Visualizing finite groups in 2D (working draft 01.09.2024)

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Abstract

This note presents a method for visualizing finite groups in two dimensions using matrix embeddings and the Frobenius inner product. The technique involves associating group elements with matrices and defining a positive semi-definite kernel function. You can find some of the visualizations as a poster at [1].

1 Introduction

Let G be a finite group with $n = |G|$ elements. In this note, we are going to describe a method to visualize such a group in the 2D plane. By Cayley's theorem for finite groups, we have an injective homomorphism of groups:

$$\pi : G \to S_n, g \mapsto \pi(g) \tag{1}$$

where each group element g is mapped to the permutation of the symmetric group S_n on $n = |G|$ elements, wich it generates by left multiplication:

$$\pi(g) : G \to G, x \mapsto g \cdot x$$

We want to associate to each group element g a matrix and then use the Frobenius inner product to define a positive semi-definite function k on the group G as follows:

It is known , see for instance "The Kendall and Mallows Kernels for Permutations" by Yunlong Jiao and Jean-Philippe Vert, that the Kendall-tau function can be made to a positive semi-definite kernel k for permutations.

Let $\mathbf{1}_{\{x\}}$ be the indicator function, which is $= 1$ if the boolean variable x is true and 0 if the boolean variable x is false, and let:

$$\phi : S_n \to M_n(\mathbb{R}), \sigma \mapsto (\mathbf{1}_{\{\sigma(i)>\sigma(j)\}} - \mathbf{1}_{\{\sigma(i)<\sigma(j)\}})_{1\leq i,j\leq n} \tag{2}$$

where $M_n(\mathbb{R})$ denotes $n \times n$ matrices over $\mathbb{R}$.
The embedding from the finite group G to $M_n(\mathbb{R})$ is then given by:

$$\psi : G \to M_n(\mathbb{R}), g \mapsto \frac{1}{\sqrt{n(n-1)}} \cdot \phi(\pi(g)) \tag{3}$$

We use the Frobenius inner product on $M_n(\mathbb{R})$, which is given by:

$$\langle A, B\rangle := \mathrm{tr}(A \cdot B^T) \tag{4}$$

to define a positive semi-definite, symmetric function k on G:

$$k : G \times G \to \mathbb{R}, (g, h) \mapsto \mathrm{tr}(\psi(g) \cdot \psi(h)^T) \tag{5}$$

2 Examples

2.1 The Klein-Four group

Let G be the Klein-Four group. The following SageMath-Code computes some of the constructions above:

```
import numpy as np
import itertools
from sklearn.decomposition import KernelPCA
from sklearn.metrics.pairwise import rbf_kernel

def regularPermutationsInSymmetricGroup(finiteGroup):
    from sage.matrix.operation_table import OperationTable
    G = finiteGroup
    O = OperationTable(G,operator.mul,names="elements")
    print(latex(O.table()))
    ll = [ Permutation([xx +1 for xx in x]) for x in O.table()]
    return ll

def embeddInMatrixSpace(sigma):
    """sigma = Permutation"""
    n = len(sigma)
    m = matrix([[0 for i in range(n)] for j in range(n)])
    for i in range(n):
        for j in range(n):
            m[i,j] = 1*(sigma(i+1)>sigma(j+1))-1*(sigma(i+1)<sigma(j+1))
    return m

def kk(p1,p2):
    m1 = embeddInMatrixSpace(p1)
```

```
    m2 = embeddInMatrixSpace(p2)
    return (m1*m2.transpose()).trace()

def grammat(group):
    perms = regularPermutationsInSymmetricGroup(group)
    return matrix([[kk(p1,p2) for p1 in perms] for p2 in perms])
```

The Cayley - Table is given by:

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

For $[1, 2, 3, 4]$, we get the following matrix:

$$\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

For $[2, 1, 4, 3]$, we get the following matrix:

$$\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

And similarily: $[3, 4, 1, 2]$

$$\begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

$[4, 3, 2, 1]$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

Here is the (normalize) Gram-matrix of k:

$$\begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & -1 \\ \frac{1}{3} & 1 & -1 & -\frac{1}{3} \\ -\frac{1}{3} & -1 & 1 & \frac{1}{3} \\ -1 & -\frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

We notice that all matrices in these examples have the following form:

- All entries are 0, -1, or $+1$.
- The diagonal entries are all 0.
- Off-diagonal entries are only $+1$ or -1.
- $M = -M^{\mathrm{T}}$ (the matrix is skew-symmetric).

