing the algebraic steps shows that (A, X) satisfies (13). □

D.2. Step 2: why the Pell equation enters

Equation (16) has the form

$$U^2 - DY^2 = N, \qquad N := \frac{D(D^2-1)}{3}.$$

This is not the classical Pell equation $u^2 - Dv^2 = 1$, but it is closely related to it.

The key fact is that if

$$u^2 - Dv^2 = 1,$$

then multiplication in the quadratic ring $\mathbb{Z}[\sqrt{D}]$ preserves the norm $U^2 - DY^2$. More precisely, if

$$\alpha = U + Y\sqrt{D}, \qquad \varepsilon = u + v\sqrt{D},$$

then

$$N(\alpha) := (U + Y\sqrt{D})(U - Y\sqrt{D}) = U^2 - DY^2,$$

$$N(\varepsilon) := (u + v\sqrt{D})(u - v\sqrt{D}) = u^2 - Dv^2.$$

Hence

$$N(\alpha\varepsilon) = N(\alpha)\, N(\varepsilon).$$

So if $N(\alpha) = N$ and $N(\varepsilon) = 1$, then also

$$N(\alpha\varepsilon) = N.$$

This is exactly the mechanism that produces infinitely many solutions.

D.3. Step 3: existence of infinitely many Pell units

Since D is divisible by 8, it is in particular positive and not a perfect square in the cases of interest. For a non-square positive integer D, the Pell equation

$$u^2 - Dv^2 = 1 \tag{17}$$

has infinitely many integer solutions. Let

$$(u_1, v_1)$$

be its fundamental positive solution, i.e. the unique solution with $u_1 > 0$, $v_1 > 0$, and $u_1 + v_1\sqrt{D} > 1$ minimal. Then every power

$$(u_n + v_n\sqrt{D}) := (u_1 + v_1\sqrt{D})^n \qquad (n \geq 1)$$

also satisfies

$$u_n^2 - Dv_n^2 = 1.$$

Thus (17) has infinitely many positive solutions.

We also need a simple parity observation.

Lemma 2. *If D is even and $u^2 - Dv^2 = 1$, then u is odd.*

Proof. Reduce the equation modulo 2. Since D is even,

$$u^2 - Dv^2 \equiv u^2 \pmod{2}.$$

But the left-hand side equals 1, so

$$u^2 \equiv 1 \pmod{2}.$$

Therefore u is odd. □

D.4. Step 4: generating infinitely many solutions from one solution

We now prove the main theorem of this section.

Theorem 2. *Assume that $8 \mid D$, and assume that there exists at least one integer solution (A_0, X_0) of*

$$A^2 + (A+1)^2 + \cdots + (A+D-1)^2 = X^2. \tag{18}$$

Then there exist infinitely many integer solutions (A_n, X_n) of (18). More precisely, one may choose them so that

$$A_n \to +\infty, \qquad X_n \to +\infty \qquad (n \to \infty).$$

Proof. Start from one solution (A_0, X_0). Define

$$U_0 := 2X_0, \qquad Y_0 := 2A_0 + D - 1.$$

By Lemma 1,

$$U_0^2 - DY_0^2 = \frac{D(D^2-1)}{3}.$$

Set

$$N := \frac{D(D^2-1)}{3}.$$

Next, let (u_1, v_1) be the fundamental positive solution of the Pell equation

$$u^2 - Dv^2 = 1,$$

and define for $n \geq 0$

$$u_n + v_n\sqrt{D} := (u_1 + v_1\sqrt{D})^n.$$

Then

$$u_n^2 - Dv_n^2 = 1 \qquad \text{for all } n \geq 0.$$

Now define

$$U_n + Y_n\sqrt{D} := (U_0 + Y_0\sqrt{D})(u_n + v_n\sqrt{D}). \tag{19}$$

We must show three things:

(i) The new pairs (U_n, Y_n) satisfy the same norm equation.

Taking norms in (19), we get

$$(U_n^2 - DY_n^2) = (U_0^2 - DY_0^2)(u_n^2 - Dv_n^2) = N \cdot 1 = N.$$

Thus

$$U_n^2 - DY_n^2 = \frac{D(D^2-1)}{3} \qquad \text{for all } n \geq 0.$$

(ii) The parity conditions are preserved.

