Visualising Finite Points of RKHS as Closed 2-D Curves

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Abstract

Let K be a positive semidefinite kernel on a finite set $X = \{x_1, \dots, x_N\}$. The usual point of view embeds X into a Euclidean or Hilbert space by a feature map $\Phi : X \to H$ satisfying

$$K(x_i, x_j) = \langle \Phi(x_i), \Phi(x_j) \rangle .$$

In this note we describe a Fourier realisation of finite RKHS point configurations: each point is represented by a closed planar curve

$$\Gamma_i(t) = \Psi(x_i)(t) \in \mathbb{C} \cong \mathbb{R}^2, \qquad t \in [0, 2\pi].$$

The construction is isometric at the Hilbert-space level: the Gram matrix of the finite point configuration is preserved exactly by the L^2 inner products of the curve-functions. We then apply the construction to the arithmetic meet matrix arising from sorted prime-factor lists. In particular, every integer in a finite truncation $\{1, \dots, N\}$ becomes a closed planar curve. We also incorporate a Farris-type congruence condition on the Fourier frequencies, which yields controlled rotational symmetry of the resulting images.

Contents

1 Introduction

Suppose that a finite set of objects $X = \{x_1, \dots, x_N\}$ is equipped with a positive semidefinite kernel

$$K : X \times X \to \mathbb{R}.$$

Then the kernel matrix

$$G = (K(x_i, x_j))_{1 \le i,j \le N}$$

is symmetric and positive semidefinite, and one may realise G as a Gram matrix of vectors in a Euclidean space:

$$K(x_i, x_j) = \langle \Phi(x_i), \Phi(x_j) \rangle .$$

This is the standard finite-dimensional RKHS or feature-space point of view.

The main idea of this note is the following.

Instead of visualising the finite feature vectors $\Phi(x_i)$ as points in $\mathbb{R}^N$, one may realise them as trigonometric polynomials and therefore as closed planar curves in $\mathbb{C} \cong \mathbb{R}^2$.

More precisely, one chooses pairwise distinct integer frequencies and places each feature coordinate into a separate Fourier mode. Because these modes are orthogonal in $L^2([0, 2\pi])$, the original Gram matrix is preserved exactly.

This leads to a new visual language:

- a finite point configuration in RKHS,
- realised as a finite family of trigonometric polynomials,
- hence as a finite family of closed curves in the plane.

The method is especially appealing when the kernel comes from arithmetic or poset structure, since the Fourier frequencies and amplitudes may then reflect arithmetic data.

Our motivating example is the arithmetic meet matrix associated with sorted prime-factor lists. There one has a finite feature map

$$n \longmapsto \Phi(n), \qquad n \in \{1, \dots, N\},$$

whose Gram matrix is

$$K(m, n) = m \wedge n$$

in the arithmetic sense. The aim is then to draw each integer n as a closed curve

$$\Gamma_n(t) \in \mathbb{C}.$$

2 Finite kernels, Gram matrices, and feature coordinates

Let $X = \{x_1, \dots, x_N\}$ be a finite set and let

$$K : X \times X \to \mathbb{R}$$

be a positive semidefinite kernel, meaning that the matrix

$$G = (K(x_i, x_j))_{1 \le i,j \le N}$$

is positive semidefinite.

Definition 2.1. *A* finite feature realisation *of K is a Hilbert space H together with a map*

$$\Phi : X \to H$$

such that

$$K(x_i, x_j) = \langle \Phi(x_i), \Phi(x_j) \rangle_H \qquad (1 \le i, j \le N).$$

For finite X, such a realisation always exists. In many situations one has an explicit coordinate system indexed by a finite set D:

$$\Phi(x) = \big(\phi_d(x)\big)_{d \in D}, \qquad K(x, y) = \sum_{d \in D} \phi_d(x)\phi_d(y).$$

This is the form most convenient for the Fourier construction below.