2.2 The cyclic 4 group C_4

Similarily to the Klein-Four group we might calculate for C_4:

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

$$[1, 2, 3, 4]$$

$$\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$[2, 3, 4, 1]$$

$$\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

$$[3, 4, 1, 2]$$

$$\begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

$$[4, 1, 2, 3]$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

Here is the (normalized) Gram-matrix of k:

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

3 Questions

1. Is $k(xg, xh) = k(gx, hx) = k(g, h)\ \forall x, g, h \in G$?
2. Is ϕ an injective mapping?
3. Let $\chi_t(g) =$ the characteristic polynomial of the matrix $\phi(g)$ in variable t. Is $\chi_t(g) = \chi_t(h) \forall g, h \in G$?
4. What properties does $\psi(g)$ adopt from properties of skew-symmetric matrices?
5. Is $\det(\phi(\pi(g))) = 1$, if $n \equiv 0 \mod (2)$ and 0 otherwise?
6. What properties can be deduced from the Pfaffian of $\phi(\pi(g))$?
7. The eigenvalues of $\phi(\pi(g))$ are, since the matrix is skew-symmetric, either $= 0$ in the $n =$ odd case and the other come in pairs of purely imaginary eigenvalues, or are all purely imaginary: $\lambda i, -\lambda i$ with $\lambda \in \mathbb{R}$