We want to recover integer values $X_n = U_n/2$ and

$$A_n = \frac{Y_n - (D-1)}{2}.$$

So we must prove that U_n is even and that $Y_n \equiv D - 1 \pmod 2$.

From (19) we have the explicit formulas

$$U_n = U_0 u_n + DY_0 v_n, \tag{20}$$

$$Y_n = U_0 v_n + Y_0 u_n. \tag{21}$$

Since $U_0 = 2X_0$ is even, and D is even, the right-hand side of (20) is even. Hence U_n is even.

Next, by Lemma 2, every u_n is odd, because each (u_n, v_n) solves $u_n^2 - Dv_n^2 = 1$ and D is even. Also U_0 is even. Therefore, reducing (21) modulo 2, we obtain

$$Y_n \equiv Y_0 u_n \equiv Y_0 \pmod 2.$$

But

$$Y_0 = 2A_0 + D - 1 \equiv D - 1 \pmod{2}.$$

Hence

$$Y_n \equiv D - 1 \pmod{2}.$$

So both parity conditions are satisfied.

(iii) The resulting solutions are infinitely many and unbounded.

By Lemma 1, each pair (U_n, Y_n) determines an integer solution

$$X_n = \frac{U_n}{2}, \qquad A_n = \frac{Y_n - (D-1)}{2}$$

of (18).

It remains to show that these solutions are infinitely many and that $A_n, X_n \to \infty$. Since

$$u_1 + v_1\sqrt{D} > 1,$$

its powers $(u_1 + v_1\sqrt{D})^n$ tend to $+\infty$ in $\mathbb{R}$, and so

$$(U_0 + Y_0\sqrt{D})(u_1 + v_1\sqrt{D})^n \to +\infty \qquad (n \to \infty)$$

provided $U_0 + Y_0\sqrt{D} > 0$, which holds if $X_0 \geq 0$ and $A_0 \geq 1$. Thus both coordinates U_n and Y_n become arbitrarily large for large n. Consequently $X_n = U_n/2 \to \infty$ and

$$A_n = \frac{Y_n - (D-1)}{2} \to \infty.$$

In particular, the solutions are all distinct for sufficiently large n, hence infinitely many. □

D.5. Step 5: explicit recurrence formulas

The proof above gives explicit recursions. If (u, v) is any fixed solution of

$$u^2 - Dv^2 = 1,$$

and (A, X) is one solution of (13), then a new solution (A', X') is obtained by

$$U' = Uu + DYv, \qquad Y' = Uv + Yu,$$

where

$$U = 2X, \qquad Y = 2A + D - 1.$$

Then

$$X' = \frac{U'}{2}, \qquad A' = \frac{Y' - (D-1)}{2}.$$

Substituting $U = 2X$ and $Y = 2A + D - 1$, we obtain

$$X' = Xu + \frac{D(2A + D - 1)}{2}v, \tag{22}$$

$$A' = Xv + \frac{(2A + D - 1)u - (D - 1)}{2}. \tag{23}$$

Because D is even and u is odd, the right-hand sides are integers.

Thus, once a single solution (A, X) is known, every Pell solution (u, v) to $u^2 - Dv^2 = 1$ produces another solution (A', X'). Iterating this process yields infinitely many solutions.

D.6. Step 6: why the condition $8 \mid D$ is natural

The divisibility condition $8 \mid D$ is not needed for the formal algebraic transformation itself, but it is natural in the lattice-theoretic applications because dimensions of even unimodular lattices are multiples of 8. Moreover, when D is even, the parity argument in the proof becomes very clean:

- $U = 2X$ is automatically even;
- $Y = 2A + D - 1$ has the correct parity to recover A;
- every Pell unit $u + v\sqrt{D}$ has u odd;
- therefore the parity conditions are preserved under multiplication.

This is exactly what allows one to pass from one solution (A_0, X_0) to an infinite sequence of further integral solutions (A_n, X_n).