3 An isometric Fourier realisation of finite RKHS points

3.1 The ambient Fourier space

Let

$$S^1 = \mathbb{R}/2\pi\mathbb{Z},$$

and consider the Hilbert space

$$L^2(S^1, \mathbb{C})$$

with inner product

$$\langle f, g \rangle_{L^2} = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, dt.$$

For each integer $k \in \mathbb{Z}$, define

$$e_k(t) = e^{ikt}.$$

Then

$$\langle e_k, e_\ell \rangle_{L^2} = \delta_{k,\ell}.$$

Thus $\{e_k : k \in \mathbb{Z}\}$ is an orthonormal set.

3.2 The Fourier curve embedding

Assume that K has a finite feature expansion

$$K(x, y) = \sum_{d \in D} \phi_d(x)\phi_d(y).$$

Choose an injective map

$$\eta : D \to \mathbb{Z}.$$

For each $x \in X$, define the trigonometric polynomial

$$\Psi(x)(t) := \sum_{d \in D} \phi_d(x) e^{i\eta(d)t}, \qquad t \in [0, 2\pi].$$

Theorem 3.1 (isometric Fourier realisation)**.** *For all $x, y \in X$ one has*

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi(x)(t)\overline{\Psi(y)(t)}\,dt = \sum_{d\in D} \phi_d(x)\phi_d(y) = K(x,y).$$

Proof. By orthogonality of the exponentials,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\eta(d)t}\overline{e^{i\eta(e)t}}\,dt = \delta_{d,e}.$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi(x)(t)\overline{\Psi(y)(t)}\,dt = \sum_{d,e\in D} \phi_d(x)\phi_e(y)\delta_{d,e} = \sum_{d\in D} \phi_d(x)\phi_d(y) = K(x,y).$$

□

Corollary 3.2. *The map*

$$x \longmapsto \Psi(x)$$

is an isometric embedding of the finite RKHS point configuration into the finite-dimensional Fourier space

$$V_\eta = \mathrm{span}_{\mathbb{C}}\{e^{i\eta(d)t} : d \in D\} \subset L^2(S^1, \mathbb{C}).$$

3.3 Closed planar curves

For each fixed $x \in X$, the function

$$t \longmapsto \Psi(x)(t)$$

is a trigonometric polynomial, hence a continuous map

$$[0, 2\pi] \to \mathbb{C} \cong \mathbb{R}^2.$$

Because every frequency $\eta(d)$ is an integer,

$$\Psi(x)(0) = \Psi(x)(2\pi),$$

so the image is a closed planar curve.

Definition 3.3. *The* curve associated with $x \in X$ *is*

$$\Gamma_x := \{\Psi(x)(t) : 0 \le t \le 2\pi\} \subset \mathbb{C}.$$

Thus every finite RKHS point becomes a closed 2-dimensional curve.

4 Farris-type rotational symmetry

A key observation, inspired by Farris, is that certain congruence conditions on the frequencies force rotational symmetry of the curves.

Proposition 4.1 (Farris symmetry criterion)**.** *Let*

$$\Psi(x)(t) = \sum_{d\in D} \phi_d(x)e^{i\eta(d)t},$$

and suppose there exist integers $m \ge 2$ and k such that

$$\eta(d) \equiv k \pmod{m} \qquad \textit{for all } d \in D.$$

Then for every $x \in X$,

$$\Psi(x)\left(t + \frac{2\pi}{m}\right) = e^{2\pi ik/m}\Psi(x)(t).$$

Hence the image curve of $\Psi(x)$ is invariant under rotation by angle

$$\frac{2\pi k}{m}.$$

If moreover $\gcd(k, m) = 1$, then the curve has full m-fold rotational symmetry.

Proof. We compute

$$\Psi(x)\left(t + \frac{2\pi}{m}\right) = \sum_{d\in D} \phi_d(x)e^{i\eta(d)(t+2\pi/m)} = \sum_{d\in D} \phi_d(x)e^{i\eta(d)t}e^{2\pi i\eta(d)/m}.$$

Since $\eta(d) \equiv k \pmod m$ for all d,

$$e^{2\pi i\eta(d)/m} = e^{2\pi ik/m},$$

hence

$$\Psi(x)\left(t + \frac{2\pi}{m}\right) = e^{2\pi ik/m} \sum_{d\in D} \phi_d(x)e^{i\eta(d)t} = e^{2\pi ik/m}\Psi(x)(t).$$