4 Visualizations

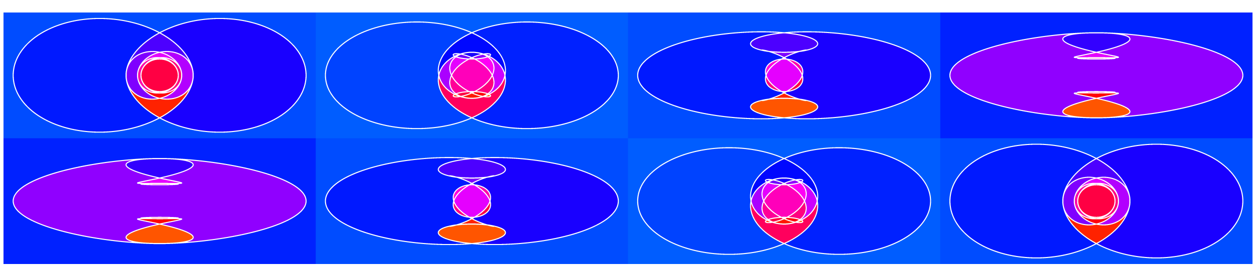

Figure 1: Colored C2xC2xC2

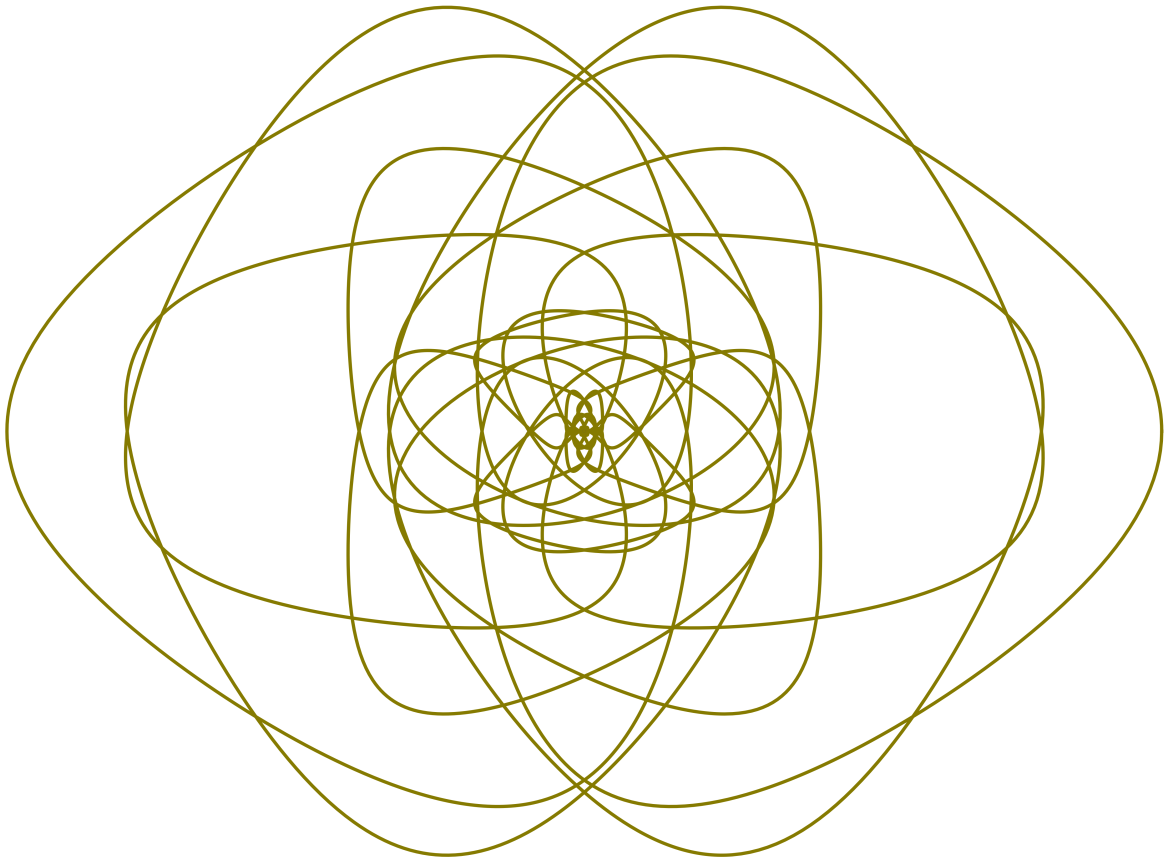

Figure 2: S4_KendalTau-Kernel_exp_dim-2.png

5 Acknowledgment

I would like to thank the MathOverflow community for answering questions which occured to me and I could not answer alone. I also would like to thank again the MathOverflow community for being honest in critique, when my questions get to strange.

References

[1] Redbubble

[2] Pfaffian

[3] Skew-Symmetric Matrix

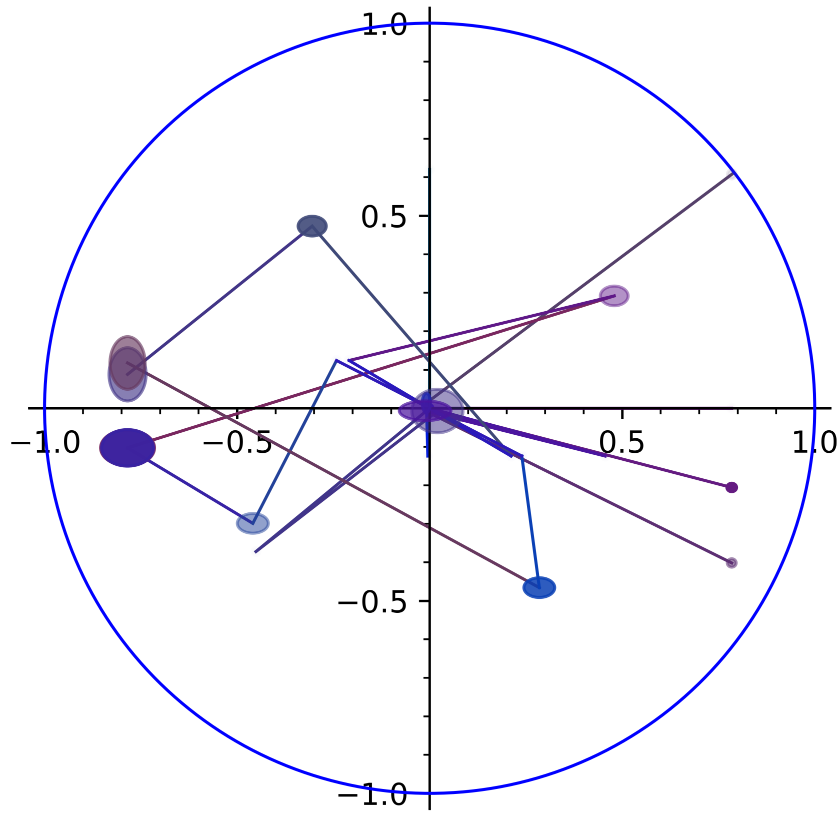

Figure 3: Quaternion KendalTau-Kernel ellipse dim-2

Keywords: finite groups, visualization, symmetric images, kendall tau kernel, skew symmetric matrices