D.7. Conclusion of the construction

We can summarize the method in one sentence:

> The equation
> $$A^2 + (A + 1)^2 + \cdots + (A + D - 1)^2 = X^2$$
> is equivalent to an inhomogeneous Pell equation
> $$U^2 - DY^2 = \frac{D(D^2 - 1)}{3},$$
> and once one integer solution is known, multiplication by infinitely many units of norm 1 in $\mathbb{Z}[\sqrt{D}]$ produces infinitely many further solutions.

This gives a completely explicit and elementary mechanism for generating infinite families of solutions in every admissible dimension D for which at least one initial solution exists.

Remark 2. Empirically, the Pell-equation construction appears to be highly selective. In dimension $D = 24$, the first computational experiments indicate that it hits exactly two Niemeier isomorphism classes, namely the Leech lattice and the Niemeier lattice with root system A_1^{24}. Both are highly symmetric: the Leech lattice is the unique rootless Niemeier lattice and has an exceptionally large automorphism group, while the lattice A_1^{24} is also very symmetric and is closely related to the Mathieu group M_{24}. In dimension $D = 88$, preliminary computer experiments seem to indicate the existence of four isomorphism classes produced by the same Pell-type mechanism. At present, however, the corresponding theta series and modular-form interpretations are still unknown, and the automorphism groups of these 88-dimensional lattices have not yet been determined. For general background on modular forms and their relation to lattice theta series, see Koecher–Krieg [1].

E. Why the next admissible dimension after 24 is 88

The classical Conway–Leech identity

$$1^2 + 2^2 + \cdots + 24^2 = 70^2$$

shows that $D = 24$ is admissible. We now prove that among the integers

$$D \equiv 0 \pmod 8, \qquad 24 < D < 88,$$

none is admissible. Hence the next admissible dimension is $D = 88$.

E.1. Reduction to an inhomogeneous Pell equation

Consider

$$A^2 + (A+1)^2 + \cdots + (A+D-1)^2 = X^2$$

with integers $A \geq 1$, $X \geq 0$, and $D \equiv 0 \pmod 8$. As usual, set

$$Y := 2A + D - 1, \qquad U := 2X.$$

Then the consecutive-square identity is equivalent to

$$U^2 - DY^2 = \frac{D(D^2-1)}{3}. \tag{1}$$

Moreover, since D is even, we have

$$Y = 2A + D - 1 \equiv 1 \pmod 2,$$

so every relevant solution of (1) must satisfy

$$Y \text{ odd}. \tag{2}$$

Thus, for a fixed D, admissibility reduces to the existence of an integer solution

$$(U, Y) \in \mathbf{Z}^2$$

of (1) with Y odd.

E.2. Excluding all multiples of 8 between 24 and 88

The only integers divisible by 8 strictly between 24 and 88 are

$$32, 40, 48, 56, 64, 72, 80.$$

We now exclude them one by one by elementary local obstructions.

Proposition 1. *There is no solution*

$$A^2 + (A+1)^2 + \cdots + (A+D-1)^2 = X^2$$

with $A \geq 1$*,* $X \geq 0$*, and*

$$D \in \{32, 40, 48, 56, 64, 72, 80\}.$$

Proof. We treat the seven values of D separately.

Case $D = 32$. Equation (1) becomes

$$U^2 - 32Y^2 = 10912.$$

Reducing modulo 11, we get

$$U^2 + Y^2 \equiv 0 \pmod{11},$$

because $32 \equiv -1 \pmod{11}$ and $10912 \equiv 0 \pmod{11}$. Since -1 is not a square modulo 11, this forces

$$11 \mid U, \qquad 11 \mid Y.$$

Hence the left-hand side is divisible by 11^2, whereas

$$10912 = 2^5 \cdot 11 \cdot 31$$

is divisible by only one factor of 11. Contradiction.