Thus the parameter shift by $2\pi/m$ corresponds to rotation in the plane by the angle $2\pi k/m$. If $\gcd(k, m) = 1$, this rotation has order exactly m. □

Remark 4.2. *This separates two types of data:*

- *the coefficients $\phi_d(x)$ encode the RKHS geometry,*
- *the congruence class of the frequencies $\eta(d)$ controls the visible symmetry.*

5 Variants of the construction

5.1 Phase parameters

One may add arbitrary phases without changing the kernel geometry. Let $\theta_d \in \mathbb{R}$ for $d \in D$ and define

$$\Psi_\theta(x)(t) = \sum_{d\in D} \phi_d(x)e^{i(\eta(d)t+\theta_d)}.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi_\theta(x)(t)\overline{\Psi_\theta(y)(t)}\,dt = \sum_{d\in D} \phi_d(x)\phi_d(y) = K(x, y),$$

since the phases cancel.

Thus phase choices modify the drawing but not the finite Hilbert geometry.

5.2 Normalisation

Since

$$\frac{1}{2\pi}\int_0^{2\pi} |\Psi(x)(t)|^2\,dt = K(x, x),$$

it is natural, when $K(x, x) > 0$, to define

$$\widetilde{\Psi}(x)(t) = \frac{\Psi(x)(t)}{\sqrt{K(x, x)}}.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} |\widetilde{\Psi}(x)(t)|^2\,dt = 1,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \widetilde{\Psi}(x)(t)\overline{\widetilde{\Psi}(y)(t)}\,dt = \frac{K(x,y)}{\sqrt{K(x,x)K(y,y)}}.$$

This removes the trivial diagonal growth and emphasises angular geometry.

5.3 Choice of frequencies

The injection

$$\eta : D \to \mathbb{Z}$$

is not canonical. Different choices produce different visual realisations while preserving the Gram matrix. Particularly interesting are:

- consecutive frequencies,
$$\eta(d) = 1, 2, 3, \dots;$$
- signed or symmetric frequencies,
$$\eta(d) \in \{\pm 1, \pm 2, \pm 3, \dots\};$$
- prime frequencies,
$$\eta(d) \in \{\pm 2, \pm 3, \pm 5, \dots\};$$
- Farris-type congruence families,
$$\eta(d) = k + m\lambda(d),$$
with $\lambda(d) \in \mathbb{Z}$ pairwise distinct and $\gcd(k, m) = 1$.

The last choice enforces global rotational symmetry while retaining substantial freedom in the individual frequencies.

6 The arithmetic meet matrix

We now specialise to the arithmetic meet kernel.

6.1 Sorted prime-factor lists and the meet

For $n \geq 1$, let

$$\pi(n) = (p_1(n), \dots, p_{\ell(n)}(n))$$

be the nondecreasing list of prime factors of n, with multiplicity. For example,

$$\pi(12) = (2, 2, 3), \qquad \pi(18) = (2, 3, 3).$$

Given

$$\pi(m) = (p_1, \dots, p_r), \qquad \pi(n) = (q_1, \dots, q_s),$$

define

$$m \wedge n := \prod_{i=1}^{\min(r,s)} \min(p_i, q_i),$$

with the convention that the empty product is 1. This yields the arithmetic meet kernel

$$K(m, n) = m \wedge n.$$

For a fixed N, set

$$X_N = \{1, 2, \dots, N\}, \qquad G_N = (K(m, n))_{1 \leq m,n \leq N}.$$

6.2 The poset description

Let $\preceq$ denote the order induced by sorted prime-factor lists:

$$d \preceq n$$

means that the sorted prime list of d has length at most that of n, and each entry is less than or equal to the corresponding entry of n.

Define the weights $g(d)$ by

$$n = \sum_{d \preceq n} g(d).$$

Equivalently, g is the Möbius-type weight determined by the finite meet-poset.