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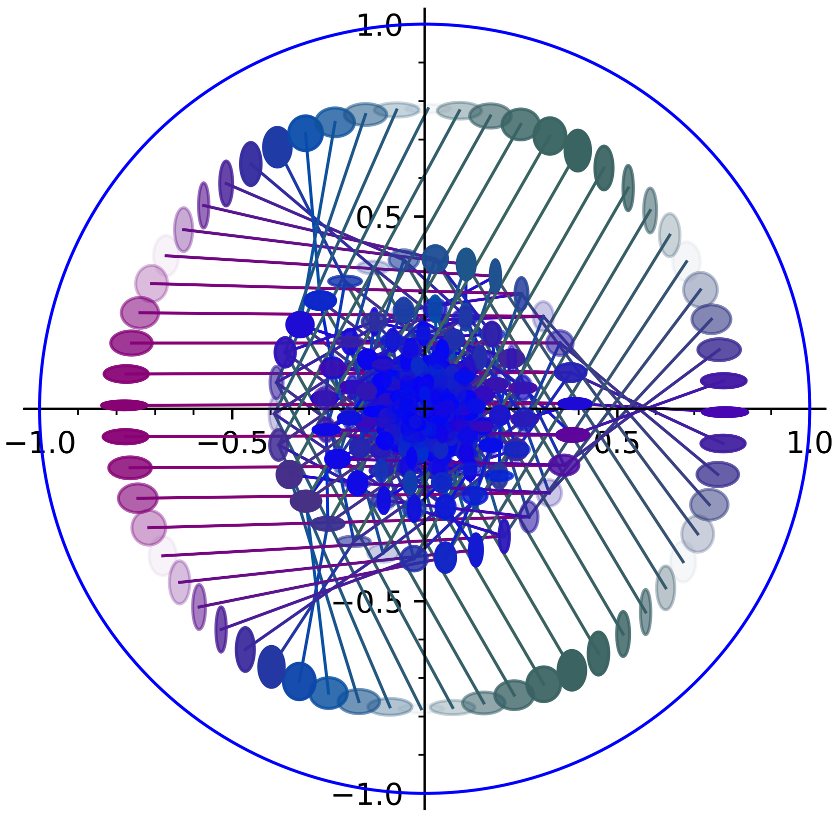

Figure 4: C60 KendalTau-Kernel ellipse dim-2

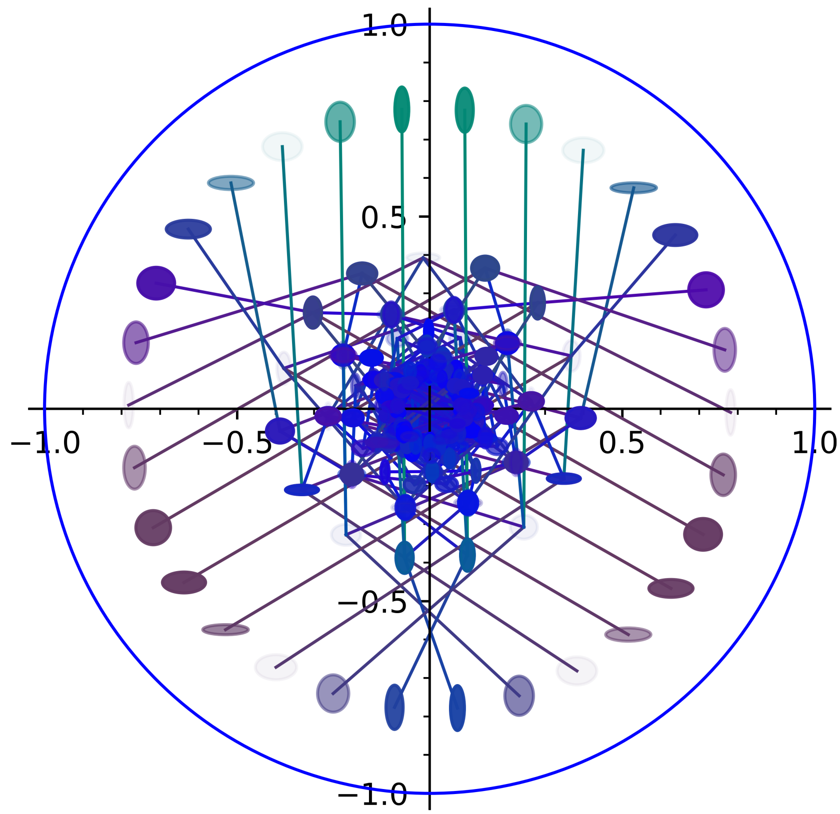

Figure 5: C30 KendalTau-Kernel ellipse dim-2