Case $D = 40$. Equation (1) becomes

$$U^2 - 40Y^2 = 21320.$$

Reducing modulo 5, we obtain

$$U^2 \equiv 0 \pmod{5},$$

hence $5 \mid U$. Write $U = 5u$. Then

$$25u^2 - 40Y^2 = 21320,$$

so

$$5u^2 - 8Y^2 = 4264.$$

Reducing modulo 5, we get

$$-8Y^2 \equiv 4 \pmod{5},$$

that is,
$$-3Y^2 \equiv 4 \pmod 5, \qquad \text{hence} \qquad Y^2 \equiv 2 \pmod 5.$$
But 2 is not a quadratic residue modulo 5. Contradiction.

Case $D = 48$. Equation (1) becomes
$$U^2 - 48Y^2 = 36848.$$
Reducing modulo 3, we get
$$U^2 \equiv 36848 \equiv 2 \pmod 3.$$
This is impossible, since the only quadratic residues modulo 3 are 0 and 1.

Case $D = 56$. Equation (1) becomes
$$U^2 - 56Y^2 = 58520.$$
Reducing modulo 7, we obtain
$$U^2 \equiv 0 \pmod 7,$$
hence $7 \mid U$. Write $U = 7u$. Then
$$49u^2 - 56Y^2 = 58520,$$
so
$$7u^2 - 8Y^2 = 8360.$$
Reducing modulo 7, this gives
$$-8Y^2 \equiv 2 \pmod 7,$$
hence
$$-Y^2 \equiv 2 \pmod 7, \qquad \text{so} \qquad Y^2 \equiv 5 \pmod 7.$$
But 5 is not a quadratic residue modulo 7. Contradiction.

Case $D = 64$. Equation (1) becomes
$$U^2 - 64Y^2 = 87360 = 64 \cdot 1365.$$
Hence $8 \mid U$. Write $U = 8u$. Then
$$u^2 - Y^2 = 1365.$$
Now Y is odd by (2), so
$$Y^2 \equiv 1 \pmod 4.$$
Therefore
$$u^2 - 1 \equiv 1365 \equiv 1 \pmod 4,$$
which implies
$$u^2 \equiv 2 \pmod 4.$$

But squares modulo 4 are only 0 and 1. Contradiction.

Case $D = 72$**.** Equation (1) becomes

$$U^2 - 72Y^2 = 124392.$$

Reducing modulo 9, we obtain

$$U^2 \equiv 124392 \equiv 3 \pmod 9.$$

But the quadratic residues modulo 9 are $0, 1, 4, 7$, so this is impossible.

Case $D = 80$**.** Equation (1) becomes

$$U^2 - 80Y^2 = 170640.$$

Reducing modulo 5, we get

$$U^2 \equiv 0 \pmod 5,$$

hence $5 \mid U$. Write $U = 5u$. Then

$$25u^2 - 80Y^2 = 170640,$$

so

$$5u^2 - 16Y^2 = 34128.$$

Reducing modulo 5, we get

$$-16Y^2 \equiv 3 \pmod 5,$$

hence

$$-Y^2 \equiv 3 \pmod 5, \qquad \text{so} \qquad Y^2 \equiv 2 \pmod 5.$$

Again 2 is not a quadratic residue modulo 5. Contradiction.

This excludes all values

$$D = 32, 40, 48, 56, 64, 72, 80.$$

□

Theorem 3. *Among all integers* $D \equiv 0 \pmod 8$ *with* $D \geq 24$*, the first admissible value is* $D = 24$*, and the next admissible value is* $D = 88$*.*

Proof. The value $D = 24$ is admissible by the Conway–Leech identity

$$1^2 + 2^2 + \cdots + 24^2 = 70^2.$$

The proposition shows that no multiple of 8 strictly between 24 and 88 is admissible. On the other hand, $D = 88$ is admissible, as witnessed by the explicit identity

$$36663588^2 + (36663589)^2 + \cdots + 36663675^2 = 343935350^2.$$

Therefore the next admissible dimension after 24 is 88. □

Remark 3. The argument in this section is entirely local and genus-0: for fixed D, the problem reduces to the inhomogeneous Pell equation (1), and all intermediate dimensions are ruled out by congruence obstructions. In particular, one does not need elliptic-curve methods for the step from $D = 24$ to $D = 88$.

References

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