Now define the feature coordinates

$$\phi_d(n) := \sqrt{g(d)}\, 1_{\{d \preceq n\}}, \qquad 1 \le d, n \le N.$$

Proposition 6.1. *For all $m, n \in \{1, \ldots, N\}$ one has*

$$K(m,n) = \sum_{d=1}^{N} \phi_d(m)\phi_d(n).$$

Proof. By definition,

$$\sum_{d=1}^{N} \phi_d(m)\phi_d(n) = \sum_{d=1}^{N} g(d)\, 1_{\{d \preceq m\}}\, 1_{\{d \preceq n\}} = \sum_{d \preceq m,\ d \preceq n} g(d).$$

Since the common lower bounds of m and n are precisely the elements below their meet,

$$\sum_{d \preceq m,\ d \preceq n} g(d) = \sum_{d \preceq m \wedge n} g(d) = K(m,n).$$

□

6.3 Fourier/Farris curves for integers

Let

$$\eta : \{1, \ldots, N\} \to \mathbb{Z}$$

be injective. Define

$$\Psi_N(n)(t) = \sum_{d=1}^{N} \phi_d(n) e^{i\eta(d)t} = \sum_{d=1}^{N} \sqrt{g(d)}\, 1_{\{d \preceq n\}}\, e^{i\eta(d)t}.$$

Theorem 6.2. *For all $m, n \in \{1, \ldots, N\}$ one has*

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi_N(m)(t)\overline{\Psi_N(n)(t)}\, dt = K(m,n).$$

Proof. This is the general Fourier realisation theorem applied to the feature system

$$\phi_d(n) = \sqrt{g(d)}\, 1_{\{d \preceq n\}}.$$

Explicitly,

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi_N(m)(t)\overline{\Psi_N(n)(t)}\, dt = \sum_{d=1}^{N} \phi_d(m)\phi_d(n) = K(m,n).$$

□

Thus the integers

$$1, \dots, N$$

are represented by a finite family of closed planar curves

$$\Psi_N(1), \dots, \Psi_N(N) \subset \mathbb{C},$$

whose L^2 Gram matrix is exactly the arithmetic meet matrix.

6.4 Symmetric arithmetic realisations

If in addition the frequencies satisfy

$$\eta(d) \equiv k \pmod{m} \qquad (1 \le d \le N),$$

with $\gcd(k, m) = 1$, then every arithmetic curve satisfies

$$\Psi_N(n)\left(t + \frac{2\pi}{m}\right) = e^{2\pi i k/m} \Psi_N(n)(t),$$

hence every curve has full m-fold rotational symmetry.

Therefore the arithmetic meet matrix admits infinitely many symmetric Fourier realisations.

7 The prime-min kernel

A particularly transparent special case is the prime layer.

Let

$$p_1 = 2 < p_2 < p_3 < \cdots$$

denote the prime numbers and fix $k \ge 1$. Set

$$M_k = (\min(p_i, p_j))_{1 \le i,j \le k}.$$

Define

$$\Delta_1 := p_1 = 2, \qquad \Delta_j := p_j - p_{j-1} \quad (j \ge 2).$$

Let L_k be the lower-triangular matrix with ones on and below the diagonal, and let

$$D_k = \operatorname{diag}(\Delta_1, \dots, \Delta_k).$$

Then

$$M_k = L_k D_k L_k^\top.$$

Indeed,

$$(M_k)_{ij} = \sum_{r=1}^{\min(i,j)} \Delta_r = p_{\min(i,j)} = \min(p_i, p_j).$$

Therefore a canonical feature system is

$$\phi_r(p_j) = \begin{cases} \sqrt{\Delta_r}, & r \le j, \\ 0, & r > j. \end{cases}$$

Equivalently,

$$\Phi(p_j) = (\sqrt{\Delta_1}, \dots, \sqrt{\Delta_j}, 0, \dots, 0).$$

Choosing pairwise distinct integers

$$\omega_1, \dots, \omega_k,$$

define

$$\Psi(p_j)(t) = \sum_{r=1}^{j} \sqrt{\Delta_r}\, e^{i\omega_r t}.$$

Proposition 7.1. *For all $1 \le i, j \le k$ one has*

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi(p_i)(t)\overline{\Psi(p_j)(t)}\,dt = \min(p_i, p_j).$$

Proof. By orthogonality,

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi(p_i)(t)\overline{\Psi(p_j)(t)}\,dt = \sum_{r=1}^{\min(i,j)} \Delta_r = p_{\min(i,j)} = \min(p_i, p_j).$$

□

This is especially attractive because the amplitudes are controlled directly by the prime gaps.

8 Visual interpretation

The Fourier curves

$$\Psi_N(n)(t) = \sum_{d=1}^{N} \sqrt{g(d)}\, 1_{\{d \preceq n\}}\, e^{i\eta(d)t}$$

were sampled numerically and plotted in the complex plane.

Different choices of the frequency map

$$\eta : \{1, \ldots, N\} \to \mathbb{Z}$$

produce different visual realisations while preserving the Gram geometry. In the most symmetric examples, the frequencies lie in a single congruence class modulo m, following the Farris criterion.

One may therefore distinguish clearly between:

- the coefficients $\sqrt{g(d)}\, 1_{\{d \preceq n\}}$, which encode the arithmetic RKHS geometry;
- the frequency design, which controls visible symmetry and aesthetic form.

In this sense the drawings can be viewed as *planar projections of the finite arithmetic Hilbert geometry associated with the meet kernel.*

8.1 Farris-type frequency design

In the present implementation, each image uses a single congruence class of frequencies. More precisely, for fixed integers $m \ge 2$ and k with

$$\gcd(k, m) = 1,$$

the frequencies are chosen in the form

$$\omega_d = n_d = k + m\lambda_d, \qquad d = 1, \ldots, M,$$

where the integers λ_d are pairwise distinct.

Equivalently,

$$n_d \equiv k \pmod{m} \qquad \text{for all } d.$$

Thus every image is associated with one Farris symmetry type (m, k), while the actual arithmetic content is encoded in the choice of the integers

$$\lambda_1, \ldots, \lambda_M.$$

The freedom in choosing the λ_d makes it possible to reflect additional properties of the numbers $1, \ldots, N$, for instance multiplicative or poset-theoretic structure.

9 Iris

The following are example images: We call those images "iris".

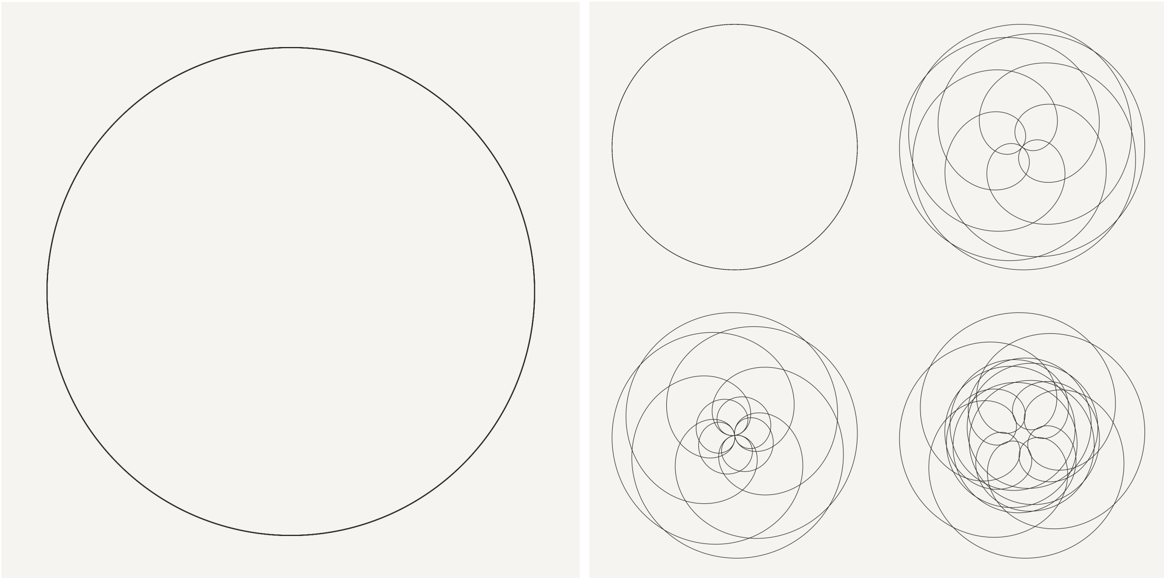

Figure 1: Fourier/Farris visualisations of finite RKHS points.

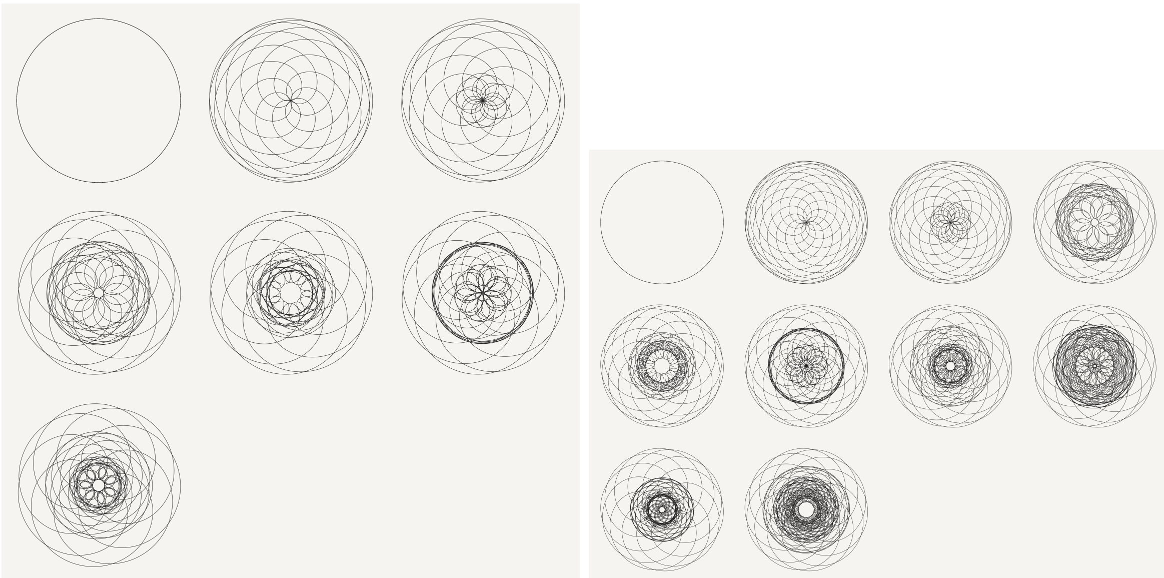

Figure 2: Further examples.

10 Remark on visual structure

Although the curves arise from a purely algebraic construction, their appearance is often highly structured, and at times strikingly artistic.

The visual complexity reflects the combinatorial structure of the prime-factor poset underlying the meet kernel, while the imposed congruence conditions on the frequencies govern visible rotational symmetry.

Thus the images may be viewed not merely as illustrations, but as geometric projections of arithmetic Hilbert space.

11 Further directions

The present construction suggests several continuations.

1. Optimise the frequency map η to enhance symmetry, reduce excessive self-intersections, or isolate arithmetic substructures.
2. Study the relation between spectral data of the Gram matrix and shape statistics of the associated curve family.
3. Compare different arithmetic kernels by comparing their Fourier/Farris curve systems.
4. Investigate prime-layer and higher poset-layer subconfigurations separately.
5. Search for canonical symmetric realisations in the sense of Farris.

12 Conclusion

A finite positive semidefinite kernel determines a finite family of RKHS points. By placing the feature coordinates into distinct Fourier modes, one can realise these points as trigonometric polynomials and hence as closed planar curves.

For the arithmetic meet matrix, this yields a concrete family of closed 2-dimensional curves attached to the integers

$$1, \dots, N,$$

with the property that their L^2 inner products recover the original meet kernel exactly.

Moreover, by imposing suitable congruence conditions on the frequencies, one may enforce Farris-type rotational symmetry. In this way finite arithmetic Hilbert geometry becomes visible as planar Fourier art.

References

[1] F. A. Farris, *Creating Symmetry: The Artful Mathematics of Wallpaper Patterns*, Princeton University Press, Princeton, NJ, 2015.